

BIRLA CENTRAL LIBRARY
PILANI (Rajasthan)

Class No. 539.11

Book No R 872 N.

Accession No 48286

NUCLEAR FORCES

BY

L. ROSENFELD

PROFESSOR OF THEORETICAL PHYSICS IN THE
UNIVERSITY OF MANCHESTER

FORMERLY PROFESSOR IN THE UNIVERSITY OF UTRECHT



1948

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM

PRINTED IN THE NETHERLANDS
Sole-Distributors for U.S.A.. Interscience Publishers, Inc., New York

Copyright 1948
N.V. NOORD-HOLLANDSCHE UITGEVERS MAATSCHAPPIJ
AMSTERDAM

TO THE MEMORY OF
JACQUES SOLOMON

Πρὸ γόων δὲ μνᾶσταις
SIMONIDES

PREFACE

...χρεὼν δέ σε πάντα πυνθέσθαι
ἤμην Ἀληθείης εὐκνκλέος ἀτρεμῆς ἥτορ
ἥδὲ βροτῶν δόξας, ταῖς οὐκ ἐν πίστις ἀληθῆς.

PARMENIDES. *Περὶ φύσεως*, Fragm. 1, 28-30.

*Now shalt thou hear of all things, of well-rounded Truth's
unmoved heart and of mortals' opinions, in which there is no
true assurance.*

The present book originated in an attempt to survey the available empirical data on various properties of nuclear systems with a view to deriving from it as much unambiguous information as possible on the character of the specific forces acting between nucleons. This enquiry had not proceeded very far, however, before it became clear that, in order to make it really useful, its scope had to be greatly enlarged. At first, I had imagined that it would have been possible to limit myself to a theoretical analysis of the experimental data without entering into detail about the methods by which such data were obtained. But I was soon confronted with the difficulty that no comprehensive and up-to-date account of the experiments was or could be expected in the next future to become available because of the rapid and unsystematic accumulation of material which had taken place during the last decade. Being forced to wade through this luxuriant but often swampy growth, I thought it would be useful to give the reader the benefit of this harassing experience, by incorporating into the account the critical surveys of different aspects of nuclear research which I had first attempted to make for my own information. It was clear that such a plan would not only involve a considerable increase in the size of the book, but would also tend to give it a somewhat hybrid character: on the one hand, it would retain its original purpose of developing a definite line of theoretical argument; on the other, by its presentation of a fairly complete summary of the experimental data, it would acquire the ephemeral quality of a reference book. This raised a problem of composition, of whose difficulties I was only too well aware; but it seemed to me that it had to be faced, as it appeared to correspond to the actual needs of the present situation.

In accordance, therefore, with the original plan, the arrangement and analysis of the material follows an inductive line of approach; but detailed, and accordingly lengthy, accounts of both experimental data and theoretical methods have been inserted at the appropriate places. This not only makes the progress of the discussion much slower, but even threatens to conceal the fairly simple thread of the argument running through the whole book; the latter defect I have tried to remedy by inserting wherever necessary

introductory remarks and summaries. Another drawback of the disposition adopted is that certain aspects of the subject have come to be treated in successive instalments, separated from each other by other considerations: for instance, the theory of the meson field is not presented in all its complication at once, but the various refinements are gradually introduced in immediate connexion with the features of nuclear forces which render them necessary or desirable. This is admittedly unsatisfactory for those readers who are looking for a self-contained survey of just one particular aspect. I have endeavoured to provide to some extent for this need by multiplying cross-references and extending the subject index. But keeping in mind the main purpose of the book, it was inevitable that the wish to bring out more sharply the relations of the different stages of the theoretical analysis to the experimental evidence should have overcome the exigencies of "elegance".

Although the general set-up is theoretical, and the emphasis laid throughout on the theoretical problems, I hope, also, that the experimental physicist — confining himself, if he so wishes, to those sections where the density of formulae is least — may find the discussion of some use. The theoretical treatment itself does not require more than elementary quantum mechanics; even so, it is in most cases fairly complete, except of course with regard to meson field theory, and in a few instances where reference is made to original papers for a fuller account of secondary points. I have always had in mind the advanced student, wishing to obtain a general view of the subject with concise introductions to its several parts, before engaging in research on some specific question. I have accordingly taken pains to unify and harmonize as much as possible the scattered contributions from various authors, the mushroom-like proliferation of which offers a bewildering variety in the choice of basic assumptions, numerical values of parameters, notations, and even terminology. I have, however, renounced unification in some cases where it would have involved an amount of numerical work out of proportion to the significance of the result. Likewise, it was not practicable to adhere strictly to the golden rule of using any letter or symbol in only one sense throughout; nevertheless it is hoped that no confusion is likely to arise in this respect. As regards terminology, I have been, of course, very sparing in the introduction of new terms; I have only to apologise for two neologisms: the one is the phrase *dichotomic variable*, adopted for didactic purposes to denote (roughly speaking) quantities with only two eigenvalues; the other is the collective name *lepton*, applied to particles of small mass (electrons and neutrinos). The need for such a collective term seems to be widely felt, and I hope that the word here proposed will meet with general acceptance; it was coined by Prof. Møller in consultation with a hellenist — a most happy instance of fruitful collaboration between science and humanities. Finally, it must be mentioned that in the compilation of the tables of empirical data, or in quoting results of numerical calculations, I was

occasionally confronted with the delicate problem of correcting the published values; it has not been found worth while, however, to make a special note of all cases in which the value appearing in this book differs slightly from that given in the original publication: the *very* distrustful reader can always check the figures if he likes.

It will not be necessary here to explain in more detail the lines along which the programme outlined above has been carried out; the main points are set forth in the Introduction. The most obviously objectionable feature of the book is undoubtedly its size, which, considering its limited scope, is certainly a bad sign. Referring to the motto borrowed from Parmenides' poem, the reader will indeed find that much more space is taken by uncertain recounting of "mortals' opinions" than by proud disclosure of "well-rounded truth"; and that our picture of nuclear forces is still very far from the ideal attained in atomic theory, where so very simple and neat a system of concepts and laws suffices to account, according to Dirac's famous phrase *, for "a large part of physics and the whole of chemistry". However, let this prove an incentive to youthful readers to try and give shape to new ideas suited to bring us salvation. In any event, I feel no regret that this laborious quest, conducted through so many heavy pages, should not end on any clear-cut conclusion. It is actually my warmest wish (a wish *not* shared by my publisher) that Part IV especially, which deals with non-central forces, should rapidly become still more obsolete than it already is.

Throughout the various stages of preparation of this work, I have been fortunate in enjoying most generous help and constructive criticism from many quarters. From the period of the first drafting, which carries us back to the chilly and famine-stricken Utrecht of the winter 1944-45, I remember, not without emotion, many discussions with my friends Opechowski and Lubanski, in which we found such comfort in those distressing times. Lubanski's premature death will long be felt as a great loss by those who had the opportunity of witnessing the unfolding of his talent and of appreciating his fine personality. Unpublished material of fundamental importance for the argument of the book has obligingly been provided by Dr. Hulthén, Mr. de Jager, Mr. Ramsey and, above all, Dr. Fröhlich and Dr. Powell; without this help it would not have been possible to treat in any reasonably adequate way the problems occupying large portions of Chapters VII, VIII and XIV. My thanks are due to Prof. Leprince-Ringuet, Dr. Powell and Dr. Roberts for supplying photographs for reproduction. The drafts of the figures, with the exception of a few prepared by Mr. D. J. Bouman, have been supplied by Mr. A. Wapstra. The figures themselves have been very skilfully drawn by the publisher's draughtsman, Mr. den Uyl, who took up his task with great devotion and industry; he contributed many helpful suggestions in difficult cases. To

* P. DIRAC, *Proc. Roy. Soc. A* **123**, 714, 1929.

Mr. Wapstra I am also indebted for the compilation of the large table of nuclei, embodying the latest available data *. Part of the manuscript was weeded of barbarisms by Miss S. Power and the Rev. O. Walsh, and this task was later completed on the proof by Mr. Huby; neither of these obliging helpers, however, can be held responsible for any linguistic errors which may have been introduced in last minute alterations. A proof has also been read with great care by Dr. Fröhlich, Prof. Heitler and Prof. Møller; to all of them I owe many important remarks and corrections. I have reserved to the last, however, mention of the collaborator to whom I am in the greatest debt: during the long drawn out process of preparing the manuscript for the press and proof-reading, Dr. Podolanski has devoted his whole time and energy to this tiresome task. It is impossible to assess the improvements which every page of the book has derived not only from his first-hand experience of typographical work, but also from his painstaking scrutiny of the text. His untiring patience and cheerfulness in the face of difficulties have been of no little comfort to me under sometimes trying circumstances. It is fitting to close this long list of acknowledgments with a word of praise for the publishers, who have bravely carried their undertaking to a good end in a most unpropitious time.

* Since this table was prepared only after the completion of the text, no references to it could be inserted to replace those to the tables of Mattauch and Flügge.

M a n c h e s t e r, December 1947.

NOTATIONS

N.1. Mathematical symbols

N.10. The following list includes such symbols as are not explained in the text, while their use is not universally standardized.

N.11. *Numbers and observables.* If A, B are any two numbers or (whenever this more general meaning is possible) two observables (operators),

$A \dots B$ denotes the interval between A and B , or some number in this interval;

A^* denotes the complex conjugate of A ;

$\Re A$ the real part of the complex quantity A ;

$A^\dagger$ the adjoint of the operator A ;

$\log A$ the natural logarithm of A ;

$\text{tr } A$ the trace of the operator A ;

$\text{av } A$ the expectation value (average value, mean value) of the observable A in some specified state;

$[A, B]_-$, or simply $[A, B]$, the commutator $AB - BA$;

$[A, B]_+$ the anticommutator $AB + BA$.

N.12. *Vectors and tensors.* When the tensor notation involving covariant and contravariant components is used, the usual rule of summation over indices occurring twice is adopted; the dummy index then occupies a covariant and a contravariant position.

Ordinary space vectors are denoted by arrows; the *vector product* of the vectors $\vec{a}, \vec{b}$ is represented by $\vec{a} \wedge \vec{b}$.

The symbol (cycl) following a relation which involves vector components means that the relations derived from that one by cyclic permutation of the vector components also hold.

N.13. *Functions of dynamical variables.* The letter Q denotes in a general way some set of coordinates pertaining to a particle; if we deal with a system of identical particles, distinguished by numbers $1, 2, \dots, i, \dots$, all quantities pertaining to the i -th particle are distinguished by the upper index (i) , e.g. $Q^{(i)}$.

If A, B are any functions of dynamical variables,

$A \approx B$ means A is approximately equal to B , or A is of the same order of magnitude as B ,

$A \sim B$ means A is proportional to B ,

$A \simeq B$ means A has asymptotic form B ;

$\dot{A}$ denotes the derivative of A with respect to ct , t being the time variable,

ΔA the Laplacian operator $\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$,

$\square A$ the D'Alembertian operator $\Delta A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2}$.

Volume integration is commonly denoted by $\int \dots dv$, the volume element $dx dy dz$ being represented by dv . If one wishes to specify the volume element attached to point P , one writes dv_P ; for that pertaining to the extremity of the radius vector $\vec{\xi}$, one writes $dv_{\vec{\xi}}$. According to this system of notation, the element of momentum space (the momentum being $\vec{p}$) will be represented by dv_p . The element of solid angle (in any space) is denoted by $d\Omega$.

A symbol of integration not accompanied by any symbol for a volume element represents an operation involving both summations over certain coordinates and integrations over other.

The Dirac δ -function with a vector as argument is used in the sense

$$\delta(\vec{k} - \vec{k}') = (2\pi)^{-3} \int e^{i(\vec{k} - \vec{k}') \cdot \vec{x}} dv.$$

N.14. Legendre polynomials and tesseral harmonics. A very convenient summary of the most useful facts and formulae of the theory of Legendre polynomials and tesseral harmonics is given by BETHE [33]. Following his example, we adopt Darwin's proposal of a uniform definition of Y_l^m for m -values of both signs. We list here for reference the general definitions:

$$P_l(x) = \frac{1}{2^l \cdot l!} \cdot \frac{d^l}{dx^l} (x^2 - 1)^l$$

$$Y_l^m(\vartheta, \varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} \cdot \sqrt{\frac{2l+1}{2} \cdot \frac{(l-m)!}{(l+m)!}} \sin^m \vartheta \left[\frac{d^m}{dx^m} P_l(x) \right]_{x=\cos \vartheta},$$

and the simplest special cases:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}};$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \vartheta, \quad Y_1^1 = \sqrt{\frac{3}{8\pi}} \sin \vartheta e^{i\varphi};$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{1}{2} \cos^2 \vartheta - \frac{1}{2} \right), \quad Y_2^1 = \sqrt{\frac{15}{8\pi}} \sin \vartheta \cos \vartheta e^{i\varphi}, \quad Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \vartheta e^{2i\varphi}.$$

N.2. System of references

N.21. Numbering of sections, etc. Each Chapter of the book is divided into sections and the latter are generally further subdivided. The subsections receive a decimal numbering, in which the first digit on the right of the dot refers to the section, and the following digits to the successive subdivisions of this section; on the left of the dot appears either a number which is that of the Chapter (with 0 for the Introduction), or (for the preliminary sections) a letter N (Notations) or U (Units and Constants), or a composite sign such as A2, referring to the Appendices. Tables and figures occurring in any subsection are designated by the number of that subsection, followed if necessary by an ordinal number; e.g.: fig. 3.221-2, table 6.22-1. Formulae are numbered through *in each section separately*; within the section they are referred to simply by their ordinal number between brackets. References to formulae of other sections include also the number of the *subsection* in which they occur; e.g.: formula (4.331-22). (Strictly, the number of the section would be sufficient, but the more precise indication of the subsection facilitates their location.)

N.22. References to literature. Reference to a book or paper is given by the name of the author, followed by the last two digits of the year of publication between square brackets; the necessary details will then be found in the author index at the end of the book, which is alphabetical with respect to authors' names and chronological for each author *. Several papers published within the same year are distinguished by letters *a*, *b*, *c*, ... In case of joint authorship, the paper is classified under the first name quoted; it has sometimes be found advisable to deviate from the order of names adopted by the authors themselves. E.g. the reference BREIT [37*b*] concerns a paper by S. SHARE and G. BREIT, which is more conveniently listed together with other papers by Breit and his collaborators, relating to similar subjects.

The abbreviations used to designate the scientific journals are the traditional ones, except in a few cases, which are explained at the beginning of the author index. Besides, some repeatedly quoted books or articles will be denoted in the text by the following signs:

B & B, H. BETHE and R. BACHER, *Rev. Mod. Phys.* 8, 82. 1936.

M & F, J. MATTAUCH and S. FLÜGGE, *Kernphysikalische Tabellen* (1941). This work is supplemented by a later article:

M & F*, S. FLÜGGE and J. MATTAUCH, *Physik. Z.* 44, 181, 1943 (with a correction *ibid.*, p. 391).

W & W, E. WHITTAKER and G. WATSON, *A course of modern analysis* (4th Ed., 1935).

* Some incidental references are directly given in full in footnotes.

UNITS AND CONSTANTS

U.1. Units

U.11. Relativistic correlation of units. According to the conceptions of relativity theory, a correlation can be established between the units of length and time on the one hand, momentum and energy, and mass and energy on the other, simply by taking the velocity of light as unit of velocity. The current practice of nuclear physics coincides only in part with this rational choice. While it is customary to measure masses and momenta in energy units, one keeps, e.g., to the second as the unit of time. In order to conform as far as possible to such usage, the following compromise has been resorted to:

(a) Mass and momentum are given the dimension of an energy and measured in energy units; i.e. the mass of a particle is identified with its rest-energy, while we call momentum the usual quantity of that name multiplied by the velocity of light.

(b) This convention involves similar modifications of derived quantities: an angular momentum will likewise be multiplied by c , a moment of inertia by c^2 ; in particular, we take as fundamental constant of "action" Dirac's constant $\hbar$ multiplied by c , which we denote by $\hbar$. In order to uphold an expression of the form $\hbar\nu$ for a quantum of energy, we characterize periodic changes (whose velocity of propagation is c) by their circular wave-number ν , i.e. the quotient of their circular frequency by c .

(c) On the other hand, we maintain the second as unit of time and (unless explicitly stated otherwise) the cm/sec as unit of velocity, so that the velocity of light will still appear explicitly in formulae involving times or velocities.

U.12. Mass and energy units. In most cases, the mega-electronvolt, MeV, is a convenient unit in which to express masses as well as energies and momenta. However, it may sometimes be advantageous to compare energies to the electron mass m , or masses to the physical mass unit, MU, defined as $1/16$ of the mass of the oxygen atom ^{16}O . The conversion factors are as follows (mMU denoting the milli-mass-unit):

$$\begin{aligned} 1 \text{ mMU} &= 0,93 \text{ MeV} \\ 1 \text{ MeV} &= 1,075 \text{ mMU} \\ m &= 0,51 \text{ MeV} = 0,548 \text{ mMU}. \end{aligned}$$

U.2. Constants

U.21. Fundamental constants. We collect in the following table the values of the fundamental constants establishing the correlation between atomic and nuclear dimensions, and between these dimensions and the macroscopic standards, together with some useful derived constants. The accuracy is limited to the present requirements of nuclear physics.

U.21. Fundamental constants		
Constant	Symbol	Value
<i>Relations to c.g.s. standards</i>		
Velocity of light	c	$3 \cdot 10^{10}$ cm sec ⁻¹
Quantum of action	$\hbar$	$1.97 \cdot 10^{-5}$ eV cm
Compton wave-length *	$\hbar/m$	$0.386 \cdot 10^{-10}$ cm
Boltzmann's constant	k	0.86 MeV/ 10^{10} degrees
<i>Dimensionless constants</i>		
Fine structure constant	$e^2/\hbar$	1/137
Mass ratio proton-electron	M_p/m	1836
<i>Derived constants</i>		
Electron radius	$d = e^2/m$	$2.82 \cdot 10^{-13}$ cm
Relativistic proton wave-length *	$\hbar/M_p$	$0.21 \cdot 10^{-13}$ cm
Inverse thereof *	$M_p/\hbar$	$4.76 \cdot 10^{13}$ cm ⁻¹
Inverse Compton wave-length *	$m/\hbar$	$2.59 \cdot 10^{10}$ cm ⁻¹
Nuclear Bohr magneton	$\mu_0 = e\hbar/2M_p$	$3.1 \cdot 10^{-11}$ eV/gauss
* Wave-length divided by 2π .		

U.22. Nuclear constants. It will be convenient to collect here the values of the parameters characterizing the main properties of nucleons and of the deuteron, which are among the best established nuclear constants.

U.22. Nuclear constants			
Constant	Symbol	Value	Reference
<i>Nucleons</i>			
Mass difference neutron-proton	$M_n - M_p$	2.47 m	(1.22-5)
Magnetic moment of proton	μ_p	2.7896	(1.21-3)
neutron	μ_n	-1.9103	(1.22-6)
proton + neutron	$\mu_d^0 = \mu_p + \mu_n$	0.8793	(6.12-10)
<i>Deuteron</i>			
Magnetic moment	μ_d	0.8565	(6.12-9)
Admixture of D state to ground state	$\sin^2 \omega$	0.04	(6.12-12)
Electric quadrupole moment	Q	$2.73 \cdot 10^{-27}$ cm ²	(6.12-7)
Binding energy of ground state	$ \epsilon_0 $	2.185 MeV	(6.11-1)
Energy of ^1S virtual level	$^1\epsilon^{(0)}$	0.065 MeV	6.431
Proton-neutron scattering radii (for zero range)	$^3a_0$	$0.437 \cdot 10^{-12}$ cm	6.431
	$^1a_0$	$-2.47 \cdot 10^{-12}$ cm	6.431

INTRODUCTION

0.1. An atomic nucleus is characterized by two integers, the *charge number* Z , expressing that its total electric charge is a multiple of the electronic charge, and the *mass number* A , according to which its total mass is approximately equal to A times the proton mass.

The interpretation of these properties which first presented itself, was to assume the nucleus to consist of A *protons* and $A - Z$ *electrons*. The phenomenon of β -disintegration, i.e. spontaneous emission of electrons, exhibited by some nuclei, seemed to support this view, although it appeared very difficult to account for the fact that the emitted electrons present a continuous energy distribution, while both the original and the product nucleus are in states of well-defined energy. In fact, the only way to save the principle of energy conservation in this case is to assume that the amount of energy necessary to restore the energy balance of the process is carried away, simultaneously with the emission of the electron, by some agent not reacting in any appreciable way with the environment; for this elusory function a particle of zero charge and zero (or very small) mass has been proposed, and given the name *neutrino*.

However, the simple conception of a proton-electron structure of nuclei proved insufficient in other respects. When a nucleus possesses a sufficiently stable fundamental state, in which it will subsist in atomic or molecular systems, it is possible to determine its *total angular momentum* * in this state, as well as its so-called *statistics*, i.e. the symmetry properties of the wave-function of any system of such nuclei with respect to the sets of coordinates pertaining to each of them. As is well-known, contradictions arose between the observed angular momentum and statistics of several nuclei and those predicted on the proton-electron structure theory. A consistent picture of nuclear constitution, embodying besides charge and mass number also the angular momentum and statistics, could only be obtained after the discovery of the *neutron*, an electrically neutral particle of about the same mass as the proton: assuming the neutron to have the same spin and statistics as the proton, it then became possible, in full agreement with the empirical data on angular momenta and statistics of nuclei, to conceive the nucleus (A, Z) as built up of Z protons and $A - Z$ neutrons. It will often be convenient, on this picture, to treat proton and neutron as two states — differing in charge and to a small extent in mass — of the nuclear constituent particle, which will be called a *nucleon*.

* The older denomination "nuclear spin" for the total angular momentum of a nucleus is liable to cause confusion and should be discarded. We shall reserve the name *spin* for the intrinsic angular momentum of an *elementary* particle.

In order, now, to account for the β -disintegration phenomena, a mechanism has to be devised, by which a nucleus, with adequate change of its charge number, would emit a (positive or negative) electron together with a neutrino. This can be achieved, in analogy with electromagnetic theory, by postulating a special kind of interaction between a nucleon and a pair of light particles, or *leptons**, consisting of an electron and a neutrino. Just as the existence of an interaction between charged particles and the electromagnetic field provides for the emission of photons by the charged particles, the interaction between nucleons and "lepton field" will give rise to the emission processes observed as β -decay.

If we now try to get a more quantitative account of the general picture of nuclear constitution just outlined, we are here — in contrast to the case of atomic constitution — left without any theoretical guidance comparable in reliability with the correspondence argument of quantum theory. The law of the forces acting between nucleons (apart from the ordinary Coulomb interaction of protons) cannot be inferred from any unambiguous theoretical reasoning and must ultimately result from the analysis of empirical data. For instance, the difference between the mass of a nucleus and the sum of the masses of the constituting nucleons, its so-called *mass-defect*, gives us directly (owing to the relativistic relation between mass and energy) the total binding energy of these constituents. On the other hand, the study of the *scattering* of nucleons yields valuable information on the law of interaction. With the help of this and other similar experimental evidence it already proves possible to restrict to a considerable extent the arbitrariness in the choice of a suitable law of nuclear force.

But in order to be really fruitful, such an analysis of the empirical material must clearly be combined with a discussion of the diverse possibilities which present themselves from more theoretical points of view. In particular, a theoretical approach to the problem of nuclear forces can be made on the line of the general *field* concept, well-known from the theory of electromagnetic forces. There is a great variety of *a priori* admissible "nuclear fields"; the field equations are conveniently assumed to be of such a form that, according to the fundamental relation between field (wave) and particle of quantum theory, the field may be associated with some kind of particle characterized by a given mass and spin, or with a pair of such particles appearing or disappearing together (like the "lepton field" discussed above in connexion with the theory of β -decay). Among these, a type of field introduced by Yukawa offers a special interest, since the associated particles (when charged) unexpectedly appear to be identical

* Following a suggestion of Prof. C. Møller, I adopt — as a pendant to "nucleon" — the denomination "lepton" (from *λεπτός*, small, thin, delicate) to denote a particle of small mass, irrespective of its charge; i.e. a lepton would be susceptible to two kinds of states, in which it appears as an electron and a neutrino, respectively. The word "electron" retains its original meaning of a particle of small mass with an elementary charge of either sign. When it is necessary to indicate the sign of the charge, the words "positon" and "negaton" may be used.

with the *mesons* which constitute the penetrating component of cosmic radiation.

0.2 When attempting to set up a general expression for the interaction energy between nucleons, one is faced with a number of questions concerning the dependence of this energy on the various parameters which characterize the state of the interacting particles.

(1) In the first place, one has to consider the dependence on the number of interacting nucleons: besides the usual interactions between pairs of particles, there may arise direct interactions between groups of more than two nucleons. In fact, such *many-body forces* are a direct consequence of any field theory; but on account of the unsatisfactory state of the quantum theory of fields, little can be said at present on the relative importance of the different many-body interactions. It seems, however, that the main features of nuclear constitution can be accounted for on the assumption that all but pair interactions play only a secondary part.

(2) The next question is, whether the nuclear forces may be regarded as predominantly *static*. In this respect, one has to examine their dependence not only on the translational velocities or momenta of the nucleons, but also on the motions of their spins and on the rate of exchange of electric charge between them (as expressed by the time variations of the coordinates characterizing their proton or neutron states). While the effect of the translational velocities can generally be assumed to be small, this is by no means so certain as regards the spin motions and charge variations, which very much depend on the form of field theory adopted. We shall in the following mostly deal with mainly static interactions.

(3) On this last assumption, the dependence of the static potential between two nucleons on the space, spin and charge coordinates has still to be investigated. Besides the distance between the two nucleons, one may expect also that the orientation of the spins with respect to the line joining the nucleons will enter into the expression for the nuclear potential, and it is a vexed question whether this potential is essentially *central* (only distance-dependent) or non-central.

Although we shall in the present work endeavour to follow as far as possible an inductive course of argument, inevitably part of the analysis of experimental evidence will be based on some definite hypothesis on nuclear interaction. We shall then treat most extensively the case of central interaction, which, on account of its greater simplicity, is especially suited to bring out the essential features of the argument. Afterwards, it will be possible to give a more cursory discussion of non-central and non-static interactions.

In this way, the book naturally divides itself into four parts. In the first one is attempted a general survey, of a more qualitative character, of the bearing of the empirical data regarding atomic nuclei on the problem of the nature of nuclear forces. This will immediately yield such features of

universal validity as the *limited range* and *limited binding power* of these forces (the latter property being quite analogous to the well-known *saturation* of the chemical valency bonds). Further, important indications can be obtained on the *charge dependence* of the nuclear interactions: the existence of forces between like nucleons * besides a proton-neutron interaction is disclosed and strong evidence is adduced of the general symmetry of nuclear force (exclusive of Coulomb repulsion between protons) with respect to the charges of the interacting nucleons.

Parts II and III are concerned with a more detailed analysis from the standpoint of central interaction. In Part II it is shown how the evidence of the deuteron and of proton-neutron and proton-proton scattering leads to a partial determination of the central potential. The chief result is the derivation of the so-called *charge independence* property of this potential: to a certain approximation, the proper nuclear interaction turns out to be the same for any pair of nucleons in the same configuration, irrespective of their charges. This fixes the analytical form of the potential as regards the dependence on spin and charge coordinates. The detailed distance dependence cannot be ascertained; but for any given law of force, the range and strength of the potential are more or less accurately fixed in magnitude. The numerical values of two constant parameters, entering into the dependence on spin and charge variables, remain as yet undetermined. They can only be fixed by the consideration of more complex nuclei, to which the following Part is devoted.

Part III, in fact, treats of the more or less adequate pictures of nuclear structure which can be outlined on the basis of central interactions. We have here especially to do with the calculation of the energies of the *stationary states of nuclei*, together with the discussion of other properties of these states needed for that purpose. It is found that the requirement that the nuclear potential exhibit the right saturation properties serves, as already mentioned, to complete the determination of the numerical values of the interaction parameters. To arrive at this result, a rough model of the nucleus suffices; it is then shown on what lines a more refined treatment can be attempted. In particular, properties of systems of three nucleons can be discussed to such extent as to afford an important corroboration of the conclusions drawn from the study of the saturation properties of the nuclear forces.

Finally, we discuss in Part IV the possible contribution of *non-central and non-static couplings* to the nuclear interaction. After a survey of the various possible types of such couplings which might be expected to occur, an analysis of the relevant empirical evidence, chiefly on the proton-neutron system, is carried out with a view to ascertaining in how far the effects concerned might either be attributable to relatively small deviations from central interaction or, on the contrary, reveal the essential inadequacy of the latter. Different theoretical possibilities are tested, but no definitive conclusion is reached.

* By *like nucleons* we always mean nucleons of like charge (protons or neutrons).

PART I

GENERAL FEATURES OF NUCLEAR FORCES

CHAPTER I

PROPERTIES OF ELEMENTARY PARTICLES

1.0. This preliminary chapter reviews in a general way the main properties of the elementary particles of different kinds, which will play a role in the following discussion.

1.1. Leptons

1.11. *Exclusion principle and anti-particles.* Electrons and neutrinos are elementary particles with a spin of half a unit, described by Dirac's well-known wave-equation. It is a consequence of very general features of quantum theory (PAULI [40]) that such particles obey the exclusion principle. This principle states that no two particles may be in the same state, i.e. have the same values for all their coordinates. It follows from this that the wave function of a system of such particles must be antisymmetric in all the sets of coordinates of the single particles; this symmetry property is expressed by the phrase "Fermi statistics". The formulation of the exclusion principle is logically possible only for particles of half-integral spin (but not for particles of integral spin); and it *must* be assumed to hold in order to prevent the occurrence of transitions of such particles to states of negative energy: this is achieved by introducing an infinite number of particles filling up all the states of negative energy, one in each such state. The particles in question are assumed not to produce any field as long as they are in the negative energy states, but to behave normally when they go over to a state of positive energy. When such a transition occurs, a "hole", i.e. an unoccupied state, is left in the distribution of particles of negative energy: such a hole will also give rise to observable effects, which can be attributed to an *anti-particle* of positive energy. Thus a hole in the distribution of negative electrons of negative energy will behave just like a positive electron. The transition of a particle from a state of negative to a state of positive energy is therefore to be interpreted as the creation of a pair consisting of a particle and an anti-particle of the kind considered: the inverse transition into an unoccupied state of negative energy corresponds to the annihilation of such a pair.

1.12. *Types of β -processes.* While electrons and anti-electrons are easily distinguished by the sign of their charge, there is no practical means of distinguishing neutrinos from anti-neutrinos. Since the expression for the interaction energy between a nucleon and a pair of leptons takes a simpler form when one of the leptons is associated with an anti-lepton of

the other kind, the process of β^- -decay is usually described as the emission of a negative electron and an anti-neutrino, while the β^+ -activity consists of the emission of a positon and a neutrino. The emission of an anti-lepton is here formally treated as the absorption of a lepton of negative energy. Besides the β^+ -decay, there is thus another competing process leading to the same kind of product nucleus: it is the absorption of a negaton, accompanied by the emission of a neutrino; as the negaton absorbed will usually be one in the K -shell of the atom, nearest to the nucleus, this phenomenon is called " K -capture".

1.13. Energy balances of β -processes. The energy balance of these different types of β -transitions is easily established in terms of the masses of the nuclei involved. The difference of these masses is equal, in the case of a β^- - or β^+ -transition, to the sum of the masses of the emitted leptons, plus their total kinetic energy, which is given by the maximum energy of the electron spectrum. From the study of the intensity distribution of this spectrum near the maximum energy, it can be concluded that the mass of the neutrino is very probably zero. Denoting by $M(A, Z)$ the mass of the nucleus of mass number A and charge number Z , by m the rest mass of the electron and by K_β the limiting kinetic energy of the β -spectrum, the energy balance may thus be written

$$M(A, Z) - M(A, Z \pm 1) = m + K_\beta. \quad (1)$$

Likewise, the balance of the K -capture process takes the form

$$M(A, Z) - M(A, Z - 1) = -m + K_\nu. \quad (2)$$

K_ν denoting the energy of the emitted neutrino; strictly speaking, one should add to this the binding energy (in absolute value) of the electron in the K -shell, but in comparison with the orders of magnitude involved in nuclear processes, such atomic binding energies are (except for very high atomic numbers) quite negligible quantities.

1.131. Stability with respect to β -transitions. It must be observed that except in the region of the heaviest nuclei, where the possibility of emission of α -particles has also to be envisaged, it is just the β -decay that conditions the distribution of stable nuclei along the whole range of mass numbers. Hence the importance of the above energy balances in allowing us to formulate the condition of stability of any nucleus with respect to β -transitions. This condition takes an especially simple form if the energy balances are written in terms of the *atomic weights*, i.e. the masses of the neutral atoms, instead of the masses of the nuclei. If the atomic weights are denoted by $W(A, Z)$, we have:

$$(A, Z) \xrightarrow{\beta^-} (A, Z + 1) : W(A, Z) - W(A, Z + 1) = K_\beta \quad (3)$$

$$(A, Z) \xrightarrow{\beta^+} (A, Z - 1) : W(A, Z) - W(A, Z - 1) = 2m + K_\beta \quad (4)$$

$$(A, Z) \xrightarrow{K} (A, Z - 1) : W(A, Z) - W(A, Z - 1) = K_\nu. \quad (5)$$

We thus see that the most stringent *stability condition* is derived from the consideration of β -decay and K -capture and can be enunciated as follows: For a nucleus to be stable against β -transitions, its atomic weight must be lower than those of the neighbouring nuclei of the same mass number, but with charge numbers larger or smaller by one.

1.132. Existence of the neutrino. Interesting evidence in support of the existence of the neutrino has recently arisen from various sides. CRANE and HALPERN [38, 39] observed the β -decay of $^{38}_{17}\text{Cl}$ in the cloud chamber by photographing the tracks a short time after the end of the expansion: the (diffuse) β -ray track yielded, by magnetic deflection, the momentum of the emitted electron, while a measure for that of the recoiling nucleus was obtained by counting the condensation droplets clustering around the spot where the disintegration had taken place. It appears that the conservation laws are not satisfied for the system consisting of β -ray and product nucleus only, the momentum taken up by the nucleus being in *all* cases too large.

Another method of detecting recoil atoms (or rather ions) consists in accelerating them by a controllable electrostatic potential towards some suitable collecting device. This method thus allows in principle of a determination of the energy distribution of the recoil atoms, but it is beset with considerable experimental difficulties. It has been tried for the first time by LEIPUNSKI [36a], who investigated the positron emitter ^{11}C ; in other forms, it has more recently been used by JACOBSEN and KOFOED HANSEN [45] with ^{88}Kr originating from Uranium fission, and by ALLEN [42] with ^7Be . This last instance is interesting inasmuch as one has to do with a K -capture transmutation, so that there is no complication due to emitted electrons and the evidence from the recoil nuclei is very direct indeed. The results of the last two experiments, although far from perfect, are in quite satisfactory agreement with the neutrino hypothesis.

Evidence on the emission of a neutrino in meson decay, which will be mentioned later (1.332), has also — when combined with the meson theory of β -activity (0.1, 1.322) — a bearing on the question and should accordingly be put on record here.

1.2. Nucleons

1.21. The proton. The proton being easily available as a hydrogen nucleus, its properties can be determined experimentally by the same methods as apply to the general investigation of nuclei. Thus it can be derived from the alternation of intensities in the band spectrum of the hydrogen molecule* that protons, like electrons, have a spin $\frac{1}{2}$ and obey the exclusion principle. The conclusion as regards the value of the spin has been confirmed by the method of magnetic deflection of atomic and molecular rays, developed by RABI and his collaborators**, — a method which at the same time yields the value of the magnetic moment.

* See, for instance, KOPFERMANN [39], Chapt. IV.

** First determinations by RABI *et al.* [36], more refined measurements by RABI *et al.* [39], MILLMAN and KUSCH [41]. A general account of the method has been given by KELLOGG and MILLMAN [46].

In view of these results, it is certainly tempting to assume for the proton a wave-equation of the same type as that established by Dirac for the electron, the only differences being the sign of the charge and the numerical value of the mass. Still, it must be admitted that such a procedure is not free from uncertainties. In fact, as has been pointed out by BOHR [34], Dirac's theory, being based on the idealisation of a point electron, may only be safely applied, for the treatment of proper relativistic effects (such as the spin properties), to particles of dimensions much smaller than the critical radius $\hbar/M$ (M being the mass of the particle) of the domain within which relativity features become preponderant. Now, the electron radius, defined by the condition that outside a region of such dimensions the electromagnetic field created by the electron approximates to that of a point charge, is of the order

$$d = \frac{e^2}{m}, \quad (1)$$

e denoting the elementary charge; the condition just enunciated is then fulfilled owing to the smallness of the fine structure constant $e^2/\hbar$. But the proton radius, determined by its proper nuclear field, turns out to be again of the order of magnitude d , which due to the smallness of the mass-ratio m/M_p (where M_p denotes the proton mass) is actually larger than $\hbar/M_p$.

From this point of view, it might be taken as an indication of the insufficiency of Dirac's theory for the proton that the value of the magnetic moment of the free proton (at rest) which follows from Rabi's measurements, mentioned above, is not that of a *nuclear magneton*

$$\mu_0 = \frac{\hbar e}{2M_p}, \quad (2)$$

predicted by the theory, but $\mu_p \cdot \mu_0$ with

$$\mu_p = 2,7896 \pm 0,0008. \quad (3)$$

It is true that this anomalous value of the magnetic moment can in principle be understood when it is realised that any nucleon, as the ultimate source of the lepton field responsible for the phenomena of β -decay, must be surrounded by a certain distribution of electricity. For instance, if we assume the lepton field to be directly produced by the nucleon, this field itself will obviously carry electric charge; we may, on the other hand, be led to assume the nucleon to be the source of some intermediate field, which in its turn gives rise to the lepton field — the meson field, as we shall see, affords an example of such an intermediate agent —; then, the intermediate field will also have to be charged. Provided the production of the field is assumed to depend on the spin of the nucleon, its distribution of electric current will be liable to give rise, on the average, to a magnetic moment in the direction of the spin. Such a magnetic moment

will be inseparably connected with the nucleon and will thus be attributed to it. On this picture, the total magnetic moment of the proton, directed along the spin axis, will in fact turn out to be larger than the normal value of one nuclear magneton by the amount $\mu_p - 1$. But this explanation of the magnetic anomaly makes it clear that a treatment based on Dirac's equation must at any rate be completed by the consideration of the effects of the proper (lepton or meson) field; and it may well be that this modification is nothing more than some convenient approximation to a more adequate description.

1.22. The neutron. The properties of neutrons are much more difficult of access than those of protons. Thus no direct determination of neutron spin or statistics can be obtained; but the indirect evidence on these points, resulting from the neutron-proton picture of nuclear constitution and well-established principles of quantum theory, is very strong indeed. In fact, if we assume the neutron also having a spin $\frac{1}{2}$ and obeying the exclusion principle, it follows from general theorems of quantum mechanics that any system of an odd number of nucleons must have a half-integral angular momentum and obey Fermi statistics, while any system of an even number of nucleons has an integral angular momentum and obeys Bose statistics (meaning that the wave function of such a system is symmetrical in the sets of coordinates pertaining to the constituent nucleons); and this rule actually holds good without exception for the numerous nuclei which have been investigated *. In particular, the deuteron, a nucleus consisting of a proton and a neutron, has been found (MURPHY and JOHNSTON [34]) to have an angular momentum 1 and to follow Bose statistics.

A precise value of the neutron mass M_n can be derived ** from a mass-spectrographic comparison (MATTAUCH [38]) of the hydrogen molecule with the deuterium atom, combined with the accurate determination of the binding energy of the deuteron (6.11-1). The previously mentioned mass difference is found to correspond to an energy of 1.434 ± 0.002 MeV, while the deuteron binding energy amounts to 2.185 ± 0.006 MeV. If the deuteron mass is denoted by M_d , we therefore have

$$2M_p - M_d + m = 1.434$$

$$M_n + M_p - M_d = 2.185,$$

whence (since $m = 0.51$ MeV)

$$M_n - M_p = 1.26 \text{ MeV} \quad (4)$$

or

$$M_n - M_p = 2.47 m. \quad (5)$$

* For the spins, see A2.21. For the statistics, see KOPFERMANN [39], p. 226.

** A critical survey of the complete evidence available on this point is given by STEPHENS [47].

In consequence of this large value of its mass, the neutron must be expected to be β -active. In fact, its spontaneous β -decay is energetically possible even if the mass of the neutrino should be as large as that of the electron. Applying to this case the general theory of β -disintegration of light nuclei, the life-time of the neutron can be estimated (A1.211) at about half an hour. This suffices to make it clear that the β -decay of the free neutron cannot be observed under the ordinary experimental conditions of production and detection of these particles*.

The fact that the neutron possesses a magnetic moment is perhaps even stranger, in view of the absence of electric charge, than the anomalous value of the proton magnetic moment. It can, however, be accounted for exactly in the same way, at least in principle. It may even be seen that the simplest assumptions as regards the interaction between a nucleon and its charged field lead, in a first approximation, to magnetic moments of the same absolute value, but opposite signs, for a proton and a neutron state not differing in other respects. The neutron will therefore be expected to exhibit a magnetic moment in the direction opposite to that of the spin, and approximately equal in absolute value to $\mu_p - 1$ nuclear magnetons. The first direct measurement, performed by ALVAREZ and BLOCH [40a], gave for the neutron magnetic moment $\mu_n = -1.935 \pm 0.02$ nuclear magnetons, the minus sign indicating that its direction is opposite to that of the spin. This value is now superseded by the much more accurate one,

$$\mu_n = -1.9103 \pm 0.0012, \quad (6)$$

obtained in the Argonne laboratory (ARNOLD and ROBERTS [46]). Comparing with (3), we find so far a satisfactory agreement with the theoretical predictions; the remaining small asymmetry can be ascribed to higher order effects. Unfortunately, in the present state of quantum field theory, it is impossible to give a more quantitative treatment of the whole effect, leading to a determination of the numerical values of μ_p and μ_n , without making use of some hazardous procedure to secure a convergent result.

1.23. The anti-proton. If we are to take the application of Dirac's theory to nucleons seriously, we have to accept the existence of *anti-nucleons*, in particular of *anti-protons*, i.e. protons of negative charge. Of course, anti-protons cannot co-exist with ordinary protons in stable configurations, owing to the tendency of proton pairs to annihilation, e.g. with emission of light quanta (like electrons). But they might penetrate the earth's atmosphere as a part of cosmic radiation or be produced in the atmosphere by other energetic particles of this radiation.

The production of a pair of protons in any process involving electromagnetic interactions will have a probability roughly $(m/M_p)^2 \approx 3 \cdot 10^{-7}$

* Negative experimental evidence on this point is afforded by GILBERT, SMITH and FREMLIN [37].

times smaller than the analogous production of an electron pair (of the same velocities); but there are other possibilities of proton pair production due to the nuclear field, especially if the latter is identified with the meson field observed in cosmic radiation (0.1). From a careful discussion by MCCONNELL [45] * it results that the cross-section for proton pair production by fast mesons, although of a larger order of magnitude than those of electromagnetic processes, always remains very small indeed. On the other hand, the annihilation in matter of anti-protons with emission of mesons or photons has also a quite small probability, unless the annihilating protons are very slow; so that more prolonged observations than hitherto performed of the proton component of cosmic radiation at low altitude should at length reveal the existence of anti-protons produced at higher levels. However, the difficulty of detecting these anti-protons is considerably increased by the occurrence of another process of relatively large probability, viz. that by which a proton, — negative as well as positive —, in traversing matter, loses its charge by emitting a meson. It is interesting to note that for very slow protons and anti-protons, the annihilation to the form of photons will become predominant; the resulting life-time of an anti-proton at rest being estimated at about 10^{-4} sec.

As to the occurrence of anti-protons in the primary cosmic radiation, a case has been presented by ARLEY [45, 46a, 46b] in favour of this eventuality. By annihilation in the atmosphere, the anti-protons would ultimately give rise to the soft component of the cosmic radiation, while the ordinary protons are the origin of the hard component. Again, it should be possible to observe at sea level some of the primary anti-protons. However, ARLEY does not pay sufficient attention to the process of meson production by the negative protons, the consideration of which would seem to upset his argument **.

The primary cosmic radiation itself has recently been the subject of an interesting hypothesis, put forward by KLEIN [45] ***. He observes that part of the galaxies may well consist of "reversed matter", i.e. of atoms built up of anti-nucleons and anti-electrons; such atoms could in any case not be distinguished from ordinary ones by spectroscopic properties. In the intergalactic regions, collisions may then occur between ordinary and reversed atoms, strayed from their respective galaxies. If the two atoms have the same number of constituent particles, their complete annihilation will result in a radiation composed of mesons, photons and leptons; if they have different constitutions, part of the annihilation energy will also liberate nucleons and anti-nucleons (the neutrons and anti-neutrons soon transforming, owing to their β -radioactivity, into protons and anti-protons). In this way, the composition of the primary cosmic radiation would at least

* See further MCCONNELL and JÁNOSSY [47].

** This has kindly been pointed out to me by Prof. HEITLER.

*** See also ARLEY's [45] criticism.

qualitatively (also as regards the order of magnitude of the energies of the different kinds of particles) be accounted for.

However this may be, we have, in the treatment of nuclear systems, only to do with an approximation involving at most effects of the first order in the nucleon velocities*; in these cases, the uncertainties connected with the use of Dirac's equation in the proper relativistic domain do not occur.

1.231. The anti-neutron. At first sight, it might be thought that the anti-neutron is undistinguishable from the ordinary neutron. But it must not be forgotten that its proper nuclear field manifests itself by a magnetic moment, which (for the simple forms of nuclear fields usually adopted) turns out to have opposite signs for neutron and anti-neutron. The theory thus provides for the creation and annihilation of pairs of neutrons of opposite magnetic moments through the interaction with the nuclear (meson) field.

1.3. Mesons

1.31. The field concept. The use of the field concept to account for the interaction between particles will most conveniently be explained with the help of the classical example of electrodynamics. It is well-known how, according to Faraday's and Maxwell's ideas, forces acting between electric charges or the Ampère force between elements of current, are derived from a process of transmission of force by means of an electromagnetic *field*, produced by the charges or elements of current and acting on them. A convenient formal representation of this process is afforded by the electromagnetic *potential*. This is a four-vector A_i , the components of which satisfy the differential equations

$$\square A_i = -4\pi s_i, \quad (1)$$

where s_i denotes the four-vector density of current and charge. Limiting ourselves to the stationary case, we have to do with a set of Poisson equations

$$\Delta A_i = -4\pi s_i, \quad (2)$$

and we can express the solutions we want with the help of the corresponding *Green's function*. This function is a solution φ_0 of the homogeneous equation

$$\Delta \varphi_0 = 0 \quad (3)$$

with a point singularity, i.e. representing physically the potential of a point source. In our case (boundary condition: potential vanishing at

* By this abbreviated expression we mean, here and in the following, the order of magnitude of the ratio between the mean value of the velocities of nucleons in the system considered and the velocity of light.

infinity) we have simply

$$\varphi_0(P'; P) = \frac{1}{r}, \quad (4)$$

if r denotes the distance between the point P and the point P' where the singularity is situated. The solutions of the Poisson equations (2) are then given by

$$A_i(P) = \int s_i(P') \varphi_0(P'; P) dv_{P'}, \quad (5)$$

expressing how each volume-element in which a source of strength $s_i(P') dv_{P'}$ is located yields an additive contribution to the potential at point P .

Now, the interaction energy between the sources and the field defined by the potential is given by

$$\mathcal{V}_{\text{el. magn.}} = -\frac{1}{2} \int A_i(P) s^i(P) dv_P. \quad (6)$$

Eliminating the potential, we get

$$\mathcal{V}_{\text{el. magn.}} = -\frac{1}{2} \int s_i(P') dv_{P'} \varphi_0(P'; P) dv_P s^i(P), \quad (7)$$

i.e. just the usual expression of the potential energy of the above-mentioned forces acting directly between distant sources. The field has served as an intermediary to connect the sources at different points P , P' and has then disappeared from the final result.

1.32. The meson field. As we shall see presently, a fundamental property of the nuclear forces, contrasting with those of electromagnetic interactions, is their *limited range*. The representation of short range forces by means of a field can of course be achieved in many ways, but one of the simplest is the following. Instead of the Laplace equation (3) for the determination of Green's function, let us consider the equation

$$\Delta \varphi - \kappa^2 \varphi = 0; \quad (8)$$

the corresponding Green's function is

$$\varphi = \frac{1}{r} e^{-\kappa r}, \quad (9)$$

which is just suited to represent a force not extending appreciably beyond distances of the order κ^{-1} . Now, the differential equations for the potential components of the most general field of this type are easily set up: they will clearly be of the form

$$\square \psi - \kappa^2 \psi = -4\pi s. \quad (10)$$

Here the covariance properties of the potential ψ and the source density s are left entirely open; an equation of the type (10) must be understood to

hold separately for each component of ψ and the corresponding component of s .

The remarkable property of this particular form of field of finite range is that the homogeneous field equations

$$\square\psi - \kappa^2\psi = 0, \quad (11)$$

which describe the pure field in absence of any source, are formally identical with the wave-equations of a particle of mass M_m , this last quantity being connected with the range constant κ by the relation

$$\kappa = \frac{M_m}{\hbar}. \quad (12)$$

In other words, the field of limited range here considered is associated, according to the principles of quantum mechanics, with a certain kind of particle, whose mass is related to the range of the field by equation (12). This is the essential point stressed by YUKAWA [35].

The range of the nuclear forces being of the same order of magnitude as the fundamental length d , defined by (1.21-1), the mass of the particles associated with the nuclear field should, by (12), be of the order of $\hbar/e^2 \approx 137$ times that of the electron, thus intermediate between the masses of electron and nucleon: hence the name *meson* proposed for such particles, and the corresponding denomination of "meson field" for the wave aspect defined by equations (11). The covariance properties of the meson field determine the intrinsic angular momentum, or spin, of the meson; apart from a spinor field (like that of the electron), corresponding to a spin $\frac{1}{2}$, the simplest possibilities are the scalar and pseudoscalar* fields of spin 0 and the vector and pseudovector fields of spin 1.

1.321. The mass-range relation. In consideration of its importance, it will be worth while discussing Yukawa's relation (12) between particle mass and field range from a less formal point of view. It will appear that this universal relation is deeply rooted in the fundamental principles of quantum theory. For this purpose it will be convenient to introduce the concept of *virtual transition*. This we can again do by starting from the case of electrodynamics.

The particles associated with the electromagnetic field are the *photons*. We can try to describe the mechanism of transmission of force by the electromagnetic field in corpuscular language, by making use of the photon concept. We should then like to say that the sources located in one definite volume-element emit photons, which are subsequently absorbed by the charges and currents contained in another volume-element and in this

* The prefix *pseudo* indicates a difference of behaviour of the quantity considered under symmetry transformations (reflexions) with respect to a point (or plane). While a scalar remains invariant under such transformations, a pseudoscalar changes sign; more generally, a pseudotensor transforms like the product of a tensor and a pseudoscalar.

way transport momentum and energy from the first volume-element to the second. But we must be very careful with such a picture. There can be no question of *actual* emission and absorption processes, because such processes are in this case not permitted by the precision with which the energies and times involved can be defined. Indeed, in order to disclose an actual transition of a photon between two volume-elements at a distance r from each other, we should need to be able to capture such a photon on its way and to identify it as such, i.e. to attribute to it a definite frequency. However, this is impossible: since the time at our disposal for the capture and identification of the photon is at most r/c , we cannot define its energy better than with a latitude ΔE given, on account of the uncertainty relation $\Delta E \cdot \Delta(ct) \approx \hbar$, by

$$\Delta E \approx \hbar/r,$$

which means for the (circular) wave-number ν an uncertainty

$$\Delta \nu \approx 1/r.$$

If, nevertheless, we wish to use the corpuscular language, we must therefore not lose sight of the symbolical character of the imagined transport of energy by photons; and of this we are reminded by the phrase "*virtual* photon transition". Such virtual transitions are to be sharply distinguished from the actual ones; the latter must always, in contrast to the former, be compatible with the law of energy conservation.

Let us now consider the transmission of force by a field associated with particles of mass M_m . We can describe this transmission process, in the corpuscular language, as a sequence of virtual emissions and absorptions of particles by the sources. But there is a condition to be fulfilled for the consistency of this picture. We have to express the fact that it must be impossible to identify the particle on its way in the time between the emission and the absorption. This time is here r/v (v being the velocity of the particle) and the corresponding uncertainty in the definition of the particle energy, $\hbar v/rc$, must be larger than M_m :

$$\hbar v/rc > M_m.$$

or, since $v \leq c$,

$$r < \frac{\hbar}{M_m}.$$

The meaning of this inequality is that our transmission mechanism cannot extend beyond a distance $\hbar/M_m$. The relation between range of the force and mass of the particle is thus seen to be a rather direct consequence of the fundamental uncertainty relation for energy and time*.

We here meet — as emphasized by the concept of virtual transition — with a "non-visualizable" feature of quantum theory of exactly the same

This argument is due to WICK [38].

kind as the fact that no mechanical orbit can be attributed to an electron in a stationary state of an atom. And this particular instance just affords a striking confirmation of the contention that the limitation of applicability of classical concepts, expressed by the uncertainty relations, in reality gives us a greater freedom to introduce new conceptions, which would not be possible from the classical point of view. We see, in particular, what a considerable extension the field idea gets in quantum theory, where every kind of particle is automatically associated with a field and thereby provides new possibilities of transmission of force, i.e. of interaction*.

1.322. Interaction of meson field with leptons. The description of nuclear interaction provided by the meson field can, as pointed out by YUKAWA [35], be completed so as to include the various kinds of β -processes. The meson field will in fact transmit an interaction also between nucleons and leptons, provided a direct coupling of this field with the leptons is introduced. The latter coupling can be chosen in such a way that the interaction between nucleons and leptons to which it gives rise will imply the occurrence of the β -transitions. The emission of a lepton pair by a nucleon, for instance, is on this view conceived as a virtual emission of a charged meson by the nucleon, followed by the virtual transmutation of the meson into a lepton pair.

It must now be observed that besides the virtual processes involved in the interpretation of nuclear forces and β -decay, the meson field theory predicts the occurrence of the corresponding actual processes as soon as the energy eventually required is available. If this theory has any physical reality, we should e.g. expect mesons to be emitted by nuclei in states of sufficient excitation. The interaction between mesons and leptons further implies that such mesons should spontaneously disintegrate into a pair of leptons, consisting of an electron and a neutrino.

1.33. The mesons of cosmic radiation. It was therefore a triumph of Yukawa's theory when charged particles of intermediate mass were actually discovered in cosmic radiation. The remarkable agreement between the general properties of these particles and those required in meson theory was further enhanced by conclusive evidence as to the spontaneous decay of the cosmic particles. It is true that as soon as we try to go into a more detailed specification of meson properties, especially as regards their charge and spin, in order to account for a larger number of peculiarities of nuclear structure, the apparent simplicity of the meson field conception

* The application of the extended field conception of quantum theory to types of quantum interactions already known, such as the chemical valency forces and the peculiar interactions connected with creation and annihilation of electron pairs, has been discussed by HUND [39, 40] (see also BLASS [44]). In this connexion it must be observed that in contrast with the case of stationary fields, discussed here, non-stationary fields will have longer ranges than κ^{-1} . It is easy to see that a field of angular frequency $c\nu$ has a range $(\kappa^2 - \nu^2)^{-1/2}$: an example is given by the chemical forces (HUND [39]).

disappears and it becomes increasingly difficult to secure complete harmony with the empirical data derived from cosmic radiation. Still, from the point of view of field theory, as outlined above, we may state that the new cosmic particles must at least partly be responsible, through their field properties, for the nuclear interactions and β -processes, and in this sense we are fully entitled to identify them with the mesons theoretically introduced by Yukawa. We must of course be prepared for the eventuality that other kinds of mesons, which for some reason escape observation under present conditions, may also play a part, perhaps even the main part, in bringing about the interactions which govern the behaviour of nucleons and leptons.

Leaving such matters for later discussion, we shall now, from the rather scanty evidence on cosmic ray mesons, only discuss the most firmly established features. We shall be concerned exclusively with *charged mesons*; charges of both signs being about equally represented*.

1.331. The meson mass. The determination of the meson mass from cloud chamber photographs of meson tracks curved in a magnetic field is a precarious affair, owing both to the errors of difficult estimation to which the measurement of the curvature of the track is subject**, and to the large uncertainties affecting the different methods of determination of the velocity of the particle. The latter quantity may be derived from an estimate of the ion density in the track, from a measurement of the range of the meson or of its velocity loss on traversing a dense screen in the cloud chamber, or finally from the analysis of an elastic collision of a meson with an atom, resulting in the expulsion of an electron from the atom. A critical discussion of these various methods (GORODETZKY [42, 46], RICHARD-FOY [45], HUGHES [46]) reveals that the best precision to be hoped for without spending an undue amount of time in waiting for a larger number of suitable photographs to accumulate would not exceed 10 % and that the error of the mass determination is in most cases far greater.

It is therefore not surprising that the results so far obtained exhibit a very wide spread. In fact, critical surveys of older measurements, carried out by WHEELER and LADENBURG [41c] and by BETHE [46], as well as the latest measurements by HUGHES [46] and FRETTER [46], all lend strong support to the conclusion that this spread is not due to an actual spectrum of meson masses, but can be wholly ascribed to experimental uncertainties. As regards the probable value of the unique meson mass, estimates differ somewhat: Bethe favours a value of 200 m ; for 26 determinations spread between 142 m and 264 m , Fretter indicates a

* The evidence establishing that cosmic ray mesons carry one elementary charge is discussed by GORODETZKY [45].

** BETHE [46] has especially emphasized the large influence of multiple scattering on the curvature of the track.

weighted average of 202 m , while 10 tracks obtained by Hughes yield an average of 229 m , with a spread extending from 140 to 340 m . One of the most reliable determinations seems to be that based on a remarkable photograph obtained by Leprince-Ringuet and his collaborators (plate I), and showing both an elastic collision with an atom and the energy loss on traversing a lead plate; the discussion of this case leads to a value

$$M_m = (240 \pm 40) m, \quad (13)$$

which, as long as no definite evidence to the contrary is forthcoming, we may assume as the unique mass value of all charged mesons *observed at sea level*.

LEPRINCE-RINGUET and LHÉRITIER [46a] have obtained an elastic collision photograph, taken at 1000 m altitude, from which a particle mass of $(990 \pm 120) m$ is deduced*. They later [46b] found some indirect confirmation of this result in four cloud chamber tracks of abnormal ionizing power, indicating a mass intermediate between those of the proton and the "ordinary" meson. Such "heavy mesons", if they exist, would give rise to nuclear interactions of correspondingly short range, which would probably not be disclosed by the evidence at present available.

1.332. The meson decay. Owing to their finite life-time, the mesons observed at sea level cannot be part of the primary cosmic radiation. But the minimum energy, of about 120 MeV, required for the creation of such a meson, is of the order of magnitude met with in the primary radiation, and there is no difficulty in accounting for the production of the observed mesons in the atmosphere. The life-time of the mesons reaching sea-level can be accurately measured by a remarkable method, utilizing to the extreme limit of their efficiency the present resources of experimental technique.

By a system of coincidence and anti-coincidence registrations of counters placed above and below a block of light material, the behaviour of *slow mesons*, penetrating into this "absorber" but not traversing it entirely, can be studied. A series of counters disposed along the sides of the absorber can record any electron emitted in the course of an eventual decay of the meson, following its entry into the absorber. It is now possible to connect the side counters with the top- and bottom counters in such a way that "retarded coincidences" between the arrival of the slow meson and the emission of the decay electron are recorded; i.e., in order to be recorded, the two events mentioned must be separated by a time interval not larger than a certain time, which can be fixed with great accuracy. From measurements performed for different values of this retardation time (of the order of the microsecond), a decay curve can clearly be derived, giving the mean life-time of the slow mesons. This method has been developed independently by RASETTI [41] and by MAZE, CHAMINADE and FRÉON [45]; it has also been applied by CONVERSI and PICCIONI [44, 46a]. A still

BETHE [46] argues, however, that this particle could be a proton.



Fig. 1.331. Cloud chamber track of a meson experiencing an elastic collision in the gas and an energy loss in a lead plate (LEPRINCE-RINGUET *et al.*; see GORODETZKY [42]).

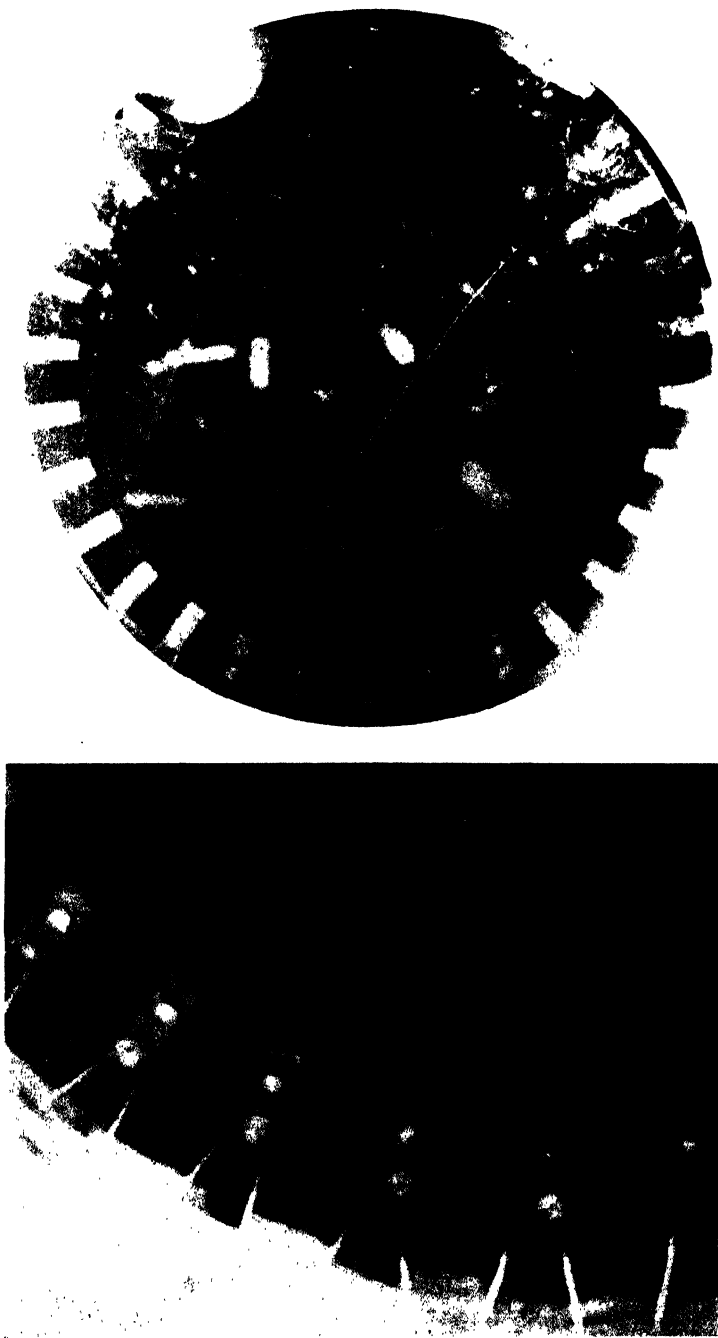


Fig. 1.332. Cloud chamber photograph of a meson decaying with emission of an electron (WILLIAMS and ROBERTS [40a]).

more refined variant of it is due to NERESON and ROSSI [42, 43]: in their apparatus, a special device permits measuring in each case the time-interval separating the recorded events. In this way, a large number of points on the decay curve is obtained and the accuracy of the life-time determination accordingly increased. The results of these difficult measurements are remarkably concordant; the mean life-time of a charged meson at rest (of the kind observed at sea level) is $(2,15 \pm 0,07) \cdot 10^{-6}$ sec.

An interesting feature of these experiments is that only about one half of the mesons stopped in the absorber are found to decay with emission of an electron (see especially CONVERSI and PICCIONI [46b]). This result is interpreted by noting that negatively charged mesons have a larger probability of being captured by nuclei in the absorber. However, further investigation of this aspect of the question has quite recently brought to light a very puzzling phenomenon. Studying the behaviour of slow mesons of both signs, separated by passage through magnetized iron plates, CONVERSI, PANCINI and PICCIONI [47] could detect no decay of the negative mesons when they were stopped in an iron absorber, but did record a production of a number of electrons nearly equal to that of the mesons slowed down, when the absorbing material was carbon. Preliminary experiments of SIGURGEIRSSON and YAMAKAWA [47] seem to disclose a similar anomaly for Be in contrast to a "normal" behaviour of S. According to a suggestion of BOHR, this effect would be due to a quite unusual mechanism of capture of the slow negative meson by atoms of small charge number: there would in fact be a large probability of its being retained on an "orbit" of large angular momentum (and consequently with small chance of capture by the nucleus) during a time sufficiently long to permit its decay into leptons.

An immediate consequence of the finite life of the meson is the dependence of the rate of absorption in the atmosphere of the hard component of cosmic radiation on the actual distance of air traversed. On this property, from which the first experimental evidence of meson decay was derived, various methods of estimating the order of magnitude of the meson life-time are based. Since we deal here with mesons of given momentum, the quantity which can be inferred from all such measurements is not the life-time t_0 of the meson at rest, but the quotient of this constant by the meson mass. Critical discussions of the relevant evidence have been given by FRÉON [45] and OPECHOWSKI [43a,b]; they must be supplemented by the recent work of COCCONI and TONGIORGI [44, 46], which would seem to dispell any doubt as to the constancy of t_0/M_m when the distance traversed by the mesons varies. This means that so far as this evidence goes, it is concerned with only one kind of mesons, for which, according to the last named authors,

$$t_0/M_m = (2,6 \pm 0,3) \cdot 10^{-6} \text{ sec}/100 \text{ MeV.}$$

Such a result is in fair agreement with the direct measurements of t_0 and M_m . It is interesting to note that the indirect life-time determinations have furnished the first direct verification of the relativistic law of time dilatation.

Cloud chamber photographs of decaying mesons, first obtained by WILLIAMS *et al.* [40a,b] * (plate I), afford very strong evidence for the emission of a neutrino together with the electron. It should be emphasized, however, that this evidence is as yet only of a qualitative character; it has not been proved that the energy and momentum balance can always be restored by the assumption of the emission of a particle with specified properties. For this purpose, a more systematic study is required.

1.333. *The meson spin.* It has been possible to derive the value of the spin of the mesons observed at sea level from the study of the cascade showers produced by such mesons in a layer of lead. Mesons can initiate showers either by accelerating an electron in a collision or by emitting photons ("Bremsstrahlung") in passing near a nucleus. The yield of these processes turns out (CHRISTY and KUSAKA [41], KUSAKA [43]) to be quite sensitive to the value of the meson spin. Mesons of spin 1 would, according to the theory, produce many more soft secondaries than actually found in experiment. The evidence (LAPP [43, 46], GEORGE [46]) definitely points to the value 0 for the spin of sea-level mesons.

It must, however, be mentioned that neither the measurements nor the theoretical calculations have as yet attained such a degree of accuracy as to exclude completely the possibility of a spin $\frac{1}{2}$. If the mesons had spin $\frac{1}{2}$, they could transmit interactions between nucleons without violating conservation of angular momentum only if they were emitted and absorbed in pairs. Pair field theories, however, have not been successful in accounting for detailed features of nuclear forces; they will not be considered in this book.

1.34. *Many-body forces.* It is clear that the mechanism of transmission of forces provided by the meson field will not only lead, as explained above, to an action at a distance between a pair of particles, but also to such direct interactions between groups of more than two particles. Take, for example, a system of three nucleons. In first approximation the interaction energy will be the sum of the interactions between the nucleon pairs (1)–(2), (2)–(3), (3)–(1). These interactions involve virtual processes of the type: nucleon (1) emits (virtually) a meson, which is absorbed by nucleon (2). The next approximation will first of all bring corrections to these pair interactions, arising from virtual processes such as: nucleon (1) emits two mesons which are both absorbed by nucleon (2). But in addition, it will also involve terms corresponding to the following kinds of virtual processes:

- (a) nucleon (1) emits two mesons, the one of which is absorbed by nucleon (2), the other by nucleon (3);
- (b) nucleon (2) and nucleon (3) each emit one meson; both mesons are absorbed by nucleon (1).

Another example has been published by SHUTT *et al.* [42].

Such terms will be typical three-body interactions, depending on the mutual distances (1)–(2) and (1)–(3).

A detailed discussion of these many-body interactions unfortunately lies outside the range of well-established validity of present field theory: like other effects of higher approximation they cannot be unambiguously separated from the divergencies occurring in these higher approximations. Nevertheless, if one disregards this fundamental objection, it is possible, on meson theory, to set up the analytical expression of the three-body interactions and to estimate their order of magnitude in stationary states of nuclear systems (PRIMAKOFF and HOLSTEIN [39], JÁNOSSY [39]); compared with the static pair interactions they are found to be at most of the first order in the nucleon velocities. While showing that the many-body forces are in general of secondary importance, this result at the same time warns us that we should not lose sight of them when discussing finer details of nuclear systems of more than two nucleons.

1.35. *Excited states of nucleons.* As already explained, a nucleon is always inseparably bound with its proper meson field, which may carry a certain quantized amount of angular momentum or electric charge, with a correspondingly large amount of energy. This means that the nucleon is in principle susceptible of a whole set of stationary states of different excitation, in which it exhibits values of spin and charge differing from those of its normal or ground state. The possibility of spontaneous emission of mesons from such a state sets an upper limit to the possible number of stable excited states. Naturally, the minimum excitation energy, or, more generally, the energy difference between successive excited states will be the smaller, the more strongly the meson field is coupled with the nucleon. In the case of very strong coupling, therefore, the excitation energies being much smaller than the rest energy of the nucleon, the excited states of this nucleon will appear as *isobars* of the ground state, with anomalous values of the spin and charge. In the opposite limit of very weak coupling, the minimum excitation energy will eventually be so large that the occurrence of excited states of the nucleons could only make itself felt in processes involving much higher energies than those hitherto studied. The nucleons may then for all practical purposes be treated as particles with just the normal, invariable values of spin and charge.

From the magnitude of the nuclear forces, it is immediately apparent that the nuclear field is at any rate much more strongly coupled to its sources than the electromagnetic field, so that the perturbation method developed in connexion with the latter cannot be applied to this case with the same confidence. Accordingly, special procedures have been developed to attack the problem from the strong coupling side; a rigorous treatment based on this assumption is, however, just as impossible as the ordinary one using perturbation theory, on account of the difficulties arising from the idealization of point sources. One is therefore obliged to introduce a

model of extended nucleon, which is necessarily unrelativistic and to a large extent arbitrary. In the same way as, in the theory of the extended electron, the field energy contributes to the inertia of the particle, the proper nuclear field gives rise to "moments of inertia" governing the rate of change of spin and charge. In fact, the excitation energy of the nucleon has just the form of a kinetic energy of rotation; the spin and charge variations can be compared to a "gyroscopic motion", the exact type of which depends on the specific assumptions made on the kinds of meson fields created by the nucleon.

Let us choose some arbitrary function $D(P)$, with $\int D(P) dv = 1$, to denote the spatial distribution of the sources of meson fields of the nucleon, and define by

$$\frac{1}{a} = \int D(P) \frac{1}{r} D(P') dv dv' \quad (14)$$

the "radius" a of the nucleon; it must, of course, be assumed to be much smaller than the range of the meson field:

$$a\kappa \ll 1. \quad (15)$$

If g represents the constant (or some average of the different constants), analogous to the elementary electric charge, which determines the intensity of the coupling of the meson field with the nucleons, the moment of inertia of spin and charge is of the order of magnitude

$$I = \frac{g^2}{a} \cdot \left(\frac{1}{\kappa} \right)^2, \quad (16)$$

corresponding to a "radius of gyration" of the order of the range and a "mass" defined by the average self-energy of the proper meson field; with the expression (9) for the potential energy and the definition (14), this self-energy is indeed $\approx g^2/a$ on account of (15). The minimum excitation energy will be of the order $\hbar^2/I$ and there will be isobars stable against emission of mesons as long as this quantity is smaller than M_m ; using (12) and (16), this gives for the dimensionless constant $g^2/\hbar$, analogous to the fine structure constant of electrodynamics, the critical value $a\kappa$, such that the condition for strong coupling is

$$\frac{g^2}{\hbar} \gg a\kappa. \quad (17)$$

Now, the roughest comparison of orders of magnitude of binding energies in atomic and nuclear systems suffices to show that $g^2/\hbar$ must be expected to be rather more than 10 times larger than the corresponding $e^2/\hbar$. Since the parameter a is not restricted otherwise than by the inequality (15), it is therefore not clear from condition (17) whether with such a value of $g^2/\hbar$ we have actually to do with a case of strong coupling, or whether we should rather describe it as one of "intermediate coupling". In

other words, it is not certain from general considerations which one of the two opposite lines of approach, characterized by the assumption of strong or weak coupling, will prove better adapted to the actual situation. Ultimately, of course, a decision as to the existence of (stable or unstable) excited states of nucleons might be reached by direct experiments in the energy region in which these states may be expected to have a marked effect on various phenomena. But even at the present stage, quite definite indications in this respect can be derived from certain consequences of the strong coupling theory concerning properties of nucleons or nuclear systems.

It must be emphasized that even in its ground state, in which its angular momentum is $\frac{1}{2}$ and its charge 0 or 1, the nucleon has very different properties according as the coupling with its proper field is treated as strong or weak. For in the first case the degrees of freedom corresponding to spin and charge, which on weak coupling are quite independent from each other, can no longer be separated, so that we cannot associate a definite spin orientation with a definite value of the charge. The result is that the magnetic moment of the "bare" proton or neutron (i.e. without the contribution due to the current distribution of the proper meson field) no longer has the "normal" value μ_0 or 0: instead, its expectation value becomes entirely symmetrical with respect to the proton and neutron states. Since the same is true of the contribution from the proper nuclear field to the total magnetic moment, the latter quantity turns out to have exactly equal and opposite values, for proton and neutron.

This is not the only prediction of the strong coupling theory which is flatly contradicted by experiment. In particular, the properties of the deuteron prove irreconcilable with the assumption of a large spin inertia of the nucleon. We are therefore amply justified in consistently adopting for all theoretical considerations in this book the point of view of *weak coupling theory*. This implies above all that we completely disregard the possible existence of excited states of the nucleons or of any coupling between spin and charge in the normal state, but keep to the usual conception as summarized in the preceding section (1.2). However, in view of the numerous and far-reaching modifications which would be necessitated by any departure from these assumptions, a brief account of the main lines of argument of strong coupling theory will be given by Appendix A3.

CHAPTER II

NUCLEAR RADII AND MASS-DEFECTS

2.0. The general character of the law of nuclear force can be inferred from the remarkable regularities exhibited by the *nuclear radii* on the one hand, and the *mass-defects* of the stable nuclei on the other. Of course, we are dealing here with strongly schematized features intended only to represent a kind of average behaviour, from which individual nuclei will deviate to some extent. The resulting information on nuclear interactions will accordingly be of qualitative, or at most roughly quantitative character.

2.1. Nuclear radius and range of nuclear forces

Already in 1911 Rutherford's celebrated experiments on the scattering of α -rays by matter revealed in a striking manner that nuclear forces, in contrast to the electrostatic ones, are of very *limited range*. It was in fact observed that the scattering followed the law derived from the assumption of simple Coulomb repulsion between α -ray and nucleus up to closest distances of approach of the order of 10^{-12} cm (for heavy nuclei). The *nuclear radius* could be defined as the critical distance from the centre at which the law of interaction with the α -particle ceases to be the electrostatic repulsion, but begins to vary rapidly with distance, the structure of the nucleus coming into play (see RUTHERFORD *et al.* [30], § 45).

More generally, any process involving a close interaction between a nucleus and a charged particle, such as a proton or an α -particle, will essentially depend on the nuclear radius. For in all such processes, the charged particle, in order to penetrate into the nucleus or to escape from it, must traverse the barrier of potential formed by the Coulomb repulsion acting at large distances and the average attraction exerted by the nuclear forces on the particle at distances smaller than the nuclear radius. The probabilities of occurrence of the processes in question will involve as a factor the transmission coefficient or *penetrability* of the barrier, which will depend, not only on the energy and charge of the particle considered, but also on the nuclear radius.

From a study of such interactions of nuclei with charged particles numerous estimates of nuclear radii can therefore be obtained. For nuclei of the radioactive families, we dispose of the data of the natural α -decay; for lighter nuclei, we can use not only the experiments on scattering of protons or α -particles of high energy, but also those on the disintegration of such nuclei by charged particles. All the evidence (see M & F, p. 58 sq., p. 72 sq., and also PRESENT [41]) is found to be compatible with the simple assumption of an approximately *constant density* of "nuclear matter"

for all nuclei. In other words, the nuclear radius is approximately proportional to the cubic root of the mass number:

$$R = r_0 A^{1/3}, \quad (1)$$

the "nucleon radius" r_0 being numerically given by

$$r_0 = 1.42 \cdot 10^{-13} \text{ cm}. \quad (2)$$

The significance of this remarkable result will presently (2.22) appear when it will be combined with the limited binding power of the nuclear forces.

Another possibility of estimating nuclear radii of light nuclei, which leads to results entirely in accordance with the above, is based on the consideration of β^+ -active nuclei with neutron excess -1 . It will be discussed later (3.3) in another connexion.

2.2. Mass-defects and saturation of nuclear forces

2.21. Mass-defects. If we except the lightest and the heaviest nuclei, we may say that, disregarding small individual fluctuations, the mass-defects of stable nuclei vary *linearly* with the mass number, i.e. with the total number of constituent nucleons*. The average binding energy per nucleon amounts to about 8 MeV.

The deviation from linearity for heavy nuclei is completely accounted for by the Coulomb repulsion of the protons. Assuming a homogeneous distribution of the charge Ze within a sphere of radius R , the Coulomb energy is given by $\frac{3}{5} \frac{Z^2 e^2}{R}$; a somewhat more accurate estimate for small Z would be

$$G = \frac{3}{5} \frac{e^2}{R} Z(Z-1). \quad (1)$$

The Coulomb energy has a marked effect on the relative proportion of neutrons in stable nuclei. There is in fact a general tendency for such nuclei to contain equal numbers of protons and neutrons; but owing to the Coulomb repulsion, the equilibrium, for heavier nuclei, is displaced towards a larger proportion of neutrons. This means that the average binding energy of a given assembly of Z protons and $N = A - Z$ neutrons will depend on the *neutron excess*

$$n = N - Z = A - 2Z \quad (2)$$

in such a way that, for given A , a minimum of the atomic weight (1.131) will occur for a certain value of n . The corresponding value Z_A of Z will in general not be an integer; stable nuclei may then exist with the nearest integral numbers of protons (3.22).

* See the graphical representation on fig. 2.21 (at the end of the book), based on the numerical data collected by M & F, M & F*.

Besides a "volume" contribution to the proper nuclear energy, proportional to A , a "surface" effect, comparable to the influence of a surface tension, must also be taken into account. This effect will be most important for lighter nuclei and the considerable irregularities presented by very light nuclei may be attributed to its preponderance.

Summing up, the average binding energy per nucleon can be put into the form

$$\varepsilon = -\varepsilon_1 \left[1 - \gamma \left(\frac{n}{A} \right)^2 \right] + \frac{4\pi R^2 O}{A} + \frac{3}{5} \frac{Z^2 e^2}{AR}, \quad (3)$$

in which the second and third terms express the surface energy (with a surface tension O) and the Coulomb repulsion, respectively. The first term, representing the volume effect, exhibits a parabolic dependence on the relative neutron excess. Inserting in this formula the law (2.1-1) of nuclear radii, we get

$$\varepsilon = -\varepsilon_1 \left[1 - \gamma \left(\frac{A-2Z}{A} \right)^2 \right] + 4\pi r_0^2 O A^{-1} + \frac{3}{5} \frac{e^2}{r_0} Z^2 A^{-1}. \quad (4)$$

Numerical values of the parameters ε_1 , γ , O and r_0 can be found to fit the empirical mass-defect curve (M & F, p. 91-92):

$$\begin{aligned} \varepsilon_1 &= 14.66 \text{ MeV}, & \gamma &= 1.40, \\ 4\pi r_0^2 O &= 15.4 \text{ MeV}, & r_0 &= 1.42 \cdot 10^{-13} \text{ cm}, \end{aligned} \quad (5)$$

the value of r_0 being in excellent agreement with the independent evidence from nuclear reactions (2.1-2). It is useful also to note the value

$$\frac{3}{5} \frac{e^2}{r_0} = 0.602 \text{ MeV}. \quad (6)$$

Equation (4) thus gives a semi-empirical representation of remarkable accuracy of the nuclear binding energies.

An idea of the relative importance of the different terms of (4), as well as of the accuracy to be expected, is afforded by the following examples:

Element	Z	A	Volume energy	Surface energy	Coulomb energy	Total binding energy	
						calculated	experimental (M & F*)
Al	13	27	-14.63	5.13	1.25	-8.25	-8.32
Ag	47	109	-14.26	3.23	2.55	-8.48	-8.47
Pb	82	208	-13.74	2.60	3.28	-7.86	-7.83
Energies expressed in MeV							

2.22. Saturation properties of nuclear forces. The linear variation with mass number of that part of the binding energy due to the proper nuclear forces brings out the fundamental *saturation* property of these forces: it implies in fact that each nucleon is only capable of binding a

fixed number of other nucleons. Considering the short range of the forces, this property throws full light on the law of constant density of nuclear matter (2.1). Indeed, as every nucleon interacts only with its nearest neighbours, the structure of a nucleus will be comparable (apart from the scale) with that of ordinary matter in a state (solid or liquid) of high density. The constituent nucleons will on an average be kept apart at a fixed distance, of the order of the range of the force. This is just the law of constant density, which thus yields a value of about $2r_0$ for the *order of magnitude of the range*.

This last inference, as we shall see, is fully confirmed by the detailed analysis of the most direct exploration of the nuclear field which can be carried out, viz. by studying the scattering of protons and neutrons by protons. It is a remarkable fact that the distance $2r_0$, according to (2.1-2), nearly coincides with the fundamental length d (1.21-1) of electron theory. The significance of this coincidence cannot be understood as long as we have no theory of the ratio of the masses of electron and nucleon. But it is at any rate convenient to make use of it, especially in rapid estimations of orders of magnitude; we shall often avail ourselves of the numerical relation

$$r_0 \approx \frac{1}{2} d. \quad (7)$$

When we now come to ask which kinds of interaction operators are liable to give rise to binding energies exhibiting the saturation property, i.e. increasing — at least in a rough approximation, for a great number of interacting particles — proportionally to this number, we have the choice between two essentially distinct possibilities. In analogy with the case of a liquid, in which the saturation of the atomic binding forces is brought about by the occurrence of a very strong repulsion between two atoms at some critical distance of approach, we might imagine the attractive force between two nucleons at distances of the order of the range to go over, at some smaller distance, into a practically infinite repulsion * (fig. 2.22).

* The analogy with a liquid might be deemed imperfect inasmuch as in the case of atoms the complicated distance dependence of the interaction is a consequence of the interplay of simpler forces between the atomic constituents, while in the nuclear case we would have to do with a fundamental law of interaction of complicated form. It must not be overlooked, however, that a relatively simple law of *field* interaction can give rise to a superposition of attractive and repulsive potentials of different ranges: an example of such a meson field theory will be mentioned on a later occasion (17.1).

In particular, it has been pointed out by SVARTHOLM [45], Chapter VI, that the many-body interactions occurring in any field theory (1.34) might, under suitable conditions, bring about the observed saturation of the nuclear bonds. Although Svartholm's assumptions are admittedly quite arbitrary and over-simplified, his argument may appropriately serve to illustrate the point under discussion. While the two-body attractions will give a nuclear binding proportional to $\binom{A}{2}$, he assumes that the n -body forces will alternatively produce a repulsion or an attraction proportional to $-(-\gamma)^{n-2} \binom{A}{n}$. The

The well-known quantum theory of chemical valency suggests, on the other hand, a mechanism of interaction, the so-called *exchange interaction*, capable of giving saturation effects without involving any explicit assumption of a change of sign of the interaction operator with distance. As a consequence of the exclusion principle, the expectation value of any inter-

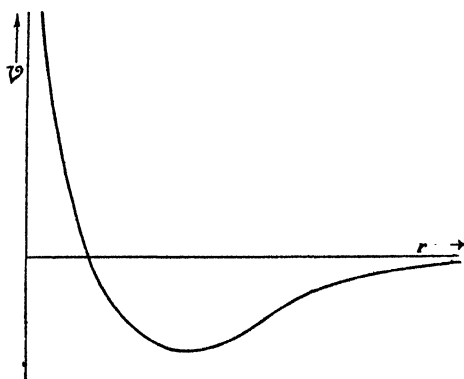


Fig. 2.22. Schematic representation of ordinary potential giving rise to saturation. Interaction energy $\mathcal{V}(r)$ between two nucleons in terms of mutual distance r .

action operator in a stationary state of a system of identical particles, besides "ordinary" terms corresponding to classical effects, involves additional "exchange" terms without classical analogy. It may be seen, quite generally, that for systems of a large number of particles, the exchange terms, in contrast to the ordinary ones, possess the property of saturation.

Let us consider a system of two particles obeying the exclusion principle and suppose that the coupling between them is so weak that it is possible to define individual stationary states $\varphi_n(Q)$ for each particle. The eigenfunctions of the stationary states of the system will then be of the approximate form

$$\psi_{n_1 n_2}(Q^{(1)}, Q^{(2)}) = \frac{1}{\sqrt{2}} \{ \varphi_{n_1}(Q^{(1)}) \varphi_{n_2}(Q^{(2)}) - \varphi_{n_2}(Q^{(1)}) \varphi_{n_1}(Q^{(2)}) \}.$$

The expectation value $\bar{\mathcal{V}}$ of the interaction operator $\mathcal{V}(Q^{(1)}, Q^{(2)})$ in this state becomes, when account is taken of the symmetry of the function $\mathcal{V}$ with respect to $Q^{(1)}$ and $Q^{(2)}$,

$$\begin{aligned} \bar{\mathcal{V}} = \int \psi_{n_1 n_2}^* \mathcal{V} \psi_{n_1 n_2} = & \int \varphi_{n_1}^*(Q^{(1)}) \varphi_{n_2}^*(Q^{(2)}) \mathcal{V}(Q^{(1)}, Q^{(2)}) \varphi_{n_1}(Q^{(1)}) \varphi_{n_2}(Q^{(2)}) \\ & - \Re \int \varphi_{n_1}^*(Q^{(1)}) \varphi_{n_2}^*(Q^{(2)}) \mathcal{V}(Q^{(1)}, Q^{(2)}) \varphi_{n_2}(Q^{(1)}) \varphi_{n_1}(Q^{(2)}), \end{aligned}$$

resulting binding energy per nucleon is

$$\varepsilon = -\varepsilon_1 \cdot \frac{(1-\gamma)^A - 1 + A\gamma}{A\gamma^2};$$

if γ is not too small and A is large, the term $(1-\gamma)^A$ may be neglected, and

$$\varepsilon = -\frac{\varepsilon_1}{\gamma} \left(1 - \frac{1}{A\gamma} \right).$$

This expression exhibits both saturation for large A and increase with A for small A ; with $\varepsilon_1 = 4.35$ MeV, $\gamma = 0.484$, it shows a remarkable (though doubtless fortuitous) agreement with the figures derived from the observed mass-defects.

clearly showing the occurrence of an ordinary and an exchange energy. The saturation properties of the exchange interactions will be extensively discussed in Chapter XI.

For the moment, it will be sufficient to mention the two possibilities of accounting for saturation effects. Later (17.1) we shall see how the decision between them is related to the question of the relative importance of non-central couplings in the total nuclear interaction.

2.3. Kinetic effects

The limited range of the nuclear forces implies the occurrence of quite considerable *kinetic energies* of the constituent particles of a nucleus. In fact, under the influence of the short-range interaction with its neighbours, any nucleon will on the average remain confined to a region, the radius of which will be of the order of the range, or somewhat larger; in other words, it will be represented by a wave-packet of linear dimensions $l \gtrsim 2d$. It will then acquire a kinetic energy of the order of magnitude *

$$\epsilon_{\text{kin}} = \frac{1}{2M} \left(\frac{2\pi\hbar}{l} \right)^2 \quad (1)$$

(this being the smallest eigenvalue of the wave-equation of a particle of mass M with an eigenfunction vanishing on the surface of a sphere of diameter l); for $l = 2d$, we get

$$\epsilon_{\text{kin}} = \frac{\pi^2}{2} \left(\frac{\hbar}{e^2} \right)^2 \frac{m}{M} \approx 25 \text{ MeV}, \quad (2)$$

and the actual value will perhaps be of the order of 20 MeV.

In spite of this high value of the kinetic energy, the mean velocity of a nucleon, owing to its large mass, is relatively small; in fact, we have, on account of (2),

$$\left(\frac{v}{c} \right)^2 = \frac{2\epsilon_{\text{kin}}}{M} = \frac{m}{M} \frac{2\epsilon_{\text{kin}}}{m} \approx \frac{1}{23}. \quad (3)$$

It thus appears that *relativistic effects* should in general not amount to more than a few percent, and that we are justified in basing the discussion of nuclear structure on a non-relativistic treatment of the constituent nucleons. On the other hand, the motions of the nucleons are sufficiently large to suggest that a nucleus should resemble a *liquid droplet* rather than a piece of crystal; as a matter of fact, the liquid phase of Helium at the lowest temperatures (the so-called He II) offers an example of a condensed state of ordinary matter, which is prevented from solidifying (at not too high pressures) by the "zero-point" motion of the atoms.

From another point of view, the kinetic energies of the nucleons are of importance in helping us to picture the interplay of the nuclear energies.

Here M denotes the mass of a nucleon, disregarding the small difference $M_n - M_p$.

Comparing the average value of 20 MeV of the kinetic energy with that of the mean binding energy of a nucleon, viz. (discounting the Coulomb repulsion) about 10 MeV, we see that the *potential energy* of the proper nuclear forces must average about 30 MeV per nucleon (a quantity 60 times larger than the Coulomb energy between two protons at distance d). The binding energy is therefore the outcome of a balance between much larger kinetic and potential energies.

This is especially apparent, as pointed out by WIGNER [33a], in the case of the deuteron. In order to keep the kinetic energy lower than the attractive potential, the system, according to (1), has to expand; but, owing to the rapid decrease of the potential for distances larger than the range, this means that the resulting binding energy is very much lowered. It is actually only about 2 MeV, thus really quite small compared to either kinetic or potential energy. As soon, however, as the number of constituent nucleons increases, the situation is considerably altered in favour of the potential energy. Already for the α -particle, the latter energy becomes roughly four or six times * that occurring in the deuteron, while the kinetic energy is only doubled. The particles are then able to settle nearer to each other, letting the nuclear potential, so to speak, display its full strength. The average binding energy per nucleon accordingly rises to a constant value of about 10 MeV.

* According as we assume no forces between nucleons of the same charge, or equal forces between all pairs of nucleons,

CHAPTER III

CHARGE DEPENDENCE OF NUCLEAR FORCES

3.0. After the general survey of the preceding Chapter, we go a step further in the analysis of nuclear properties by looking more closely into the distribution of the stable nuclei according to mass and charge number, and their relation to the neighbouring β -active nuclei. This, combined with the general stability condition 1.131, will give us important information on the charge dependence of the nuclear forces.

3.1. Proton and neutron pairing

3.11. Classification of nuclei. If we classify the stable nuclei according to their mass and charge numbers, we are at once struck by the difference in behaviour between those of even and odd mass number. While the latter are about equally distributed as regards even or odd charge number, the former are divided into two classes of very unequal population: most of them are *even*, i.e. have even numbers both of protons and neutrons; the only *odd* nuclei (i.e. nuclei with odd numbers of protons and neutrons) which are stable are the four lightest of that type, ${}^2_1\text{H}$, ${}^6_3\text{Li}$, ${}^{10}_5\text{B}$, ${}^{14}_7\text{N}$.

A nucleus of mass number A and charge number Z , which therefore contains $N = A - Z$ neutrons, will be denoted either by (A, Z) or by $[N, Z]$. The following table presents the classification just discussed, with the numbers of nuclei of each class, including the α -active ones, according to recent data (M & F):

Class of nuclei	Z	N	Total number *
Even mass nuclei { even nuclei odd nuclei	even	even	177 (15)
	odd	odd	4
Odd mass nuclei	even	odd	60 (8)
	odd	even	52 (1)
* This number includes both stable and α -active nuclei, the number of the latter being indicated between brackets.			

3.12. Proton and neutron pairing. Apart from the exceptional case of the light odd nuclei, it therefore appears that the successive binding of two nucleons of the same kind (protons or neutrons) sets free a greater amount of energy than that of a proton and a neutron: in other words, there is a tendency for the building up of stable nuclei by successive addition of nucleons to proceed by completion of *pairs* of protons or neutrons. The further empirical fact that all even nuclei have zero angular

momentum receives a simple interpretation by assuming that the two nucleons forming a pair have *opposite spins*. On this view, the "pairing" of like nucleons is thus related to a "*saturation*" of their spins.

If we supposed that individual stationary states could be attributed to the constituent nucleons (9.21), we should actually expect a grouping of these states into pairs corresponding to the same specification of all "quantum numbers" (including charge) but to different spin orientations. The structure of a stable nucleus in its ground state would then be visualized by filling up the successive pairs of individual states with saturated spins by the available protons and neutrons. In general, the "last" pair of neutron states occupied would, on account of the neutron excess, be very different from the last pair of proton states.

3.2. Stability of isobars

3.21. Stable isobars. There is another important difference between nuclei of odd and even mass number: while for any odd value of the mass number there is just one stable nucleus, we meet among the even nuclei with a large number of pairs of stable *isobars* (i.e. nuclei of the same mass number). There are in fact 56 pairs and 4 trebles of such isobaric nuclei. In every case, the charge numbers of neighbouring stable isobars differ by two units.

Now, the existence of such isobars cannot be understood solely on the basis of nucleon pairing*. For if two paired nucleons were subject to exactly the same forces (apart from the Coulomb interaction), there could only be one stable nucleus for any even value of the mass number. In fact, if we start from the even nucleus ($2a, 2z$) and suppose that the stable nucleus of mass $2a + 1$ is obtained by adding a neutron, the next stable nucleus would necessarily be obtained by completing the neutron pair, which gives the nucleus ($2a + 2, 2z$): the odd isobar ($2a + 2, 2z + 1$) would have a higher atomic weight, but the even isobar ($2a + 2, 2z + 2$) a still higher one (fig. 3.21). In order to reconcile the pairing conception with the empirical evidence, we must therefore, in the first place, assume that two nucleons forming a pair attract each other, so that the atomic weight of the unstable even isobar ($2a + 2, 2z + 2$) can eventually be sufficiently depressed to reverse the stability relation of this nucleus to the intermediate odd isobar ($2a + 2, 2z + 1$). This result will only be achieved, however, if it is further assumed that, in the latter isobar, the odd proton and the odd neutron have a much smaller interaction (fig. 3.21).

The last assumption appears to be generally fulfilled, with the notable exception of the four lightest nuclei of the odd type, which have exactly as many neutrons as protons. In the latter case, the odd proton and the odd neutron are in much the same situation with respect to the other constituent nucleons and may consequently be thought to form a pair entirely comparable to the spin-saturated pairs of nucleons of like charge

* The following argument is taken from B & B, § 10.

discussed above. The roles of spin and charge variables would simply be interchanged. (The angular momenta of the odd nuclei in question * have

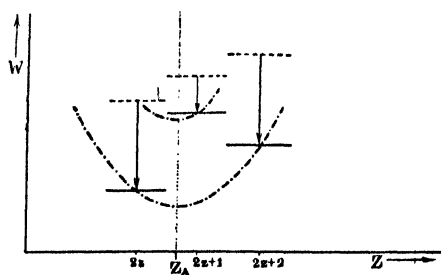


Fig. 3.21. Atomic weight of isobaric nuclei, before (---) and after (—) taking account of the interaction between the two last nucleons added in the successive building up of the nuclei.

indeed the value 1, in conformity with the idea that the odd nucleons would have the same spin orientation). We are thus led to assume that such pairs of nucleons also exert a strong attraction on each other, so that the odd isobar then becomes more stable than the neighbouring even nuclei.

Using the more definite picture of 3.12, based on the conception of individual states of the constituent nucleons, we may express the difference between the light and the heavier odd nuclei by stating that the odd nucleons experience a strong mutual attraction when they belong to states of identical specification except for the charge variable, whereas they do not interact appreciably when, owing to the neutron excess, they occupy different states.

On the whole, we thus arrive, at least qualitatively, at a complete account of the main features of the distribution of stable nuclei; and the analysis has disclosed that *besides the proton-neutron forces, there are also forces between like particles, which are attractive for "paired" nucleons.*

3.22. Atomic weights of isobars. To get some idea of the order of magnitude of the energies involved in the preceding considerations on isobar stability, we first observe that our general formula (2.21-4) must be made more precise in order to take account of the differences between odd and even mass and charge numbers. Consider an *even* nucleus (A, Z); according to formula (2.21-4), its atomic weight may be written in the form

$$W(A, Z) = W_A + w_A (Z - Z_A)^2, \quad (1)$$

with

$$Z_A = \frac{4\varepsilon_1\gamma + (M_n - M_p - m)}{2w_A}, \quad w_A = \frac{4\varepsilon_1\gamma}{A} + \frac{3e^2}{5r_0} A^{-1}; \quad (2)$$

the points representing stable isobars of mass number A and *even* Z in a graph like fig. 3.21 with coordinates Z, W lie on a parabola defined by

* The angular momentum of $^{10}_5\text{B}$ is not known.

equation (1). The *odd* isobars of the same mass number, however, correspond to another parabola, given by

$$W(A, Z+1) = W_A + \delta_A + w_A (Z+1 - Z_A)^2 \quad (3)$$

with a suitably chosen additive constant δ_A . The difference

$$\zeta = Z+1 - Z_A \quad (4)$$

between the charge number of the odd isobar and the abscissa Z_A of the apices of the parabolas is obviously comprised between $+1$ and -1 ; with this notation, we have

$$\begin{aligned} W(A, Z) &= W_A + w_A (\zeta - 1)^2 \\ W(A, Z+1) &= W_A + \delta_A + w_A \zeta^2 \\ W(A, Z+2) &= W_A + w_A (\zeta + 1)^2. \end{aligned} \quad (5)$$

Now, if there were no like-particle forces, $W(A, Z+1)$ would be just midway between $W(A, Z)$ and $W(A, Z+2)$, i.e. δ_A would be equal to w_A . The effect of the like-particle forces is therefore expressed by $\delta_A - w_A$. The conditions for the odd isobar to be unstable with respect to both even isobars can be put into the form

$$\delta_A - w_A > 2 w_A |\zeta|. \quad (6)$$

This inequality defines a critical value

$$\zeta_{\text{crit}} = \frac{\delta_A - w_A}{2 w_A}, \quad (7)$$

such that there is more than one stable isobar of mass number A if and only if

$$|\zeta| < \zeta_{\text{crit}}. \quad (8)$$

If we assume that the values of ζ are distributed at random along the whole range of even mass numbers, the statistics of isobaric pairs (or trebles) will immediately give us an estimate of some average value $\bar{\zeta}_{\text{crit}}$. Since we encounter 60 isobaric pairs or trebles among the 97 even mass numbers between 16 and 208, we may take

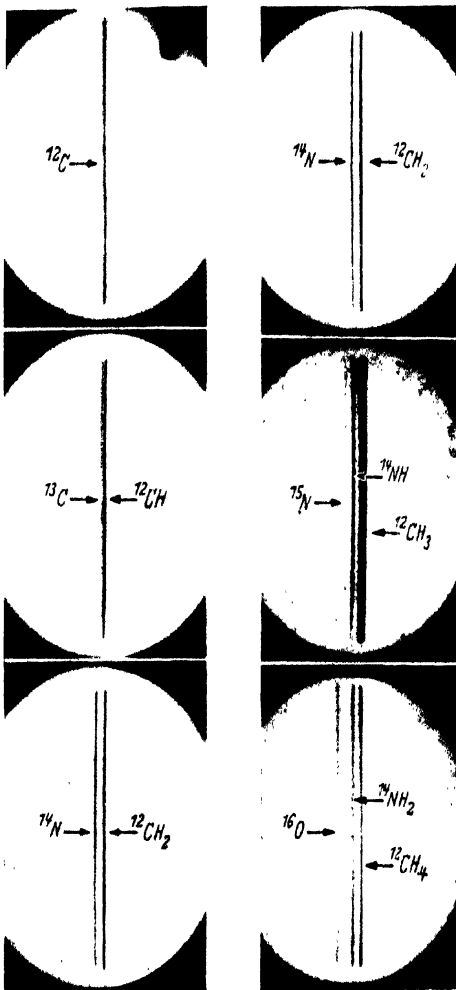
$$\bar{\zeta}_{\text{crit}} \approx 0.6. \quad (9)$$

With the numerical values (2.21-5, 6), giving

$$w_A = \left(\frac{82.1}{A} + \frac{0.6}{A^{\frac{1}{3}}} \right) \text{MeV}, \quad (10)$$

we accordingly find

$$\delta_A - w_A \approx 2 w_A \bar{\zeta}_{\text{crit}} \approx 1.2 w_A. \quad (11)$$



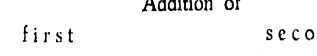


Fig. 3.221-3. Mass-spectrograms showing the successive addition of two neutron-proton pairs to the ^{12}C nucleus (MATTAUCH [37]).

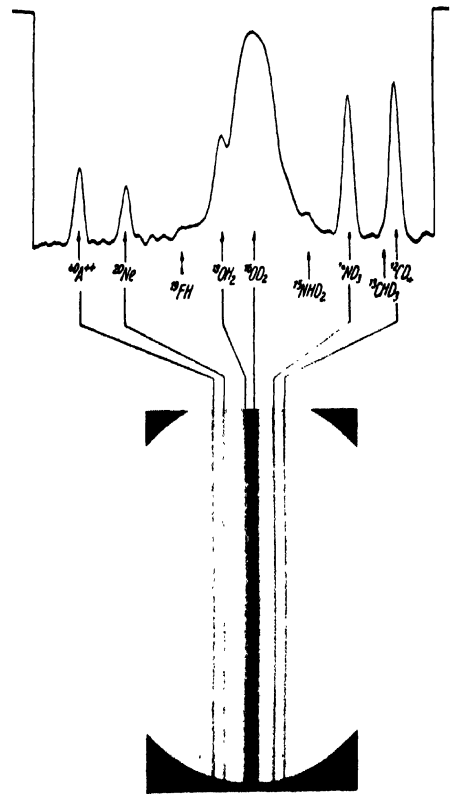


Fig. 3.221-4. Mass-spectrogram, pertaining to mass-number 20, showing binding energies of various aggregates of protons and neutrons (Mattauch [39]).

This gives, e.g.:

A	w_A MeV	$\delta_A - w_A$ MeV	δ_A MeV
64	1,4	1,7	3,1
125	0,8	0,9	1,7
216	0,5	0,6	1,1

3.221. Comparison with measurements of mass-defects. Our knowledge of accurate atomic weights is still so fragmentary that there is as yet only one instance of isobaric nuclei for which the relations between atomic weights embodied in fig. 3.21 can be checked with reasonable accuracy. It concerns the mass-number $A = 64$, for which we have (M & F*, table I):

Z		$W - A$ 10^{-4} MU
28	Ni (stable)	$-525,6 \pm 5,6$
29	Cu (unstable)*	$-507,4 \pm 5,6$
30	Zn (stable)	$-513,6 \pm 5,6$

(The *relative* accuracy of these three figures is better than the indicated absolute uncertainty).

As to the atomic weights of lighter nuclei (up to ${}^{40}_{18}\text{A}$), which have been determined with high accuracy for almost all unstable as well as stable isotopes, they exhibit such large individual fluctuations that it is not possible at this stage to draw from them other than qualitative conclusions. The mass-defects of the stable nuclei strikingly bring out the successive completion of neutron and proton pairs, with the resulting periodic occurrence of nuclei of maximum binding energy, composed of equal even numbers of protons and neutrons.

We may further consider sequences of (stable and unstable) *isotopes* (i.e. nuclei with the same number of protons) or *isotones* (i.e. nuclei with the same number of neutrons) and derive from the comparison of their mass-defects the binding energy of the "last" nucleon serving to build up any one of these nuclei from the preceding one of the sequence. The result is given (at the end of the book) in fig. 3.221-1, taken from M & F*, and further illustrated by a diagram (fig. 3.221-2). Both directly show how in all cases the second nucleon of a given pair is more tightly bound than the first **.

3.3. Charge symmetry of nuclear forces

A further feature of nuclear interaction which can be established with fair accuracy is the following: *the proper nuclear forces are symmetrical with respect to the charges of the interacting nucleons.* This property can

* ${}^{64}\text{Cu}$ transmutes as well into ${}^{64}\text{Ni}$ (with positron emission) as into ${}^{64}\text{Zn}$ (with negaton emission).

** Plate II reproduces some beautiful mass-spectrograms, obtained by MATTAUCH [37, 39], to the same effect. A more detailed discussion of this and other general evidence is given by JENSEN [39].

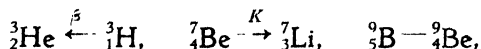
be inferred, in the first place, from the tendency of stable nuclei toward equality of proton and neutron numbers, especially apparent in the light nuclei from ${}^4_2\text{He}$ to ${}^{16}_8\text{O}$, where it is not yet disturbed by the influence of the Coulomb force. The nuclei in question are formed by alternate addition of a neutron and a proton, which corresponds to an alternation of the two types of nucleon pairing discussed above (3.21). The only effect of the Coulomb repulsion is here to give the binding of a neutron to an even nucleus a slight preference over that of a proton.

Still more definite evidence for the symmetry property under consideration is provided by the light nuclei with a neutron excess -1 , which transmute with positron emission into isobaric nuclei of neutron excess $+1$; there is a long sequence of known instances of this type of β -transitions, beginning with ${}^{11}_6\text{C} \rightarrow {}^{11}_5\text{B}$, ${}^{13}_7\text{N} \rightarrow {}^{13}_6\text{C}$, etc. and extending up to ${}^{41}_{21}\text{Sc} \rightarrow {}^{41}_{20}\text{Ca}$. In fact (FOWLER, DELSASSO and LAURITSEN [36]), if we assume the symmetry of the nuclear forces with respect to the charges of the nucleons, the binding energies of isobaric nuclei such that the number of protons of the one is the number of neutrons of the other will differ only by the electrostatic contributions, which may easily be estimated; and in the special case of the above mentioned isobaric pairs with neutron excesses ± 1 , this energy difference can directly be deduced from the maximum kinetic energy K_β of the emitted positons. If we take for the Coulomb energy the expression (2.22-1), equation (1.13-1) for the mass difference of the isobaric pair takes the form

$$\frac{3}{5} \frac{e^2}{R} (A-1) = K_\beta + (M_n - M_p + m). \quad (1)$$

We can use this equation either in connexion with the expression (2.1-1, 2) for the nuclear radius to predict the limiting energy K_β of the positron spectrum, or conversely to calculate the nuclear radius with the help of the observed K_β values. In any case, the remarkable consistency of the results* affords strong support to the underlying assumption about the symmetry of the nuclear forces.

The same argument also applies to the pairs**



in which the unstable partner is not a positron emitter; the mass differences can in these cases be determined by other methods. However, a more precise calculation of the Coulomb energy is required in these cases to account for fluctuations in the observed mass differences with the parity of Z : these may be due, partly to a contribution from the exchange term

* The available evidence, including the comparison with formula (1) is collected in table A 1.21. Systematic investigations on this point are due, above all, to WHITE *et al.* [39, 41]; see also ELLIOT and KING [41].

** The nucleus ${}^9_5\text{B}$ disintegrates into two α -particles and a proton.

(2.22) of Coulomb energy, which, according to the parity of Z , enhances or counteracts the ordinary term (PHILLIPS and FEENBERG [41]) *, partly to variations of nuclear dimensions according to the more or less loose binding of the "last" nucleon (BETHE [38a], ELLIOT and KING [41]). When account is taken of these corrections, a further confirmation of the above conclusion is obtained. It is interesting to note that apart from the mentioned fluctuations, the nuclear radius is assumed to satisfy the law (2.1-1) of constant nuclear density, implying that there is no "blowing up" of the nucleus by Coulomb repulsion.

We must finally mention the pair of unstable isobars ^{10}C – ^{10}Be (which both go over by β^- - or β^+ -decay into the stable odd nucleus ^{10}B) as the one known example of a pair of isobars with neutron excesses — 2 and + 2, for which it could be verified that the energy difference is entirely accounted for by electrostatic interactions (WHITE *et al.* [40]). In this case, the mass of ^{10}C is deduced from that of ^{10}B and the maximum energy of the positron spectrum, while that of ^{10}Be is brought back to that of ^9Be by the study of the reaction $^9\text{Be}(d, p)^{10}\text{Be}$.

Summing up, we are led to a precise formulation of the symmetry of proper nuclear forces with respect to charge: *the neutron-neutron interaction operator is the same as that for two protons (exclusive of the Coulomb force)*. This conclusion is important inasmuch as there is as yet no more direct approach to the force between two neutrons (14.0).

3.4. Building up of stable nuclei

3.40. It is a remarkable fact, especially stressed by FUCHS [39], that the mainly qualitative considerations developed in this first Part are already sufficient to convey a general understanding of the apparently capricious order in which stable nuclei of the different classes — even nuclei, odd nuclei, odd mass nuclei — successively appear when the numbers of constituent protons and neutrons are increased. In this section, we shall analyse the distribution of the known stable nuclei from the point of view put forward in Fuchs' paper.

3.41. The energy surface. As already pointed out in the case of even and odd nuclei (3.22), the atomic weights, and therefore also the binding energies $\mathcal{E}(N, Z)$ (< 0), of the nuclei of the three classes are quite distinct functions of the numbers N, Z of neutrons and protons. In a graphical representation of the binding energy in terms of N and Z , the points pertaining to nuclei of these various classes accordingly lie on three distinct surfaces, which may be regarded as three sheets of the *energy surface* $\mathcal{E}(N, Z)$. The sheet corresponding to odd nuclei lies above that for even nuclei, except in the region of the lightest nuclei, where a crossing over of the two sheets occurs. The sheet pertaining to odd mass nuclei lies somewhere in between the other two.

A more general discussion of this effect will be given in 10.34.

We shall restrict the following discussion to the region in which the neutron excess of the stable nuclei is larger than 2, thus excluding the region of the lightest nuclei, for the treatment of which we refer the reader to Fuchs' paper. In the region of heavier nuclei, all three sheets of the energy surface (see fig. 3.41 at the end of the book) have similar properties: each one is moulded into a "valley", sloping down from the high mass number values, and in or near the bottom of which lie the stable nuclei. The projection of the bottom of the valley on the (N, Z) -plane — the *line of maximum binding* — is (as first noted by GAMOW [34]) a rather winding line. An explanation of such irregularities would necessitate a much closer investigation of nuclear structure than is at present possible. Here, however, we shall only be concerned with the general orientation of this line, defined by an angular coefficient

$$\frac{dN}{dZ} \approx 1.5 \quad (1)$$

(dN, dZ denoting the increments of N and Z along the line).

Cross-sections of the energy surface by planes $N + Z = \text{const.}$ i.e. loci of isobaric nuclei, are curves turning their concavity upward; their lowest parts are approximated by parabolas similar to those of 3.22 (which referred to atomic weights instead of binding energies). At the lowest point, projecting on the line of maximum binding, one has

$$\frac{\partial \mathcal{E}}{\partial Z} = \frac{\partial \mathcal{E}}{\partial N}, \quad \frac{\partial^2 \mathcal{E}}{\partial N^2} - 2 \frac{\partial^2 \mathcal{E}}{\partial Z \partial N} + \frac{\partial^2 \mathcal{E}}{\partial Z^2} > 0. \quad (2)$$

Now, the saturation character of the nuclear bonds conditions further general features of the sheets of the energy surface. First, the slope of the valley will be nearly uniform along the whole line of maximum binding (in the region of heavy nuclei we are considering), so that

$$\frac{\partial^2 \mathcal{E}}{\partial N^2} dN + \frac{\partial^2 \mathcal{E}}{\partial N \partial Z} dZ = 0 \quad , \quad \frac{\partial^2 \mathcal{E}}{\partial Z \partial N} dN + \frac{\partial^2 \mathcal{E}}{\partial Z^2} dZ = 0. \quad (3)$$

Combined with the inequality (2), this shows, in particular, that

$$\frac{\partial^2 \mathcal{E}}{\partial N^2} > 0 \quad , \quad \frac{\partial^2 \mathcal{E}}{\partial Z^2} > 0 \quad , \quad \frac{\partial^2 \mathcal{E}}{\partial N \partial Z} < 0 \quad (4)$$

along the line of maximum binding. Secondly, since the relative increase of the binding of a neutron (say) when another nucleon is added to the nucleus is very small,

$$\frac{\partial^2 \mathcal{E}}{\partial N^2} \ll \left| \frac{\partial \mathcal{E}}{\partial N} \right|, \quad \left| \frac{\partial^2 \mathcal{E}}{\partial N \partial Z} \right| \ll \left| \frac{\partial \mathcal{E}}{\partial N} \right|. \quad (5)$$

As a matter of fact, the rough estimate allowed by the experimental data yields

$$\frac{\partial^2 \mathcal{E}}{\partial N^2} \approx 0.5 \dots 1 \text{ MeV.} \quad (6)$$

while according to (3), (1),

$$\left| \frac{\partial^2 \mathcal{E}}{\partial Z \partial N} \right| \approx 1.5 \frac{\partial^2 \mathcal{E}}{\partial N^2}; \quad (7)$$

these values are to be compared with $|\partial \mathcal{E} / \partial N| \approx 8 \text{ MeV}$.

3.42 Stability conditions. We now go over, following Fuchs, to a closer discussion of the stability condition 1.131. Fuchs' argument is essentially based on the consideration of the amounts of energy released when a specified nucleon is added to the nucleus. We have to distinguish four such energy quantities, according as the added nucleon is a neutron or a proton and according as it is the first or the second of the pair which is being formed; for an initial nucleus $[N, Z]$ containing N neutrons and Z protons, we use the self-explanatory notations $\partial_N^{(1)}(N, Z)$, $\partial_N^{(2)}(N, Z)$, $\partial_Z^{(1)}(N, Z)$, $\partial_Z^{(2)}(N, Z)$, with the convention that the ∂_Z 's do not include the change in Coulomb energy. Just as we did with $\mathcal{E}(N, Z)$, we regard the ∂ 's as continuous functions defined for all real values of their arguments. We list here some general properties of the ∂ 's:

(1) On account of (5), ∂_N is not very different from $-\partial \mathcal{E} / \partial N$ and consequently, by (4), decreases with increasing N when Z remains unchanged. Similarly, ∂_Z is found to vary in the opposite direction; summing up:

$$\text{for fixed } Z, \begin{array}{l} \partial_N(N, Z) \text{ decreases} \\ \partial_Z(N, Z) \text{ increases} \end{array} \text{ with increasing } N \text{ (or } n). \quad (8)$$

(2) The attraction between paired nucleons implies

$$\partial_N^{(2)} > \partial_N^{(1)}, \quad \partial_Z^{(2)} > \partial_Z^{(1)}. \quad (9)$$

(3) Considering that the attraction just mentioned will mainly depend on the distance of the two interacting nucleons and that this distance will on the average remain constant whatever the total number of constituent nucleons, the differences $\partial_N^{(2)} - \partial_N^{(1)}$, $\partial_Z^{(2)} - \partial_Z^{(1)}$ will have the same property:

$$\partial_N^{(2)} - \partial_N^{(1)}, \partial_Z^{(2)} - \partial_Z^{(1)} \text{ are practically independent of } N, Z. \quad (10)$$

(4) On account of the symmetry of nuclear forces with respect to charge, one has moreover

$$\partial_N^{(2)} - \partial_N^{(1)} \approx \partial_Z^{(2)} - \partial_Z^{(1)}. \quad (11)$$

For the order of magnitude, we may roughly take

$$\partial_N^{(2)} - \partial_N^{(1)} \approx 5 \text{ MeV}. \quad (12)$$

With the help of the ∂ 's, the stability condition 1.131 is readily brought into the following form: the former or the latter of the nuclei $[N, Z]$, $[N-1, Z+1]$ is stable against β -transitions according as the difference

$\partial_Z(N-1, Z) - \partial_N(N-1, Z)$ is smaller than a critical value defined by the equation

$$\partial_Z(N-1, Z) - \partial_N(N-1, Z) = \frac{6e^2}{5r_0} \frac{Z}{(N+Z)^2} - (M_n - M_p - m_e). \quad (13)$$

If we regard the charge number Z as given, this equation defines a corresponding critical value (generally not integral) of N , or of the neutron excess n . Choosing now *even* values for N and Z , and grouping together the nuclei of charge numbers Z , $Z+1$ and mass numbers $N+Z-1$, $N+Z$, we thus get for each group* four critical values of the neutron excess, viz.

n_1		$[N-1, Z+1]$	,	$[N-2, Z+2]$
n_2	separating the domains	$[N-1, Z]$	,	$[N-2, Z+1]$
n_3	of stability of the nuclei	$[N-2, Z+1]$	,	$[N-3, Z+2]$
n_4		$[N, Z]$	,	$[N-1, Z+1]$

Because of (8), the nuclei in the

left	right
------	-------

 column are stable when the actual neutron excess is

smaller	larger
---------	--------

 than the corresponding critical n_i , and we shall presently see that

$$n_1 < n_2 < n_3 < n_4. \quad (14)$$

With increasing neutron excess, we therefore successively meet, for the charge numbers Z , $Z+1$, $Z+2$, the following types of stable nuclei:

n	$[N-1, Z]$	—	—
$\downarrow$	$[N, Z]$	—	—
n_1			
	$[N-1, Z]$	—	—
	$[N, Z]$	—	$[N-2, Z+2]$
n_2			
	—	$[N-2, Z+1]$	—
	$[N, Z]$	—	$[N-2, Z+2]$
n_3			
	—	—	$[N-3, Z+2]$
	$[N, Z]$	—	$[N-2, Z+2]$
n_4			
	—	—	$[N-3, Z+2]$
	—	—	$[N-2, Z+2]$

(Vertical columns contain isotopes, horizontal rows isobars.) A representation of all known stable (and α -active) nuclei, according to this scheme, will be found at the end of the book (fig. 3.42).

* Equation (13) gives rise to positive critical values of n only when the Coulomb energy variation on the right becomes sufficiently large, i.e. when Z exceeds a certain limit. For this point, and the treatment of lighter nuclei, see Fuchs' paper. Our discussion remains restricted to the heavier nuclei.

3.43. Properties of the critical values of the neutron excess. With a view to establishing the inequalities (14), we observe that the critical n -value resulting from an equation of the type (13) is larger, the smaller the left-hand side; on the other hand, the right-hand side is quite insensible to changes of Z or $N + Z$ by one or two units: it will therefore suffice to order the left-hand side quantities

$$\begin{aligned} \partial_1 &= \{\partial_Z^{(2)} - \partial_N^{(1)}\}_{N-2, Z+1}, & \partial_2 &= \{\partial_Z^{(1)} - \partial_N^{(1)}\}_{N-2, Z} \\ \partial_3 &= \{\partial_Z^{(2)} - \partial_N^{(2)}\}_{N-3, Z+1}, & \partial_4 &= \{\partial_Z^{(1)} - \partial_N^{(2)}\}_{N-1, Z}, \end{aligned} \quad (15)$$

corresponding to the various n_i 's. Comparing (11), (12) with (6), (7), we conclude that the sign of any difference $\partial_i - \partial_j$ involving a difference $\partial_Z^{(2)} - \partial_Z^{(1)}$ or $\partial_N^{(2)} - \partial_N^{(1)}$ will be fixed by the sign of this last difference, according to (9), irrespective of other small differences of $\partial^{(1)}$'s or $\partial^{(2)}$'s with different arguments. This yields immediately

$$\partial_1 > \partial_2 > \partial_4, \quad \partial_1 > \partial_3 > \partial_4. \quad (16)$$

In order to compare ∂_2 and ∂_3 , we first observe that

$$\partial_2 \approx \partial_3(N+2), \quad (17)$$

where the notation $\partial_3(N+2)$ is used to represent the value $\{\partial_Z^{(2)} - \partial_N^{(2)}\}_{N-1, Z+1}$ of the function $\partial_3(N)$ for the value $N+2$ of its argument. Relation (17) follows from (10), (11) with the help of the identity

$$\partial_Z^{(1)}(N-2, Z) + \partial_N^{(1)}(N-2, Z+1) = \partial_N^{(1)}(N-2, Z) + \partial_Z^{(1)}(N-1, Z), \quad (18)$$

expressing that the energy change due to the addition of a proton and a neutron to the nucleus is independent of the order in which the particles are added. In virtue of (8), one gets from (17) the inequality

$$\partial_2 > \partial_3, \quad (19)$$

which, together with (16), leads to the inequalities (14).

Moreover, using (10) and (11), one readily finds

$$\begin{aligned} \partial_1 - \partial_2 &\approx \partial_3 - \partial_4 \\ \partial_1 - \partial_4 &\approx 2 \left[\partial_N^{(2)} - \partial_N^{(1)} - \left| \frac{\partial^2 \mathcal{E}}{\partial Z \partial N} \right| \right]. \end{aligned} \quad (20)$$

From (17) and (20) estimates of the differences $n_i - n_j$ can readily be obtained, if one again disregards possible differences between the Coulomb energy changes occurring on the right-hand side of (13). One finds

$$\begin{aligned} n_2 - n_1 &\approx n_4 - n_3 \\ n_3 - n_2 &\approx 2 \\ n_4 - n_1 &\approx 2 \frac{\partial_N^{(2)} - \partial_N^{(1)} - \left| \frac{\partial^2 \mathcal{E}}{\partial Z \partial N} \right|}{\frac{\partial^2 \mathcal{E}}{\partial N^2} + \left| \frac{\partial^2 \mathcal{E}}{\partial Z \partial N} \right|}; \end{aligned} \quad (21)$$

with the help of (7), (6), the last approximate equality yields

$$n_4 - n_1 \approx 1,2 \left(\frac{\partial_N^{(2)} - \partial_N^{(1)}}{\left| \frac{\partial^2 \mathcal{G}}{\partial Z \partial N} \right|} - 1 \right) \quad (22)$$

$$n_4 - n_1 \approx 3 \dots 7; \quad (23)$$

then, from the first two relations (21) and (23),

$$n_2 - n_1 \approx n_4 - n_3 \approx 0,5 \dots 2,5. \quad (24)$$

The difference $n_3 - n_2$ characterizes the "isotopic breadth" (number of stable isotopes of given Z) of nuclei of odd charge; $n_4 - n_1$ indicates, for any given Z , how many pairs of stable isobars of charge numbers Z , $Z + 2$ can exist.

The relations (21), (23), (24) are on the whole in good agreement with the empirical n_i 's, which can be determined from the table of known nuclei, for the different (even) values of Z , with a margin of 2 (or in some cases only 1) *. In particular, the number of stable isotopes of any odd charge number is always 1 or 2, corresponding to $2 \leq n_3 - n_2 \leq 4$. The difference $n_4 - n_1$ is found to increase with Z up to $Z \approx 60$, and then to decrease; the first trend may be understood from (22), while the explanation of the decrease at higher Z is not clear (cf. Fuchs' paper).

* The ∂_i 's can then be deduced in terms of Z from the equations (13). Owing to the slowness of the variation of the right-hand sides of these equations with n , all four ∂_i 's come to lie practically on the same smooth curve, slowly rising with increasing Z . The orders of magnitude range from ≈ 3 MeV ($Z \approx 10$) to ≈ 11 MeV ($Z \approx 80$).

PART II

TWO-NUCLEON SYSTEMS ON THE HYPOTHESIS OF CENTRAL INTERACTION

CHAPTER IV

DYNAMICAL VARIABLES AND FUNDAMENTAL EQUATIONS

4.1. Dichotomic variables

4.11. Definition and main properties. When we have to distinguish between two possible states of some system, such as, e.g., the two states of different charge (proton and neutron) of a nucleon, we may conveniently characterize them by the eigenvalues, say $+1$ and -1 , of some quantal variable τ . A discrimination between the two states in question then corresponds to a "measurement" of the variable τ , such a measurement yielding the answer $+1$ or -1 . In a matrix representation, such a *dichotomic variable* will appear as a Hermitian matrix with 2 rows and 2 columns, and will satisfy the equation

$$\tau^2 = 1. \quad (1)$$

From these conditions, it is easily deduced that the most general form for τ is *

$$\tau = \begin{pmatrix} a & \sqrt{1-a^2} \cdot e^{-i\varphi} \\ \sqrt{1-a^2} \cdot e^{i\varphi} & -a \end{pmatrix}, \quad (2)$$

where a is a real number with $|a| \leq 1$, and $e^{\pm i\varphi}$ represents an arbitrary phase factor. Putting $a = \cos \vartheta$, we may also write

$$\tau = \begin{pmatrix} \cos \vartheta & \sin \vartheta \cdot e^{-i\varphi} \\ \sin \vartheta \cdot e^{i\varphi} & -\cos \vartheta \end{pmatrix}. \quad (3)$$

This form discloses a remarkable connexion between our dichotomic variables and the directions in ordinary space. In fact, we may interpret (ϑ, φ) as the polar coordinates (with respect to an arbitrarily chosen system of reference) of a unit vector $\vec{n}$, the cartesian components of which will thus be

$$\begin{aligned} n_x &= \sin \vartheta \cos \varphi \\ n_y &= \sin \vartheta \sin \varphi \\ n_z &= \cos \vartheta. \end{aligned} \quad (4)$$

Instead of (3) we may then write

$$\tau = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}. \quad (5)$$

* We exclude, of course, the unit matrix, which is also a solution of (1).

Introducing now the matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (6)$$

we see from (5) that the special dichotomic variables which correspond to the directions of the x -, y - and z -axis, respectively, and which we will denote by τ_x , τ_y , τ_z , are represented just by these matrices:

$$\tau_x = \tau_1, \quad \tau_y = \tau_2, \quad \tau_z = \tau_3; \quad (7)$$

further the relation

$$\tau = \tau_x n_x + \tau_y n_y + \tau_z n_z \quad (8)$$

holds between the variable τ and the unit vector $\vec{n}$.

If we choose another system of reference to represent the unit vector $\vec{n}$, the matrices τ_x , τ_y , τ_z will no longer be given by (7); in fact, as is immediately apparent from the relation (8), they will transform *like the components of a vector*. The variable τ , as expressed by (8), may thus be regarded as the projection of the fundamental vector $\vec{\tau}$ with components τ_x , τ_y , τ_z on the direction $\vec{n}$; as such, it will be denoted as $\tau(\vec{n})$. For the product $\tau(\vec{n}) \tau(\vec{n}')$ we derive from (5), by a straightforward calculation, the invariant relation

$$\tau(\vec{n}) \tau(\vec{n}') = \vec{n} \cdot \vec{n}' + i \tau(\vec{n} \wedge \vec{n}'), \quad (9)$$

from which it follows that

$$[\tau(\vec{n}), \tau(\vec{n}')]_+ = 2 \vec{n} \cdot \vec{n}' \quad (10)$$

$$[\tau(\vec{n}), \tau(\vec{n}')]_- = 2i \tau(\vec{n} \wedge \vec{n}'). \quad (11)$$

Applied to the cartesian (orthogonal) components of the vector $\vec{\tau}$, these relations become

$$\tau_x \tau_y = i \tau_z \quad (\text{cycl.}) \quad (12)$$

$$[\tau_x, \tau_y]_+ = 0 \quad (\text{cycl.}) \quad (13)$$

$$[\tau_x, \tau_y]_- = 2i \tau_z \quad (\text{cycl.}). \quad (14)$$

On account of the special representation (6), analogous formulae may be written down for the matrices τ_1 , τ_2 , τ_3 ; they may of course, be directly verified with the help of the definitions (6).

4.111. Spin. The correspondence here discussed between dichotomic variables and spatial directions is, in the general case, purely symbolical. But it acquires a real significance in the theory of the *spin*, or intrinsic

angular momentum, of the electron or nucleon. For the establishment of this concept the existence of a set of dichotomic variables forming a vector, together with the relations (14) between them, is of essential importance. As is well-known, spectroscopic evidence showed the necessity of attributing to the electron an intrinsic angular momentum, such that its component in any direction can only have two values of equal magnitudes and opposite signs. This spin component must thus clearly be proportional to a dichotomic variable like $\tau(\vec{n})$, which we shall call $\sigma(\vec{n})$. The coefficient expressing its magnitude can further be determined by the condition that the commutation rules

$$[S_x, S_y]_- = i\hbar S_z \quad (\text{cycl.}), \quad (15)$$

characteristic for the components of any angular momentum, be satisfied. Comparing (15) with (14), we see that this is just the case for the components of any dichotomic vector $\vec{\tau}$, apart from the factor $\frac{1}{2}\hbar$. The spin angular momentum $\vec{S}$ may thus be expressed as

$$\vec{S} = \frac{1}{2}\hbar\vec{\sigma}. \quad (16)$$

4.12. Transformation of eigenfunctions. When the description of a system involves a dichotomic variable τ , the wave-functions $\psi(\tau')$ depend on the eigenvalue τ' of τ ; since τ' is susceptible of the two values $+1$ and -1 , we may conceive the wave-function as consisting of two "components"

$$u \equiv \psi(+1) \quad , \quad v \equiv \psi(-1),$$

which, in connexion with the matrix representation of the dichotomic variables, are conveniently arranged as a matrix

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \quad (17)$$

with two rows and one column. When operators involving dichotomic variables are written as matrices, the result of their operating on ψ is then obtained by applying the usual rule for matrix multiplication. The quantities $|u|^2$, $|v|^2$ give the relative probabilities for finding the state corresponding to $\tau' = +1$ or $\tau' = -1$, respectively (all other conditions being the same). The eigenfunctions of these states are therefore of the form

$$\psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix} \quad , \quad \psi_- = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad (18)$$

respectively; the matrix τ itself has the diagonal form denoted, according to (6), by τ_3 . The functions u and v depend, of course, on the other variables of the system; in particular, there are many different eigenfunctions of the type ψ_+ or ψ_- .

Since τ has the form τ_3 , the operator which, applied to any wave-function ψ , transforms it into an eigenfunction ψ_+ is

$$T_+ = \frac{1}{2}(1 + \tau_3); \quad (19)$$

similarly,

$$T_- = \frac{1}{2}(1 - \tau_3) \quad (20)$$

transforms ψ into an eigenfunction ψ_- . Especially, we have, by (18),

$$T_+ \psi_- = 0, \quad T_- \psi_+ = 0, \quad (21)$$

and conversely, these relations characterize the eigenfunctions of the states $\tau' = +1$ and $\tau' = -1$, respectively. Let now $\tau_\perp$ be the dichotomic variable corresponding to an arbitrary direction perpendicular to that relating to the variable τ ; in our representation, in which $\tau = \tau_3$, this will be, according to (7),

$$\tau_\perp = \tau_1 \cos \varphi + \tau_2 \sin \varphi, \quad (22)$$

with arbitrary φ . Since, by (10), $\tau_\perp$ anticommutes with τ , we have

$$\tau_\perp T_+ = T_- \tau_\perp. \quad (23)$$

Observing that $T_+ T_- = 0$, we get from this

$$T_+ \tau_\perp T_+ = 0,$$

which means, according to (21), that if we apply the operator $\tau_\perp T_+$ to any eigenfunction ψ_+ , the result will be an eigenfunction ψ_- (while the application of $\tau_\perp T_+$ to an eigenfunction ψ_- gives zero). In other words, the operator $\tau_\perp T_+$ effects the transition from the state $\tau' = +1$ to the state $\tau' = -1$. Denoting it by Π , we have by (22)

$$\Pi = \tau_\perp T_+ = \frac{1}{2}(\tau_1 - i\tau_2) e^{i\varphi},$$

or more simply, if we choose $\varphi = 0$,

$$\Pi = \frac{1}{2}(\tau_1 - i\tau_2). \quad (24)$$

Similarly, the operator transforming an eigenfunction ψ_- into one of the type ψ_+ is $\tau_\perp T_-$ or $\Pi^\dagger$.

4.13. Exchange operators for two identical particles. Let us now consider two identical systems (1), (2), such as, for instance, two electrons or two nucleons; if a dichotomic variable τ , and consequently the dichotomic vector $\vec{\tau}$, is attached to the system, let $\vec{\tau}^{(1)}$ be the vector pertaining to system (1) and $\vec{\tau}^{(2)}$ the analogous vector pertaining to system (2). Operators involving both $\vec{\tau}^{(1)}$ and $\vec{\tau}^{(2)}$ may be written as matrices with 4 rows

and 4 columns, the elements of which are numbered by the eigenvalues ± 1 of $\tau_3^{(1)}$ and $\tau_3^{(2)}$; thus, for instance,

$$\tau_1^{(1)} \tau_1^{(2)} = \begin{pmatrix} 0 & \tau_1^{(2)} \\ \tau_1^{(2)} & 0 \end{pmatrix} = \begin{matrix} \tau_3^{(1)'} & +1 & +1 & -1 & -1 \\ & \tau_3^{(2)'} & +1 & -1 & +1 & -1 \\ +1 & +1 & 0 & 0 & 0 & 1 \\ +1 & -1 & 0 & 0 & 1 & 0 \\ -1 & +1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \end{matrix} \quad (25)$$

The wave-functions of the total system (1) + (2) will then consist of 4 "components", numbered in the same way and forming a 1-column matrix.

Especially interesting are the operators $\vec{\tau}^{(1)} \vec{\tau}^{(2)}$ and

$$P = \frac{1}{2} (1 + \vec{\tau}^{(1)} \vec{\tau}^{(2)}). \quad (26)$$

Using (1) and (12), we readily find

$$(\vec{\tau}^{(1)} \vec{\tau}^{(2)})^2 = 3 - 2 (\vec{\tau}^{(1)} \vec{\tau}^{(2)}), \quad (27)$$

from which it follows that

$$P^2 = 1. \quad (28)$$

The operator P is thus unitary (since it is Hermitian and its own inverse) and it effects a canonical transformation of the $\vec{\tau}^{(1)}, \vec{\tau}^{(2)}$. The result of this transformation is, as a simple calculation shows,

$$\begin{aligned} P \vec{\tau}^{(1)} P &= \vec{\tau}^{(2)} \\ P \vec{\tau}^{(2)} P &= \vec{\tau}^{(1)}, \end{aligned} \quad (29)$$

i.e. an interchange of the dichotomic variables of the two systems (1), (2); P is often called the *exchange* operator for the variables τ .

According to (28), P has two distinct eigenvalues $+1, -1$. Eigenfunctions of the eigenvalues $+1$ are symmetrical in $\tau_3^{(1)'}, \tau_3^{(2)'}$, eigenfunctions of -1 antisymmetrical in these arguments; further, they may, of course, be taken as bilinear combinations of the eigenfunctions $\psi_+^{(1)}, \psi_-^{(1)}$ of $\tau_3^{(1)}$ and $\psi_+^{(2)}, \psi_-^{(2)}$ of $\tau_3^{(2)}$. The three distinct symmetrical combinations

$$\psi_+^{(1)} \psi_+^{(2)}, \quad \psi_-^{(1)} \psi_-^{(2)}, \quad \psi_+^{(1)} \psi_-^{(2)} + \psi_-^{(1)} \psi_+^{(2)} \quad (30)$$

imply a threefold degeneracy of the eigenvalue $+1$; the eigenvalue -1 is simple, with the antisymmetrical eigenfunction

$$\psi_+^{(1)} \psi_-^{(2)} - \psi_-^{(1)} \psi_+^{(2)}. \quad (31)$$

4.14. Equivalent formulations of the exclusion principle. When the specification of a particle involves a dichotomic variable, we are at liberty

to treat a system of such particles as a mixture of two distinct kinds of particles, characterized by the two eigenvalues of the dichotomic variables. Thus a system of particles of spin $\frac{1}{2}$ can be regarded as composed of particles of "left" spin and particles of "right" spin; a system of nucleons as a system of protons and neutrons. In the original conception, each particle is defined by an eigenvalue τ_3 of the dichotomic variable and a set Q of other coordinates; in the new conception, we have two distinct sets of coordinates Q .

The exclusion principle can accordingly be formulated in two equivalent ways, by stating either that any wave-function of the arguments $Q^{(i)}, \tau_3^{(i)}$ must be antisymmetrical in *all* coordinates, or that any wave-function of the arguments $Q^{(i)}, Q^{(j)}$ pertaining to the two different kinds of particles must be antisymmetrical *separately* with respect to the $Q^{(i)}$ and the $Q^{(j)}$. The equivalence of these two formulations may readily be seen if one observes that the most general wave-function of a system of A particles, satisfying the exclusion principle as first enunciated, may be expanded in terms of a complete orthogonal system of "Slater-determinants" *, constructed in the following way. Take any operator (such as the total kinetic energy of the system) which depends additively on the individual particles and does not involve any coupling between the dichotomic variable and the other coordinates; the eigenfunctions of the corresponding operator for a single nucleon will then be of the form $\psi_n(Q)u_{\pm}(\tau)$, where τ is put for the eigenvalue τ_3 of the dichotomic variable and $u_{\pm}(\tau)$ are the eigenfunctions of τ_3 :

$$\begin{aligned} u_+ (+1) &= u_- (-1) = 1 \\ u_+ (-1) &= u_- (+1) = 0; \end{aligned} \quad (32)$$

and the eigenfunctions of the total operator, satisfying the exclusion principle, may be written as determinants of the type

$$\begin{vmatrix} \psi_{n_1}(Q^{(1)}) u_+(\tau^{(1)}) & \dots & \psi_{n_1}(Q^{(A)}) u_+(\tau^{(A)}) \\ \dots & \dots & \dots \\ \psi_{n_N}(Q^{(1)}) u_+(\tau^{(1)}) & \dots & \psi_{n_N}(Q^{(A)}) u_+(\tau^{(A)}) \\ \psi_{n_{N+1}}(Q^{(1)}) u_-(\tau^{(1)}) & \dots & \psi_{n_{N+1}}(Q^{(A)}) u_-(\tau^{(A)}) \\ \dots & \dots & \dots \\ \psi_{n_A}(Q^{(1)}) u_-(\tau^{(1)}) & \dots & \psi_{n_A}(Q^{(A)}) u_-(\tau^{(A)}) \end{vmatrix}, \quad (33)$$

the upper indices pertaining to the different particles, while $n_1, \dots, n_A$ represent a permutation of the sequence $1, 2, \dots, A$. The Slater-determinant (33) refers to N particles in "+1" states and $Z = A - N$ particles in "-1" states; on account of (32), it is immediately apparent that it is different from zero only when N of the τ 's take the value +1 and the Z

other the value -1 , in which case it reduces to the form

$$\pm \begin{vmatrix} \psi_{n_1}(Q^{(m_1)}) & \dots & \psi_{n_1}(Q^{(m_N)}) \\ \vdots & \ddots & \vdots \\ \psi_{n_N}(Q^{(m_1)}) & \dots & \psi_{n_N}(Q^{(m_N)}) \end{vmatrix} \cdot \begin{vmatrix} \psi_{n_{N+1}}(Q^{(m_{N+1})}) & \dots & \psi_{n_{N+1}}(Q^{(m_A)}) \\ \vdots & \ddots & \vdots \\ \psi_{n_A}(Q^{(m_{N+1})}) & \dots & \psi_{n_A}(Q^{(m_A)}) \end{vmatrix}, \quad (34)$$

where $m_1, \dots, m_A$ denotes some permutation of $1, 2, \dots, A$. This, now, is just the type of products of Slater-determinants which would have been obtained for the eigenfunctions of a system of N “+1” particles and Z “-1” particles, considered as two distinct kinds of particles, each obeying the exclusion principle.

4.141. An alternative proof, due to KLEIN [38], will be given here for the sake of its elegance. We would even have given it in preference, were it not that it is based on the so-called “second quantization” method, the knowledge of which we do not wish to require from the reader. The quantized amplitudes of a system of particles satisfying the exclusion principle in the form stated above are functions of the Q 's and τ , say $a(Q, \tau)$, $a^+(Q, \tau)$, which obey the commutation rules

$$\begin{aligned} [a(Q_1, \tau_1), a(Q_2, \tau_2)]_+ &= [a^+(Q_1, \tau_1), a^+(Q_2, \tau_2)]_+ = 0 \\ [a(Q_1, \tau_1), a^+(Q_2, \tau_2)]_+ &= \delta(Q_1, Q_2) \delta_{\tau_1 \tau_2}, \end{aligned} \quad (35)$$

where $\delta(Q_1, Q_2)$ is the ordinary or the Dirac δ -function, according as the Q 's have a discrete or continuous range of values. We recall that the product $a^+(Q, \tau) a(Q, \tau) = N(Q, \tau)$ represents the number (1 or 0) of particles in the state (Q, τ) ; as may readily be deduced from (35), the N 's commute with all a 's and a^+ 's, except that

$$a(Q, \tau) N(Q, \tau) = [1 - N(Q, \tau)] a(Q, \tau). \quad (36)$$

Now, we intend to show the equivalence of this mode of description of the system with another, characterized by two sets of amplitudes $a_+(Q)$, $a_-(Q)$ satisfying the following requirements:

- (1) the expressions $a_+^\dagger a_+$, $a_-^\dagger a_-$ represent the numbers of “+1” particles and “-1” particles in the state Q , respectively;
- (2) the $(a_+, a_+^\dagger)$, $(a_-, a_-^\dagger)$ satisfy separately commutation rules of the same type as (35);
- (3) any of the $(a_+, a_+^\dagger)$ commutes with any of the $(a_-, a_-^\dagger)$.

The two first conditions are obviously satisfied by identifying a_+ , a_- with $\pm a(\tau = +1)$, $\pm a(\tau = -1)$, respectively; the third condition can then be fulfilled by suitably fixing the double sign. It suffices to put

$$\begin{aligned} a_+(Q) &= a(Q, +1) \\ a_-(Q) &= (-1)^{\sum Q} a(Q, -1). \end{aligned} \quad (37)$$

From (36), it follows, in fact, that

$$[a(Q, +1), (-1)^{\sum Q} a(Q, +1)]_+ = 0, \quad (38)$$

whence, taking account of (35),

$$\begin{aligned} [a(Q, +1), (-1)^{\sum Q} a(Q, +1)]_+ &= 0 \\ [a(Q, -1), (-1)^{\sum Q} a(Q, +1)]_- &= 0; \end{aligned} \quad (39)$$

from (39) and (35), the relations

$$[a_+(Q_1), a_-(Q_2)]_- = [a_+(Q_1), a_-^\dagger(Q_2)]_- = 0 \quad (40)$$

are immediately deduced, on account of the definition (37).

4.2. Wave-equation of the nucleon

4.21. Linearization of wave-equation. A very remarkable use of dichotomic variables has, as is well-known, been made by DIRAC [35] in order to "linearize" the fundamental equation

$$E^2 = M^2 + p^2 \quad (1)$$

between the energy E , rest mass M and momentum p (multiplied by c) of a free particle of spin $\frac{1}{2}$, i.e. to find a rational function of M and of the components of p , the square of which would be equal to the right-hand side of (1). From (4.11-1) and (4.11-8) we first deduce, for any vector p with commutable components, and commuting with $\vec{\tau}$, the identity

$$p^2 = (\vec{\tau} \vec{p})^2, \quad (2)$$

whence the expression $\vec{\tau} \vec{p}$ is seen to yield the desired result as regards p^2 . We may also decompose the vector $\vec{p}$ into two perpendicular components $p_{||}, p_{\perp}$; with the help of the dichotomic variables $\tau_{||}, \tau_{\perp}$ associated with the directions of these components, we may then write, on account of the invariant form of (2),

$$p^2 = p_{||}^2 + p_{\perp}^2 = (\tau_{||} p_{||} + \tau_{\perp} p_{\perp})^2. \quad (3)$$

This gives us a purely algebraical identity, expressing that the square of the rational function $\tau_{||} p_{||} + \tau_{\perp} p_{\perp}$, in which $p_{||}, p_{\perp}$ are any two commutable quantities, while $\tau_{||}, \tau_{\perp}$ refer to two perpendicular directions, is just $p_{||}^2 + p_{\perp}^2$.

Turning now to equation (1), we can effect its linearization by means of two dichotomic variables $\vec{\sigma}, \vec{\varrho}$; the first of these will allow us, according to (2), to replace p^2 by $(\vec{\sigma} \vec{p})^2$, while the second will be used, according to (3), to express the right-hand side of (1) as the square of $\varrho_{||} M + \varrho_{\perp} (\vec{\sigma} \vec{p})$. This last expression Dirac then assumes as the Hamiltonian of the free particle:

$$E = \varrho_{||} M + \varrho_{\perp} (\vec{\sigma} \vec{p}). \quad (4)$$

On closer discussion, it turns out that the vector $\vec{\sigma}$ can be interpreted as the spin vector of the particle. For this point, as well as for the further development of the theory, we must refer the reader to Dirac's book. Here we shall only mention that the other dichotomic variables $\varrho_{||}$ and $\varrho_{\perp}$ also receive simple and important interpretations.

We may first enumerate the four components of the wave-function by the eigenvalues of σ_3 and of $\varrho_{||}$, which thus takes the diagonal form* ϱ_3 ; for $\varrho_{\perp}$ we may then conveniently choose the form ϱ_1 . The variable $\varrho_{||}$ divides these components into two pairs, which we shall denote by $\psi_+ \equiv \psi(\varrho_{||} = +1)$ and $\psi_- \equiv \psi(\varrho_{||} = -1)$, respectively. We now write down the wave-equation for ψ_- separately. To find this, first multiply (4) on the right by $P_- = \frac{1}{2}(1 - \varrho_{||})$; remembering the property of the operator $\varrho_{\perp} P_-$, discussed above (4.12), of transforming ψ_- into ψ_+ , we then get

$$(E + M) \psi_- = (\vec{\sigma} \vec{p}) \psi_+, \quad (5)$$

whence, for a state of given momentum $\vec{p}$ and positive energy E , we find for the order of magnitude of the ratio of the amplitudes of ψ_- and ψ_+

$$\left| \frac{\psi_-}{\psi_+} \right| \approx \frac{p}{E + M} \approx \frac{v}{c}, \quad (6)$$

v being the velocity of the electron in the state considered. In view of the limiting case of slow velocities, the pairs of components ψ_+ , ψ_- are therefore usually called the "large" and the "small" components, respectively. We thus see that $\varrho_{||}$ discriminates between large and small components of the wave-function.

If, on the other hand, we choose $\varrho_{\perp}$ as dichotomic variable to enumerate the components of the wave-function instead of $\varrho_{||}$ (so that now $\varrho_{\perp} = \varrho_3$ and $\varrho_{||} = \varrho_1$, say), we get another separation into two pairs, called "spinors", each of which transforms according to a two-dimensional irreducible representation of the Lorentz group (excluding reflections). We shall, however, not further discuss this aspect of the theory, of which no use is made in the book.

4.22. Isotopic variable. In conformity with the conception of proton and neutron as states of different charge of the nucleon, we introduce a dichotomic variable to distinguish these states. It will be called the *isotopic variable***, we make the convention that its eigenvalues $+1$, -1 denote neutron and proton states, respectively. To the isotopic variable corresponds, according to the general theory, a fundamental vector in a symbolical space, which will be denoted (to distinguish it from vectors in ordinary space) by a clarendon type τ , while its components τ_1 , τ_2 , τ_3 , as defined by (4.11-6), will be distinguished by clarendon type indices. The corresponding symbolical space will be called the *isotopic space*. The wave-function of the nucleon will then be considered as depending also on the eigenvalues τ_3 of the isotopic variable τ_3 . With the help of the operators

$$T_+ = \frac{1}{2}(1 + \tau_3) \quad , \quad T_- = \frac{1}{2}(1 - \tau_3), \quad (7)$$

* Dirac takes $\varrho_{||} = -\varrho_3$, $\varrho_{\perp} = -\varrho_1$, which only amounts to a permutation of the respective roles of the eigenvalues $+1$ and -1 of $\varrho_{||}$.

** The usual name is "isotopic spin".

the mass M of the nucleon is expressed as an operator

$$M = M_n T_+ + M_p T_- \quad (8)$$

If we adopt (1.21) for the nucleon the linearization procedure (4.21) of Dirac, the Hamiltonian of a free nucleon takes the form (4), in which M has to be replaced, strictly speaking, by its expression (8); it will, however, mostly be permissible to neglect the effects of the mass difference between proton and neutron and treat M as a constant*.

4.221. Interaction with electromagnetic field. In order to account for the interaction of a nucleon with an external electromagnetic field, we must add to the Hamiltonian not only the usual terms connected with the charge of the proton, but also other terms arising from the anomalous magnetic moment (1.21, 1.22). The Hamiltonian representing this interaction correctly to the first order of approximation in the nucleon velocity is the following:

$$\begin{aligned} \mathcal{H} = e_1 \vec{\sigma} (\vec{p} - T_- e \vec{A}) + T_- e V + e_3 M \\ - [\mu_n T_+ + (\mu_p - 1) T_-] \mu_0 \vec{\sigma} \vec{H}, \end{aligned} \quad (9)$$

$\vec{A}$, V denoting the electromagnetic (vector and scalar) potential, and $\vec{H}$ the magnetic field, taken at the point $\vec{x}$ occupied by the nucleon. From this Hamiltonian the wave-equation is derived by interpreting the operators $\mathcal{H}$, $\vec{p}$ as

$$\mathcal{H} \rightarrow \frac{i \hbar}{c} \frac{\partial}{\partial t}, \quad \vec{p} \rightarrow \frac{\hbar}{i} \text{grad.} \quad (10)$$

More rigorously, we should include into the Hamiltonian the interaction of the nucleon with the nuclear field (1.21), which should automatically give rise, in first approximation, to the supplementary terms of equation (9). In this way, the magnetic anomalies are related to the property of nucleons of going over from neutron to proton states, and conversely, with emission of mesons or leptons; this may certainly be regarded as an attractive feature of the field conception. From a formal point of view, considered as a method for correcting Dirac's equation, when applied to nucleons, without upsetting its main properties, such a procedure, however, is admittedly wrong in so far as it partakes of all the evils of quantum field theory, and it remains to be seen whether a much more radical departure from the present formalism will not prove unavoidable.

* From the point of view of a fundamental theory of elementary particles, one has to keep in mind the possibility that the mass parameter occurring in the wave equation of a nucleon should actually be a constant M , irrespective of the charge of the nucleon, while the difference between the observed masses of proton and neutron should be attributed to the energy of the proper electromagnetic and nuclear fields created by these particles.

4.3. Reduction of the wave-equation of the two-nucleon system

4.31. The fundamental wave-equation. The wave-equation of the two-nucleon system can be derived, to the first order in the nucleon velocities, from the Hamiltonian

$$\mathcal{H}' = e_1^{(1)} \vec{\sigma}^{(1)} \vec{p}^{(1)} + e_1^{(2)} \vec{\sigma}^{(2)} \vec{p}^{(2)} + (e_3^{(1)} + e_3^{(2)}) M + \mathcal{V}, \quad (1)$$

in which the nucleon mass M is considered as a constant and the interaction operator $\mathcal{V}$ includes the Coulomb energy

$$\frac{e^2}{r} \mathbf{T}^{(1)} \mathbf{T}^{(2)}.$$

Strictly speaking, a Hamiltonian of the form (1) is Lorentz invariant only when the potential energy $\mathcal{V}$ corresponds to a "contact" interaction, i.e. is proportional to $\delta(\vec{x}) \cdots \delta(x) \delta(y) \delta(z)$. In fact, this mode of interaction is the only one allowed by relativity requirements as long as one has to do with a *direct* interaction, without any intermediary field. For the only admissible expressions of the interaction energy, from this point of view, are of the type

$$\int s_{ijk\dots}(\vec{x}) s^{ijk\dots}(\vec{x}) dv,$$

involving products of source densities $s_{ijk\dots}$ taken at the same point of space (and the same time); this just corresponds to an interaction potential of the form $\delta(\vec{x})$. Still, the Hamiltonian (1) may be used with any other form of distance dependence of $\mathcal{V}$ up to the first order of approximation in the nucleon velocities. This will be shown in a later section (15.22), in which second order corrections will also be summarily discussed.

The operator $\mathcal{V}$, which is symmetrical in the sets of coordinates of the two interacting nucleons, has to satisfy the following general requirements of invariance:

- (a) conservation of energy demands that it be Hermitian;
- (b) conservation of momentum and Galilean relativity, that it depend on the position and momentum variables of the nucleon pair only through the relative coordinates and momenta *;
- (c) conservation of angular momentum, that it be invariant under spatial rotations;
- (d) $\mathcal{V}$ must also be invariant under spatial reflections (this property does not correspond to any classical conservation law **);

* Conservation of momentum means invariance with respect to translations, the infinitesimal transformation operators of which are proportional to the components of the total momentum. The operators of infinitesimal Galilei transformations are proportional to the coordinates of the centre of gravity.

** One might speak of "conservation of parity" (in a statistical sense); cf. E. WIGNER, *Gött. Nachr.* 1927, p. 375. The concept of *parity* is defined in 4.331.

- (e) invariance under a change of sign of the time variable implies that $\mathcal{V}$ remains invariant when the momenta and spin variables change sign, while at the same time any operator $\Pi \equiv \frac{1}{2} (\tau_1 - i\tau_2)$, effecting a neutron-proton transition (4.12-24), is replaced by the corresponding $\Pi^\dagger$, which effects the reverse transition, and vice versa;
- (f) conservation of charge, that $\mathcal{V}$ commute with $\tau_3^{(1)} + \tau_3^{(2)}$, in other words that it be invariant under rotations about the 3-axis in symbolical isotopic space.

In addition, we shall now assume that

- (g) $\mathcal{V}$ does not contain either the momenta $\vec{p}^{(1)}, \vec{p}^{(2)}$ or the variables $\varrho_i^{(1)}, \varrho_i^{(2)}$ ($i = 1, 2, 3$);

this means that we have to do with a *static interaction*. This is the essential restriction underlying the following developments. Combining it with the invariance requirements (b), (c) and (d) above, we conclude that

- (h) $\mathcal{V}$ is symmetric with respect to the spin variables $\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}$.

4.311. Reduction to the barycentric system of reference. Let us introduce the coordinates

$$\vec{x}_b = \frac{\vec{x}^{(1)} + \vec{x}^{(2)}}{2}, \quad \vec{p}_b = \vec{p}^{(1)} + \vec{p}^{(2)} \quad (2)$$

of the centre of gravity of the system and the relative coordinates

$$\vec{x} = \vec{x}^{(1)} - \vec{x}^{(2)}, \quad \vec{p} = \frac{1}{2} (\vec{p}^{(1)} - \vec{p}^{(2)}). \quad (3)$$

Applying to the Hamiltonian (1) the canonical transformation (2), (3), we get, on account of assumption (b) about $\mathcal{V}$,

$$\mathcal{H}' = \frac{1}{2} (\varrho_1^{(1)} \vec{\sigma}^{(1)} + \varrho_1^{(2)} \vec{\sigma}^{(2)}) \vec{p}_b + \mathcal{H}, \quad (4)$$

with

$$\mathcal{H} = (\varrho_1^{(1)} \vec{\sigma}^{(1)} - \varrho_1^{(2)} \vec{\sigma}^{(2)}) \vec{p} + (\varrho_3^{(1)} + \varrho_3^{(2)}) \mathbf{M} + \mathcal{V}. \quad (5)$$

From this it appears that $\vec{p}_b$, commuting with $\mathcal{H}'$, is a constant of the motion. We may therefore restrict ourselves to such states of the system for which

$$\vec{p}_b = 0, \quad (6)$$

i.e. the centre of gravity is at rest. In this *barycentric system of reference*, the Hamiltonian, expressed in the relative coordinates, is simply given by $\mathcal{H}$, as defined by (5). The total angular momentum (multiplied by c) is, in this system,

$$\hbar \vec{J} \equiv \vec{x} \wedge \vec{p} + \frac{\hbar}{2} (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}); \quad (7)$$

by assumption (c) about $\mathcal{V}$, its three components are constants of the motion.

4.32. Classification of eigenstates by angular momentum. In the usual way, let us consider the commutable operators J^2 and J_z (z denoting any fixed direction in space): the eigenfunctions of the Hamiltonian $\mathcal{H}$ will be linear combinations of the simultaneous eigenfunctions of these two operators. The latter may be obtained as follows. For the square of the spin angular momentum (in units $\hbar$), we get in the first place

$$\frac{1}{4}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})^2 = \frac{1}{2}(3 + \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) = P_\sigma + 1, \quad (8)$$

P_σ being the exchange operator (4.13) for the spin variables. The spin eigenstates therefore consist of two systems, corresponding to the eigenvalues 0 and 2 of the quantity (8): a non-degenerate or *singlet* system and a trebly degenerate or *triplet* system. Their respective eigenfunctions may be written, if v_+ , v_- denote in the usual way the eigenfunctions of σ_z with respective eigenvalues $+1$, -1 :

$$\begin{aligned} \text{Singlet:} \quad {}^1(\sigma)_0 &= \frac{1}{\sqrt{2}} [v_+(1) v_-(2) - v_-(1) v_+(2)] \\ \text{Triplet:} \quad {}^3(\sigma)_0 &= \frac{1}{\sqrt{2}} [v_+(1) v_-(2) + v_-(1) v_+(2)] \\ {}^3(\sigma)_1 &= v_+(1) v_+(2) \\ {}^3(\sigma)_{-1} &= v_-(1) v_-(2); \end{aligned} \quad (9)$$

these are at the same time eigenfunctions of the z -component of the spin $\frac{1}{2}(\sigma_z^{(1)} + \sigma_z^{(2)})$, corresponding to the eigenvalues given by the lower indices of the symbols (σ) . It is easy to see that for the singlet states, we actually get

$$\frac{1}{2}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) {}^1(\sigma)_0 = 0 \quad (10)$$

for all spin components.

On the other hand, the square and the z -component of the orbital angular momentum (in units $\hbar$)

$$\vec{L} = \frac{1}{\hbar} \vec{x} \wedge \vec{p} \quad (11)$$

have as eigenfunctions the tesseral harmonics $Y_l^m(\vartheta, \varphi)$, corresponding to the eigenvalues $l(l+1)$ of L^2 and m of L_z . Since, by (7), (8),

$$J^2 = L^2 + P_\sigma + 1 + (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \cdot \vec{L} \quad (12)$$

and

$$J_z = L_z + \frac{1}{2}(\sigma_z^{(1)} + \sigma_z^{(2)}), \quad (13)$$

* The definitions and notations adopted for the functions are explained in N.14. Here, r , ϑ , φ are the polar coordinates of particle (1) with respect to particle (2), the polar axis being in the z -direction.

we see directly that for the singlet system, the eigenvalues of J^2 and J_z are, on account of (10), just the same as those of L^2 and L_z , so that the eigenfunctions are simply

$$^1Z_l^{(l)m} = ^1(\sigma)_0 Y_l^m. \quad (14)$$

For the triplet system, we may, according to (13), write the eigenfunctions corresponding to a given value l of the orbital angular momentum and to a given eigenvalue m of J_z in the form

$$^3Z_j^{(l)m} = a_j \cdot ^3(\sigma)_0 Y_l^m + b_j \cdot ^3(\sigma)_1 Y_l^{m-1} + c_j \cdot ^3(\sigma)_{-1} Y_l^{m+1}. \quad (15)$$

The eigenvalues λ_j of J^2 as well as the coefficients a_j , b_j , c_j result, by (12), from the eigenvalue-problem

$$[l(l+1) + 2 - \lambda_j] \cdot ^3Z_j^{(l)m} + \langle \sigma^{(1)} + \sigma^{(2)} | \vec{L} \cdot ^3Z_j^{(l)m} = 0. \quad (16)$$

Using the well-known formulae for the matrix-elements of $\vec{L}$, this is readily reduced to a set of three homogeneous linear equations in a_j , b_j , c_j . Putting $\lambda_j = j(j+1)$, the zeros of the secular determinant are

$$j = l \quad \text{and} \quad j = l \pm 1;$$

the normalized eigenfunctions of the form (15) are found to be

$$\begin{aligned} ^3Z_l^{(l)m} &= \frac{1}{\sqrt{2l(l+1)}} \left[\sqrt{2} \sqrt{m \cdot ^3(\sigma)_0} \cdot Y_l^m \right. \\ &\quad \left. + \sqrt{(l+m)(l-m+1)} \cdot ^3(\sigma)_1 \cdot Y_l^{m-1} - \sqrt{(l-m)(l+m+1)} \cdot ^3(\sigma)_{-1} \cdot Y_l^{m+1} \right] \\ ^3Z_{l+1}^{(l)m} &= \frac{1}{\sqrt{2(l+1)(2l+1)}} \left[\sqrt{2} \sqrt{(l+m+1)(l-m+1)} \cdot ^3(\sigma)_0 \cdot Y_l^m \right. \\ &\quad \left. - \sqrt{(l+m)(l+m+1)} \cdot ^3(\sigma)_1 \cdot Y_l^{m-1} - \sqrt{(l-m)(l-m+1)} \cdot ^3(\sigma)_{-1} \cdot Y_l^{m+1} \right] \\ ^3Z_{l-1}^{(l)m} &= \frac{1}{\sqrt{2l(2l+1)}} \left[\sqrt{2} \sqrt{(l+m)(l-m)} \cdot ^3(\sigma)_0 \cdot Y_l^m \right. \\ &\quad \left. + \sqrt{(l-m)(l-m+1)} \cdot ^3(\sigma)_1 \cdot Y_l^{m-1} + \sqrt{(l+m)(l+m+1)} \cdot ^3(\sigma)_{-1} \cdot Y_l^{m+1} \right]. \end{aligned} \quad (17)$$

4.33. Reduction to "large" components. The distribution between "large" and "small" components of the wave-functions (4.21) is effected by the operator $\frac{1}{2}(\varrho_3^{(1)} + \varrho_3^{(2)})$, the eigenfunctions of which form a set

$$^1(\varrho)_0; \quad ^3(\varrho)_0, \quad ^3(\varrho)_1, \quad ^3(\varrho)_{-1}$$

built up in a way entirely analogous to (9) with the eigenfunctions w_+ , w_- of the dichotomic variable ϱ_3 . Any wave-function of our problem may be written as

$$\begin{aligned} \psi &= \psi^{(0)} + \psi^{(1)} + \psi^{(2)}, \\ \psi^{(0)} &= ^3f_1 \cdot ^3(\varrho)_1 \\ \psi^{(1)} &= ^3f_0 \cdot ^3(\varrho)_0 + ^1f_0 \cdot ^1(\varrho)_0 \\ \psi^{(2)} &= ^3f_{-1} \cdot ^3(\varrho)_{-1}; \end{aligned} \quad (18)$$

$\psi^{(0)}$ is then the "large" component, $\psi^{(1)}$ is of the first order in the nucleon velocities, $\psi^{(2)}$ of the second order in these velocities. In the following we shall, in conformity with our initial assumptions (4.31), perform the reduction of the wave-equation only to the first order of approximation in the nucleon velocities; the extension of the method to a higher degree of accuracy would, however, be quite straightforward*.

Inserting (18) into the equation $\mathcal{H}\psi = E\psi$, taking account of assumption (g) about $\mathcal{V}$, and observing that the "non-relativistic" energy

$$\varepsilon = E - 2M \quad (19)$$

and $\mathcal{V}$ are also negligible in comparison with $2M$, we get

$$\begin{aligned} \frac{1}{2} (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) \vec{p} \cdot {}^3f_0 - \frac{1}{2} (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \vec{p} \cdot {}^1f_0 + \mathcal{V} \cdot {}^3f_1 &= \varepsilon \cdot {}^3f_1 \\ \frac{1}{2} (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) \vec{p} \cdot {}^3f_1 &= 2M \cdot {}^3f_0 \\ -\frac{1}{2} (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \vec{p} \cdot {}^3f_1 &= 2M \cdot {}^1f_0, \end{aligned} \quad (20)$$

whence, by eliminating ${}^3f_0$ and ${}^1f_0$ and using (4.11-9), the wave-equation for the large component

$$\left(\frac{p^2}{M} + \mathcal{V} \right) \cdot {}^3f_1 = \varepsilon \cdot {}^3f_1 \quad (21)$$

follows in the form of the ordinary Schrödinger equation with the reduced mass $\frac{1}{2}M$. Once ${}^3f_1$ is determined, the small components ${}^3f_0$ and ${}^1f_0$ may be calculated from the last two relations (20).

4.331. Properties of the large components. Owing to the property (h) of $\mathcal{V}$ and to formula (8), the potential commutes with the square of the total spin and thus conserves the multiplicity of the large components ${}^3f_1$: these eigenfunctions separate into two distinct groups, the *singlet* and the *triplet systems*. Likewise, on account of the rotation invariance (c) of $\mathcal{V}$, the functions ${}^3f_1$ correspond to definite values of the quantum numbers j and m . In the singlet system, the orbital momentum is hereby also fixed ($l = j$). But in the triplet system, it will in general not have a definite eigenvalue: the functions ${}^3f_1$ will be linear combinations of the ${}^3Z_j^{(l)m}$ with $|j - 1| \leq l \leq j + 1$.

The character of this "mixture" of states of different orbital momenta can be more closely determined by taking into consideration the *parity* of the eigenstates. When a spatial symmetry transformation with respect to the origin is performed, the eigenfunction may either resume its original value or change sign: in the former case, we have to do with an *even*

* A rigorous treatment, involving the use of group theory, is due to KEMMER [37a]. The discussion given here is quite elementary.

state; in the latter, with an *odd* state. This transformation property is called the *parity* of the state; it can be designated either by the words "even", "odd" or by the corresponding eigenvalues $+1$, -1 of the spatial symmetry operator. Any state of orbital momentum l has the parity $(-1)^l$. Now, by property (d) of the potential, the large components ${}^3f_1$ must also have a definite parity; they can therefore only involve states with either even or odd values of l . Of the three possible l -values for a given j , viz. $|j-1|$, j , $j+1$, the two extreme ones are of the same parity, while the middle one is of opposite parity. For every value of j , there are thus two distinct triplet states, the one of which has a definite orbital momentum $l=j$, while the other is a mixture of states of orbital momenta $l=j\pm 1$. Using the well-known spectroscopic notation ${}^2L_j$ for states of definite orbital momentum, we may express the result of this discussion by the following table:

j	Even states	Odd states
0		${}^3P_0$
1	${}^3S_1 + {}^3D_1$	${}^3P_1$
2	${}^3D_2$	${}^3P_2 + {}^3F_2$
3	${}^3D_3 + {}^3G_3$	${}^3F_3$
	etc.	

(22)

The coefficients of the Z 's in the ${}^3f_1$ may further be decomposed into a radial factor and an eigenfunction of the total isotopic variable $\frac{1}{2}(\tau_3^{(1)} + \tau_3^{(2)})$, characterizing the given total charge of the system. In a notation entirely analogous to (9), the latter eigenfunctions form a set

$$\tau(\tau)_{m_t}; \quad {}^1(\tau)_0; \quad {}^3(\tau)_{-1}, \quad {}^3(\tau)_0, \quad {}^3(\tau)_{+1}$$

in which the ν 's occurring in (9) are replaced by the eigenfunctions u_+ , u_- of τ_3 . The eigenvalue m_t of the total isotopic variable is also a quantum number of the eigenstate; the values $m_t = -1, 0, +1$ refer to the two-proton, proton-neutron and two-neutron system, respectively.

The symmetry of the charge eigenfunction is indicated by $\tau = 1$ or 3 ("charge singlet", "charge triplet"). The particular eigenfunction $\tau(\tau)_{m_t}$ occurring in the coefficient of a given Z is uniquely fixed by the exclusion principle in the following way. When space and spin coordinates of the two particles are interchanged, the functions ${}^1Z_i^{(l)m}$, ${}^3Z_j^{(l)m}$ are multiplied by $(-1)^{l+1}$ and $(-1)^l$, respectively. In order that the total eigenfunction be antisymmetrical when we also interchange the isotopic variables, we must therefore take:

	Even states	Odd states
Singlet	${}^3(\tau)$	${}^1(\tau)$
Triplet	${}^1(\tau)$	${}^3(\tau)$

(23)

For a system of two protons or two neutrons, the eigenfunction ${}^\tau(\tau)_{m_t}$ has necessarily the form ${}^3(\tau)_{\pm 1}$ and the possible eigenstates are accordingly restricted to the "charge symmetrical" states. The proton-neutron system, on the other hand, possesses the full set of possible eigenstates, with the isotopic eigenfunctions ${}^3(\tau)_0$ and ${}^1(\tau)_0$.

Summing up, the "large" eigenfunctions may thus be written in the general form

$${}^3f_1 = \sum_{l'} \frac{{}^\tau R_j^{(l')}(r)}{r} \cdot {}^\tau Z_j^{(l')m} \cdot {}^\tau(\tau)_{m_t}, \quad (24)$$

the summation over l' reducing to a single term $l' = j$ or extending over $l' = j \pm 1$ according to the above rules; the value of τ' is fixed in terms of σ and the parity of the state by (23). Inserting this in equation (21), it is easily deduced that the radial functions $R(r)$ satisfy the set of differential equations *

$$-\frac{\hbar^2}{M} \left[\frac{d^2}{dr^2} {}^\sigma R^{(l)} - \frac{l(l+1)}{r^2} {}^\sigma R^{(l)} \right] + \sum_{l'} \langle l | {}^\sigma \mathcal{V} | l' \rangle \cdot {}^\sigma R^{(l')} = {}^\sigma \epsilon \cdot {}^\sigma R^{(l)}, \quad (25)$$

in which the matrix-elements of $\mathcal{V}$ are defined by

$$\langle l | {}^\sigma \mathcal{V}_{j, m_t} | l' \rangle = \sum_{\sigma_z, \tau_z} \int d\Omega {}^\sigma Z_j^{(l)m*} \cdot {}^\tau(\tau)_{m_t} \cdot \mathcal{V} \cdot {}^\sigma Z_j^{(l')m} \cdot {}^\tau(\tau)_{m_t}; \quad (26)$$

the summations are to be performed over the isotopic and spin variables $\tau_{\mathbf{3}}^{(1)'}, \tau_{\mathbf{3}}^{(2)'}, \sigma_z^{(1)'}, \sigma_z^{(2)'}$, the integration over the angle variables ϑ, φ ; due to the rotation invariance of $\mathcal{V}$, the quantities (26) are independent of the quantum number m ; the charge multiplicity τ is the same for l and l' , since these numbers have the same parity.

4.34. Central potential. A considerable simplification occurs if the interaction operator $\mathcal{V}$ does not depend on the angle variables. In this case, it has only diagonal matrix elements of the type (26) and the differential system (25) always reduces to separate differential equations for the ${}^\sigma R^{(l)}$. This means that the large eigenfunctions ${}^3f_1$ all belong to definite orbital quantum numbers. The diagonal matrix element of the central potential $\mathcal{V}$ is, of course, independent of j ; in fact, it depends only on the parity of the state, not on the value of l . It is therefore a function ${}^\sigma \mathcal{V}_{m_t}^{(l)}$ of r , which may be regarded as the *effective central potential* in the states of the system with given charge, multiplicity and parity. On account of (23), there are, for the proton-neutron system, four independent effective potentials, pertaining to even or odd states of the singlet or triplet system; we may denote these types of states by the symbols ${}^1S, {}^1P, {}^3S, {}^3P$. The system of two like nucleons, on the other hand, has only eigenstates

* Indices j, m_t affecting the symbols $R, \mathcal{V}, \epsilon$ have been omitted from formula (25).

of the types ${}^1S$ and ${}^3P$ and accordingly two distinct effective potentials.

The radial equation, of which ${}^\tau R^{(l)}$ is a proper solution, has the form *

$$\left\{ -\frac{\hbar^2}{M} \left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right] + {}^\tau \mathcal{V}^{(l)}(r) \right\} {}^\tau R^{(l)} = {}^\tau \epsilon^{(l)} \cdot {}^\tau R^{(l)}, \quad (27)$$

and the large component of the eigenfunction may be written

$${}^3f_1 = \frac{{}^\tau R_{m_t}^{(l)}(r)}{r} \cdot {}^\tau Z_j^{(l)m}(\vartheta, \varphi; \sigma_z^{(1)'}, \sigma_z^{(2)'}) \cdot {}^\tau(r)_{m_t}, \quad (28)$$

τ being determined by (23).

The calculation of the small components ${}^3f_0, {}^1f_0$ by the last two of formulae (20) can be shortened by the observation that the operators on the left-hand side, being rotation invariant, transform any ${}^\tau Z_j^{(l)m}$ into a linear combination of other Z 's with the same j and m ; in order to determine the coefficients of these linear expressions, it is therefore sufficient to consider only one term of the ${}^3Z$'s as given by (15), e.g. $a_j \cdot {}^3(\sigma)_0 \cdot Y_l^m$. Further the operator $(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \cdot \vec{p}$ in the equation for ${}^1f_0$ does not alter the multiplicity σ , while that in ${}^3f_0$ exchanges triplet and singlet states: and according to well-known formulae, the effect of the operator $\vec{p}$ is to transform a $Z^{(l)}$ into a combination $\alpha Z^{(l-1)} + \beta Z^{(l+1)}$. For the singlet system, we thus get

$${}^3f_0 \sim \alpha \cdot {}^3Z_l^{(l-1)m} + \beta \cdot {}^3Z_l^{(l+1)m}$$

and, by (10),

$${}^1f_0 = 0;$$

for the triplet system, we may write

$${}^3f_0 \sim {}^1Z_l^{(l)m}$$

and

$$\text{for } j=l: \quad {}^1f_0 \sim \alpha' \cdot {}^3Z_l^{(l-1)m} + \beta' \cdot {}^3Z_l^{(l+1)m}$$

$$\text{for } j=l \pm 1: \quad {}^1f_0 \sim {}^3Z_j^{(j)m}.$$

The remaining explicit calculation is very short and finally yields, with (18), the following results;

Singlet system:

$$\psi^{(0)} = {}^\tau(r)_{m_t} \cdot {}^3(\varrho)_1 \cdot {}^1Z_l^{(l)m} \cdot {}^1R_{m_t}^{(l)}(r)/r$$

$$\begin{aligned} \psi^{(1)} = {}^\tau(r)_{m_t} \cdot {}^3(\varrho)_0 & \left[\sqrt{\frac{l}{2l+1}} {}^3Z_l^{(l-1)m} \mathcal{D}_{-1} \right. \\ & \left. + \sqrt{\frac{l+1}{2l+1}} {}^3Z_l^{(l+1)m} \mathcal{D}_{l+1} \right] {}^1R_{m_t}^{(l)} \end{aligned} \quad (29a)$$

* The index m_t should be added to the symbols $\mathcal{V}, R, \epsilon$ in (27).

Triplet system:

$$\begin{aligned}
 \psi^{(0)} &= \tau(\tau)_{m_t} \cdot {}^3(\varrho)_1 \cdot {}^3Z_j^{(l)m} \cdot {}^3R_{m_t}^{(l)}(r)/r \\
 l=j: \quad \psi^{(1)} &= \tau(\tau)_{m_t} \cdot {}^1(\varrho)_0 \left[\sqrt{\frac{j+1}{2j+1}} {}^3Z_j^{(j-1)m} \mathcal{D}_{-j} \right. \\
 &\quad \left. - \sqrt{\frac{j}{2j+1}} {}^3Z_j^{(j+1)m} \mathcal{D}_{j+1} \right] {}^3R_{m_t}^{(j)} \\
 l=j-1: \quad \psi^{(1)} &= \tau(\tau)_{m_t} \left[{}^1(\varrho)_0 \sqrt{\frac{j+1}{2j+1}} {}^3Z_j^{(j)m} \right. \\
 &\quad \left. + {}^3(\varrho)_0 \sqrt{\frac{j}{2j+1}} {}^1Z_j^{(j)m} \right] \mathcal{D}_j {}^3R_{m_t}^{(j-1)} \\
 l=j+1: \quad \psi^{(1)} &= \tau(\tau)_{m_t} \left[-{}^1(\varrho)_0 \sqrt{\frac{j}{2j+1}} {}^3Z_j^{(j)m} \right. \\
 &\quad \left. + {}^3(\varrho)_0 \sqrt{\frac{j+1}{2j+1}} {}^1Z_j^{(j)m} \right] \mathcal{D}_{-(j+1)} {}^3R_{m_t}^{(j+1)}.
 \end{aligned} \tag{29b}$$

In these formulae, the symbol $\mathcal{D}_j$ denotes the radial operator

$$\mathcal{D}_j = -\frac{i\hbar}{\sqrt{2M}} \cdot \frac{1}{r} \left[\frac{d}{dr} - \frac{j}{r} \right]. \tag{30}$$

4.341. Exchange potentials. Besides ordinary central forces, described by a potential operator $J(r)$, other types of central interactions, known as exchange interactions, have played a considerable part in the historical development of the study of nuclear forces. Although they are now superseded by more general kinds of nuclear potentials, it will not be superfluous to mention them here for future reference.

Let us first consider a potential operator involving the product of a general central interaction operator and the exchange operator P_x of the position coordinates of the two interacting particles:

$$\mathcal{V}' = \mathcal{V}(r; \sigma, \tau) P_x, \tag{31}$$

the function $\mathcal{V}$ depending in an arbitrary way on the spin and charge coordinates $\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}; \tau^{(1)}, \tau^{(2)}$. Although the operator P_x affects the angular variables, as it transforms ϑ, φ into $\pi - \vartheta, \varphi + \pi$, the potential $\mathcal{V}'$ can still be regarded as a central potential, because the interchange of position coordinates, on account of the exclusion principle, is equivalent to the interchange of spin and charge coordinates and change of sign of the wavefunction; in other words, we may write

$$P_x = -P_\sigma P_\tau. \tag{32}$$

provided the operators are only applied to completely antisymmetrical functions.

As regards the corresponding effective potentials (4.34), the effect of the factor P_x , which is obviously the same as the operator of spatial symmetry transformation, is simply to change the sign of the effective potential

for odd states, while it has no effect on the effective potential for even states:

$${}^{\sigma}\mathcal{V}'^{(l)} = (-1)^l \cdot {}^{\sigma}\mathcal{V}^{(l)}. \quad (33)$$

Other possibilities are clearly the introduction, as a factor, of one the exchange operators P_{σ} or P_{τ} . Also in these cases, analogous properties of the signs of the effective potentials can be enunciated. On the whole, we may distinguish four fundamental types of exchange potentials, respectively defined by

$$\begin{aligned} \mathcal{V}_W &= J(r) \quad , \quad \mathcal{V}_M = J(r) P_x = -J(r) P_{\sigma} P_{\tau} \\ \mathcal{V}_B &= J(r) P_{\sigma} \quad , \quad \mathcal{V}_H = -J(r) P_{\tau} = J(r) P_x P_{\sigma} \end{aligned} \quad (34)$$

and designated by the names of their promotor: Wigner, Majorana, Bartlett and Heisenberg.

4.4. Electromagnetic interactions

4.41. The exchange operators. In a field theory of nuclear forces, the operators of charge and current density of a system of nucleons cannot be expected to be expressed simply by the sum of the corresponding operators pertaining to the individual nucleons. Since at least part of the nuclear field must be assumed to carry electric charge (1.21), there will in fact occur virtual transitions (1.321) involving the exchange of an elementary charge between a proton and a neutron and this will imply a redistribution of the charges and currents of the system, depending on these virtual exchange processes. The corresponding terms in the charge and current density of the system (just as well as those in the energy operator) will therefore involve the characteristic operators $\Pi^{(i)}$, $\Pi^{(i)\dagger}$ effecting the transformation of a nucleon from neutron to proton state or vice versa (4.12-24); more precisely, since a pair of nucleons is involved, they may depend on the operators $\Pi^{(i)} \Pi^{(k)\dagger}$ and $\Pi^{(k)} \Pi^{(i)\dagger}$, or on the Hermitian combinations

$$\begin{aligned} T^{(ik)} &= \Pi^{(i)} \Pi^{(k)\dagger} + \Pi^{(i)\dagger} \Pi^{(k)} \\ &= \frac{1}{2} (\tau_1^{(i)} \tau_1^{(k)} + \tau_2^{(i)} \tau_2^{(k)}), \end{aligned} \quad (1)$$

$$\begin{aligned} \Pi^{(ik)} &= \frac{1}{i} (\Pi^{(i)} \Pi^{(k)\dagger} - \Pi^{(i)\dagger} \Pi^{(k)}) \\ &= \frac{1}{2} (\tau_1^{(i)} \tau_2^{(k)} - \tau_2^{(i)} \tau_1^{(k)}) = \frac{1}{2} (\tau^{(i)} \wedge \tau^{(k)})_3. \end{aligned} \quad (2)$$

Now, from the general law of formation of the energy-momentum tensor on the one hand, and the electric charge and current density on the other, from the given Lagrangian of the system (BELINFANTE [39, 40], ROSENFELD [40], MØLLER and ROSENFELD [43]), it appears that the former quantity involves a sum, and the latter a difference of adjoint quantities: the interaction energy will consequently embody exchange terms of the

form $T^{(ik)}\omega^{(ik)}$, while in the charge and current density the exchange terms will contain factors $\Pi^{(ik)}$.

The operator $T^{(12)}$ has the eigenfunctions ${}^3(\tau)_0$, ${}^1(\tau)_0$ and ${}^3(\tau)_{\pm 1}$ with the respective eigenvalues $+1$, -1 and 0 (double). As regards $\Pi^{(12)}$, it has, of course, the same eigenvalues, but it does not commute with $T^{(12)}$; applied to the eigenfunctions ${}^3(\tau)_{\pm 1}$ of a system of two like nucleons, it also gives, of course, zero, but with respect to the deuteron system, it interchanges charge singlet and charge triplet:

$$\Pi^{(12)} {}^3(\tau)_0 = i {}^1(\tau)_0, \quad \Pi^{(12)} {}^1(\tau)_0 = -i {}^3(\tau)_0. \quad (3)$$

One finds further

$$[T^{(12)}, \tau_3^{(1)}] = -[T^{(12)}, \tau_3^{(2)}] = 2i \Pi^{(12)}. \quad (4)$$

4.42. The exchange moments. The interaction of a system containing charges and currents with a slowly varying external electromagnetic field $\vec{E}$, $\vec{H}$ can, as is well-known (A2.11-6), be expanded in the form

$$\mathcal{V}_{\text{el.magn}} = -\vec{E}\vec{P} - \vec{H}\vec{M} - (Q \text{ grad}) \vec{E} - \dots \quad (5)$$

in terms of the successive 2^n -pole moments; in formula (5) we have omitted the term involving the total charge and written only those depending on the electric dipole moment $\vec{P}$, the magnetic dipole moment $\vec{M}$ and the electric quadrupole moment represented by the tensor Q . For the deuteron system, we must expect, according to the above considerations, such moments also to involve exchange terms (there will be further terms depending explicitly on the nuclear field variables, but these do not play any role as long as no free mesons take part in the process studied); for instance, we shall have

$$\vec{M} = \vec{M}_{\text{nuc}} + \vec{M}_{\text{exch}} \quad (6)$$

with an exchange term of the form

$$\vec{M}_{\text{exch}} = \Pi^{(12)} \vec{m}^{(12)}, \quad (7)$$

while $\vec{M}_{\text{nuc}}$ denotes the sum of the moments of the nucleons; if there is no orbital motion (*S*-state), we have simply

$$\vec{M}_{\text{nuc}} = \vec{\mu}^{(1)} + \vec{\mu}^{(2)}, \quad (8)$$

with (1.21, 1.22, 4.22)

$$\vec{\mu}^{(i)} = \mu_0 (T_+^{(i)} \mu_n + T_-^{(i)} \mu_p) \vec{\sigma}^{(i)}. \quad (9)$$

The exchange moments give no contribution to the expectation values of these quantities in stationary states of the deuteron, but they may give

rise to transitions between such states under the influence of an external field. Moreover, they will generally contribute to the expectation value of the electromagnetic energy of the system in a stationary state, since this depends quadratically on the moments, and $(\Pi^{(12)})^2 = \frac{1}{2}(1 - \tau_3^{(1)} \tau_3^{(2)})$, i.e. $= 1$ in any state of the deuteron.

4.43. The electric dipole moment and Siegert's theorem. The explicit calculation of the exchange moments falls outside the scope of this work; a systematic survey has been given by MØLLER and ROSENFELD [43]. It is found, in particular, that the exchange magnetic moment of the deuteron, defined with respect to the centre of gravity of the system, is due solely to the spin dependent part of the meson field. As to the exchange electric dipole moment, it can be shown for any nuclear system to give rise only to effects of the second order in the nucleon velocities, and it can therefore be neglected. There remains thus just the moment due to the separate protons:

$$\vec{P}_{\text{nuc}} = e \sum_i T_i^{(1)} \vec{x}^{(i)}, \quad (10)$$

which for the deuteron becomes (the moment being again defined with respect to the centre of gravity)

$$\vec{P}_{\text{nuc}} = -\frac{e}{2} \frac{\tau_3^{(1)} - \tau_3^{(2)}}{2} \vec{x}, \quad (11)$$

in terms of the relative coordinates $\vec{x}$ (4.311-3).

It is interesting to notice that even the interaction operator $-\vec{E} \vec{P}_{\text{nuc}}$ embodies exchange effects; this may be seen by observing that this operator is equivalent, for the calculation of expectation values in stationary states or probabilities of processes involving conservation of the total energy, to

$$\text{grad } V \cdot \vec{P}_{\text{nuc}} - \vec{A} \cdot \dot{\vec{P}}_{\text{nuc}},$$

where $\vec{A}$, V denotes the electromagnetic potential and the dot signifies derivation with respect to ct . The two operators differ in fact only by a time derivative, which has vanishing matrix elements in the cases mentioned. Now,

$$\dot{\vec{P}}_{\text{nuc}} = e \sum_i T_i^{(1)} \dot{\vec{x}}^{(i)} + e \sum_i \dot{T}_i^{(1)} \vec{x}^{(i)};$$

while the first term just represents the ordinary contribution from the total proton current, the second term may be written, in first approximation,

$$= \frac{e}{2} \sum_i \frac{i}{\hbar} [\tau^{(12)} w^{(12)}, \tau_3^{(i)}] \vec{x}^{(i)},$$

i.e., by (4),

$$\frac{e}{\hbar} \Pi^{(12)} w^{(12)} (\vec{x}^{(1)} - \vec{x}^{(2)}), \quad (12)$$

which is of the exchange type. The expression (12) exhibits a remarkable connexion between the exchange part of $\vec{P}_{\text{nuc}}$ and that of the nuclear potential. This relation, known as Siegert's theorem (SIEGERT [37], LAMB and SCHIFF [38]), could be used, at an early stage of development of the field theory, to write down approximately the exchange part of $\vec{P}$ when the exchange nuclear potential was known. A more exact calculation (MØLLER and ROSENFELD [43]) shows that even when effects of the first order in the velocities are taken into account, the exchange part of $\vec{P}$ is represented by an expression of the form (12), in which, however, $w^{(12)}$ denotes the *static* exchange potential only. It is this somewhat surprising result which authorizes the above-mentioned conclusion that any explicit exchange term in the electric dipole moment can only yield contributions of higher order in the nucleon velocities.

CHAPTER V

THE STATIONARY STATES OF THE PROTON-NEUTRON SYSTEM

5.0. In the following Chapters, the properties of stationary states of two-nucleon systems will be investigated, starting from the radial equation (4.34–27) valid on the assumption of a central interaction. Apart from the state of binding of the deuteron, we shall chiefly be concerned with scattering processes, and the results of the theoretical treatment will be confronted with the experimental data. It will be convenient to examine separately the proton-neutron and the proton-proton systems. In particular, the Coulomb interaction in the latter case entails an essential complication, because (as is well-known) such an interaction influences even the asymptotic behaviour of the radial wave-functions, which determines the features of the scattering process. We therefore begin with the proton-neutron system, in which only the nuclear field is operative.

5.1. Properties of short-range central potentials

5.11. *Types of short-range central potentials.* On account of the limited range of the nuclear potential (2.1), we shall have to study the behaviour of the radial wave-equation when the function representing the effective potential energy decreases (in absolute value) very rapidly with increasing distance. Considering the behaviour at small distances, on the other hand, we can at once exclude singularities at the origin stronger than a pole of the second order. For if the coefficient of the unknown function in an equation of the form (4.34–27) possessed such a singularity, the equation would have no regular integrals, and thus no eigensolutions*. The case of a pole of the second order does not unrestrictedly admit of eigensolutions; we shall leave it aside**. In the following, we shall thus assume that the potential has at most a simple pole at the origin. We propose to show how a discussion based essentially on these assumptions only, leads to a fairly complete survey of the general behaviour of the eigenfunctions.

As regards the analytical form of the potential, we may in the first place think of simple functions such as that resulting from the meson field

* This is an immediate consequence of the general theory of linear differential equations (W & W, Chapter X, especially 10.3), valid for any linear equation of the second order such that the coefficient of the first derivative of the unknown has at most a simple pole at the critical point considered. In fact, under such conditions, the indicial equation breaks down.

** See N. F. MOTT and H. S. W. MASSEY, *The theory of atomic collisions* (1933), Chapter II, 3.2, p. 30.

theory, which involve the product κr of the distance with some constant, the inverse of which can then be regarded as defining the *range* of the potential. But we cannot exclude the possibility of more than one such constant occurring in the expression of $\mathcal{V}$; if there are, for instance, different kinds of mesons with different masses, we shall meet with nuclear potentials of the form $(a_1 e^{-\kappa_1 r} + a_2 e^{-\kappa_2 r} + \dots)/r$. Besides, there is no *a priori* reason why the effective potentials pertaining to different types of stationary states should not have different ranges. In any case, we can refer all distances to one of the range constants occurring in the expression for the potential; this particular constant we shall denote by κ and, for brevity, call κ^{-1} "the" range of the nuclear force. We then introduce as dimensionless variable

$$\xi = \kappa r. \quad (1)$$

Simple types of short-range potentials which have been extensively studied are the following (see especially B & B, § 12 and *passim*):

$$(a) \text{ the potential "well"} \quad \begin{cases} w = 1 & \text{for } 0 \leq \xi \leq 1 \\ w = 0 & \text{for } \xi > 1 \end{cases} \quad (2)$$

$$(b) \text{ the exponential potential } w = e^{-2\xi} \quad (3)$$

$$(c) \text{ the Morse* potential } w = 2e^{-2\xi} - e^{-4\xi} \quad (4)$$

$$(d) \text{ the „Gauss” potential } w = e^{-\xi^2} \quad (5)$$

$$(e) \text{ the meson potential } w = e^{-\xi}/\xi. \quad (6)$$

The last one can, as especially emphasized by HULTHÉN [42a] (5.22, 5.231) conveniently be approximated by

$$w = \frac{e^{-\xi}}{1 - e^{-\xi}}; \quad (7)$$

also the type

$$w = 1 - \tanh^2 \xi \quad (8)$$

has been considered (HYLLERAAS [37]). In all the preceding formulae, the function $w(\xi)$ has been normalized in such a way that it reduces either to ξ^{-1} or to 1 for $\xi \rightarrow 0$. The actual effective potentials are then of the form $\mathcal{V} = -Jw(\xi)$; the constant J will be called the *strength* of the corresponding effective potential. (In the most general case, both J and $w(\xi)$ must carry indices σ, l referring to the type of stationary state considered **.)

This convention and terminology has a clear-cut physical meaning only if the potential is attractive (or repulsive) at *all* distances. Still, we shall

* This potential has been introduced and especially studied by Morse and his collaborators; their first paper (MORSE *et al.* [36]) contains the fundamental formulae and graphs.

** In the following, we shall, when convenient without obscurity, omit such indices σ, l .

retain them, for the sake of uniformity, also in the more complicated case of a potential repulsive at small distances and going over into an attraction at larger distances (fig. 2.22); the "strength" J is then negative, but the function $w(\xi)$, positive at $\xi = 0$, changes sign and passes through a minimum at some distance ξ_m , after which it rapidly tends to zero: it is, of course, the (positive) value $J w(\xi_m)$ which in such a case is of physical importance, at any rate for states of negative or not too large positive energies (5.133).

In order to write the radial equation (4.34-27) in a dimensionless form, we put

$$\begin{aligned}\eta &= \frac{M}{\hbar^2 \kappa^2} \varepsilon \\ B &= - \frac{M}{\hbar^2 \kappa^2} V \\ b &= + \frac{M}{\hbar^2 \kappa^2} J\end{aligned}\quad (9)$$

and accordingly get

$$\frac{d^2}{d\xi^2} {}^\sigma R^{(l)} + \left({}^\sigma B^{(l)} + \eta - \frac{l(l+1)}{\xi^2} \right) {}^\sigma R^{(l)} = 0. \quad (10)$$

Owing to the minus-sign in (9), a *positive* sign of $B = b w(\xi)$ means an *attractive* force. To get an idea of the orders of magnitude involved, we note that the unit of energy introduced in (9) amounts, if κ^{-1} is taken $\approx d$ (2.22), to $\left(\frac{\hbar}{e^2} \right)^2 \frac{m}{M} m \approx 5 \text{ MeV}$.

5.111. Normalization of radial eigenfunctions. The eigenfunctions of the discrete stationary states will be normalized with respect to the variable ξ according to the condition

$$\int_0^\infty |R|^2 d\xi = 1; \quad (11)$$

to go over to the usual normalization with respect to r , one has thus to multiply the function $R(\xi)$ by the factor $\sqrt{\kappa}$. For the eigenfunctions of the continuum, we have the condition of normalization in the reduced energy scale η

$$\int_0^\infty d\xi R_\eta \int_{\eta-\Delta}^{\eta+\Delta} R_{\eta'} d\eta' = 1; \quad (12)$$

we go over to the normalization in the usual energy scale (and variable r) by means of the factor $\frac{1}{\hbar} \sqrt{\frac{M}{\kappa}}$. In the latter case, it is often convenient to introduce, instead of the energy ε , the momentum p of the relative motion or, better still, the quantity k representing 2π times the de Broglie wave-number (inverse of the wave-length λ); between these quantities, we have

the relations

$$k = \frac{2\pi}{\lambda} = \frac{p}{\hbar} = \frac{\sqrt{M\xi}}{\hbar} = \sqrt{\eta} \cdot \kappa. \quad (13)$$

5.12. Behaviour of radial eigenfunction at small and large distances. In order that the eigenfunction ψ be finite at the origin, the boundary condition

$${}^{\sigma}R^{(l)}(0) = 0 \quad (14)$$

must be imposed on the solutions of the radial equation (10). The behaviour of these solutions for small values of ξ is conditioned by the highest singularity of the coefficient of the unknown; from the general theory, as developed in W & W, 10.32, it results that the solution which vanishes at $\xi = 0$ behaves $\sim \xi^{l+1}$ for small ξ .

Since the nuclear potential becomes negligible at large ξ , the asymptotic form of the function R will be expressible as a linear combination of the independent solutions of the equation

$$\frac{d^2 W}{d\xi^2} + \left(\eta - \frac{l(l+1)}{\xi^2} \right) W = 0. \quad (15)$$

For the latter we can take either the confluent hypergeometric functions (W & W, 16.12) $W_{0,l+\frac{1}{2}}(2iz)$ and $W_{0,l+\frac{1}{2}}(-2iz)$, with $z \equiv \sqrt{\eta}\xi$, or the functions related to those of Bessel (W & W, 17.5):

$$\sqrt{z} J_{l+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi}} e^{i(l+1)\pi i} \{ W_{0,l+\frac{1}{2}}(2iz) + (-1)^{l+1} W_{0,l+\frac{1}{2}}(-2iz) \} \quad (16)$$

$$\sqrt{z} J_{-(l+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi}} e^{-i l \pi i} \{ W_{0,l+\frac{1}{2}}(2iz) + (-1)^l W_{0,l+\frac{1}{2}}(-2iz) \}.$$

For half integer values of the index m , the asymptotic series (W & W, 16.3) for $W_{0,m}(z)$ reduces to a finite sum:

$$W_{0,l+\frac{1}{2}}(2iz) = e^{-iz} \left[1 + \sum_{n=1}^l \frac{H_{\nu=0}^{n-1} [l(l+1) - \nu(\nu+1)]}{n! (2iz)^n} \right]. \quad (17)$$

The asymptotic behaviour of ${}^{\sigma}R^{(l)}$ will consequently be essentially different according to the sign of η . For negative η , i.e. stationary states of binding, $z = i\sqrt{-\eta}\xi$, and on account of (17), the eigensolution, which must vanish at infinity, will decrease exponentially at large distances:

$${}^{\sigma}R^{(l)} \simeq e^{-\sqrt{-\eta}\xi}. \quad (18)$$

For positive η , i.e. stationary states representing a proton and a neutron in relative motion, the formula

$${}^{\sigma}R^{(l)} \simeq \alpha \sqrt{z} J_{l+\frac{1}{2}}(z) + \beta \sqrt{z} J_{-(l+\frac{1}{2})}(z), \quad z \equiv \sqrt{\eta}\xi \quad (19)$$

can be combined with the following, deduced from (16) and (17):

$$\begin{aligned}\sqrt{z} J_{l+\frac{1}{2}}(z) &\simeq \sqrt{\frac{2}{\pi}} \sin\left(z - l \frac{\pi}{2}\right) \\ \sqrt{z} J_{-(l+\frac{1}{2})}(z) &\simeq \sqrt{\frac{2}{\pi}} (-1)^l \cos\left(z - \frac{\pi}{2}\right);\end{aligned}\quad (20)$$

this leads to the asymptotic form

$${}^{\sigma}R^{(l)} \simeq \frac{1}{\sqrt{\pi} \sqrt{\eta}} \sin\left(\sqrt{\eta} \xi - l \frac{\pi}{2} + {}^{\sigma}\delta^{(l)}\right), \quad (21)$$

characterized by the *phase constant* ${}^{\sigma}\delta^{(l)}$; the factor $(\pi \sqrt{\eta})^{-\frac{1}{2}}$ normalizes ${}^{\sigma}R^{(l)}$ to unity in the reduced energy scale, in conformity with (12)*.

While, according to (17), $W_{0,l+\frac{1}{2}}(\pm 2iz) \sim z^{-l}$ for small $|z|$, and (W & W, 17.24) the same is true of $\sqrt{z} J_{-(l+\frac{1}{2})}(z)$, one has $\sqrt{z} J_{l+\frac{1}{2}}(z) \sim z^{l+1}$, so that the latter function represents the solution of equation (15) satisfying the boundary condition (14). In other words, the solution ${}^{\sigma}R^{(l)}$ of our problem (10) for a positive value of η goes over into $\sqrt{z} J_{l+\frac{1}{2}}(z)$ when the potential ${}^{\sigma}B^{(l)}$ vanishes. Comparison of the asymptotic form (21) with the first formula (20) illustrates the significance of the phase ${}^{\sigma}\delta^{(l)}$: this constant expresses the influence of the interaction on the behaviour of the moving nucleons at large distances from each other. The importance of the distortion of the wave-function by the nuclear potential will depend on the magnitude of this function in the region where the potential is effective. Considering the behaviour for small values of $\sqrt{\eta} \xi$ of the wave-functions of different orbital momenta, we shall therefore, for a given energy, expect the phases to decrease with increasing l . The main effect will always be that on S-waves, while the distortion of the waves of higher orbital momentum will become increasingly important as the energy increases.

* From equation (10) we derive by partial integration, for two states with the same σ, l

$$\int_0^{\xi_0} R_{\eta} R_{\eta'} d\xi = \left\{ R_{\eta} \frac{dR_{\eta'}}{d\xi} - R_{\eta'} \frac{dR_{\eta}}{d\xi} \right\}_{\xi=\xi_0} \cdot \frac{1}{\eta - \eta'},$$

so that the normalization condition (12) takes the form

$$\lim_{\xi_0 \rightarrow \infty} \int_{\eta-\Delta}^{\eta+\Delta} \left\{ R_{\eta} \frac{dR_{\eta'}}{d\xi} - R_{\eta'} \frac{dR_{\eta}}{d\xi} \right\}_{\xi=\xi_0} \cdot \frac{d\eta'}{\eta - \eta'} = 1,$$

depending only on the asymptotic form of the solution; it is easily verified that (21) satisfies this condition.

5.121. Some useful identities. Let ${}^{\sigma}R^{(l)}$ be a proper solution of equation (10) and W any solution of equation (15) for the same values of l and η . The integral $\int_0^{\infty} W \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi$ can easily be shown to depend only on the behaviour of W and ${}^{\sigma}R^{(l)}$ for $\xi = 0$ and $\xi = \infty$. In fact, if we take ${}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}$ from equation (10), we get

$$\int_0^{\infty} W \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi = - \int_0^{\infty} W \left[\frac{d^2}{d\xi^2} + \eta - \frac{l(l+1)}{\xi^2} \right] {}^{\sigma}R^{(l)} d\xi,$$

whence, by partial integration, and taking account of (15)

$$\int_0^{\infty} W \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi = \left[{}^{\sigma}R^{(l)} \frac{dW}{d\xi} - W \frac{d{}^{\sigma}R^{(l)}}{d\xi} \right]_0^{\infty}. \quad (22)$$

This formula can be specialized in various ways*. For a state of binding ($\eta < 0$), we get, using (17) and (16),

$$\begin{aligned} \int_0^{\infty} W_{0,l+\frac{1}{2}}(2\sqrt{-\eta}\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi \\ = \frac{2l+1}{l+1} \left[\frac{d{}^{\sigma}R^{(l)}}{d\xi} \cdot W_{0,l+\frac{1}{2}}(2\sqrt{-\eta}\xi) \right]_{\xi=0} \end{aligned} \quad (23)$$

$$\begin{aligned} \int_0^{\infty} \sqrt{\sqrt{-\eta}\xi} \cdot I_{l+\frac{1}{2}}(\sqrt{-\eta}\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi \\ = \frac{(-1)^{l+1}}{\sqrt{2\pi}} \lim_{\xi \rightarrow \infty} \left[{}^{\sigma}R^{(l)} \frac{dW}{d\xi} - W \frac{d{}^{\sigma}R^{(l)}}{d\xi} \right]; \end{aligned} \quad (24)$$

in (24), we have put $I_n(z) = i^{-n} J_n(iz)$ and $W \equiv W_{0,l+\frac{1}{2}}(-2\sqrt{-\eta}\xi)$. For a state of relative motion ($\eta > 0$), (20) and (21) give

$$\frac{\pi}{\sqrt{2}} \int_0^{\infty} \sqrt{\xi} J_{l+\frac{1}{2}}(\sqrt{\eta}\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi = \sin {}^{\sigma}\delta^{(l)} \quad (25)$$

$$(-1)^{l+1} \frac{\pi}{\sqrt{2}} \int_0^{\infty} \sqrt{\xi} J_{-(l+\frac{1}{2})}(\sqrt{\eta}\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi = \cos {}^{\sigma}\delta^{(l)} + \left[\frac{\sqrt{2} {}^{\sigma}R^{(l)}}{\sqrt{\xi} J_{l+\frac{1}{2}}(\sqrt{\eta}\xi)} \right]_{\xi=0}; \quad (26)$$

in deriving the last term of (26) use has further been made of the property of Bessel functions (W & W, 17.2)

$$\lim_{z \rightarrow 0} J_n(z) J_{-n}(z) = \frac{1}{\Gamma(1+n) \Gamma(1-n)} = \frac{\sin n\pi}{n\pi}.$$

5.13. General behaviour of eigenfunctions of S-states. The wave-

* The following identities have been derived, as well as slightly more general ones, by HULTHÉN [44c, 45a].

equation (10) gives directly the curvature of the integral curve $R(\xi)$ in each point:

$$\frac{d^2 R}{d\xi^2} = -(\eta + B) R, \quad B = bw(\xi). \quad (27)$$

Starting with a solution which behaves like $a\xi$ at the origin (5.12) (where, for definiteness, we take $a > 0$), we are thus able to follow the whole course of the integral curve in its dependence on the potential $B(\xi)$.

5.131. Repulsive potential. If the potential is repulsive at all distances ($B < 0$), it is clear that no negative eigenvalue η is possible; since otherwise we would have $d^2 R/d\xi^2 > 0$, i.e. the curve would always bend upwards, and consequently exhibit an exponential increase at large distances. For positive values of η , the curve (fig. 5.131) will generally* begin by bending upwards, until the point ξ_0 is reached, for which $B(\xi_0) + \eta = 0$; at ξ_0 the curve has an inflexion point and bends downwards until $R(\xi)$ itself becomes zero; and from this point it goes on oscillating in a fashion rapidly approximating the asymptotic form (21); the phase $\delta^{(0)}$ is seen to be negative.

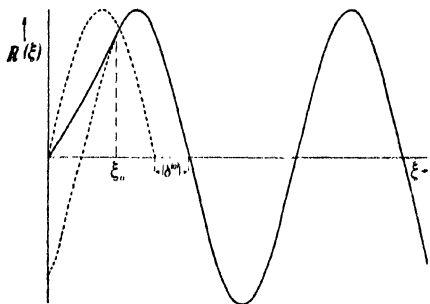


Fig. 5.131. S-state; repulsive potential.

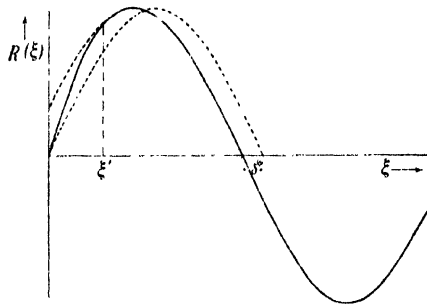


Fig. 5.132-1. S-state of relative motion ($\eta > 0$); attractive potential.

5.132. Attractive potential. If the potential is attractive at all distances ($B > 0$), the integral curve begins by bending downwards**, so that $R(\xi)$, after passing through a maximum, will begin to decrease. Everything now depends, however, on the rapidity of this variation, which in its turn is conditioned by the strength of the potential. Let ξ' be some value of ξ for which the potential becomes so small that from this point on the wave-function may be considered to be sufficiently approximated by its asymptotic form. If when reaching ξ' the value of R is still increasing, the

* The only exception would be, for a potential finite at the origin, the case $\eta > |b|$. But this case can readily be discussed in the same way as the others and shows the same general behaviour.

** The reader will readily recognize that the only exceptional case, viz. that of a potential finite at the origin and a negative eigenvalue with $|\eta| > b$, can be excluded.

integral curve cannot, of course, go over smoothly into an exponentially decreasing one, but must be of the periodic type (fig. 5.132-1); this will necessarily happen for sufficiently small values of b (and of η , if $\eta > 0$).

But if b is large enough, it will become possible (fig. 5.132-2, curve (a)) for the wave-function to begin decreasing before the point ξ' and to go over at this point into the exponential decrease required for the existence of a stationary state with negative eigenvalue; the latter quantity will then be determined by the slope of the integral curve at the point of junction ξ' . We thus come to the conclusion that if the potential constant b remains below a certain critical value b_1 , there can be no state of binding of the deuteron. As soon as b exceeds b_1 there will be one such state; the case $b = b_1$ is a limiting case, for which the binding energy is zero.

As b increases beyond b_1 , the eigenvalue corresponding to the state of binding under discussion must increase in absolute value, in order to prevent a too rapid bending of the integral curve*. At the same time, however, the possibility arises of another type of integral curve, in which the curve, after passing its first maximum, slopes down till it crosses the ξ -axis on the left of ξ' , and further proceeds bending upwards on the

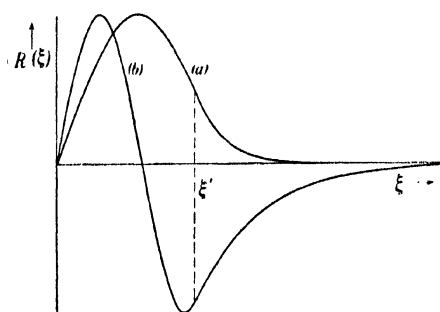


Fig. 5.132-2 S-states of binding ($\eta < 0$); attractive potential.

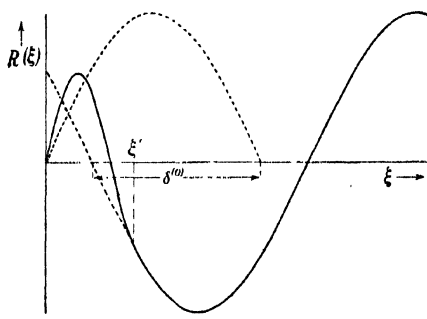


Fig. 5.132-3. S-state of relative motion ($\eta > 0$); strong attractive potential.

negative side: if by the point ξ' it has not yet reached its minimum (fig. 5.132-3), again only a periodic asymptotic behaviour, with positive eigenvalue, is possible: but there will clearly be a second critical value $b_2 (> b_1)$ of b , beyond which the minimum will have been passed when ξ' is reached and an exponential decrease (on the negative side) becomes possible (fig.

* It is clear that if b tends to infinity, $|\eta|$ likewise will become infinite. From this fact we may infer the impossibility of "contact" interaction (4.31). For a "contact"

potential of the form $\delta(x)$ may be approximated by potential "wells" of decreasing width κ^{-1} and increasing depth J , in such a way that $\lim_{\kappa \rightarrow \infty} J \cdot (\kappa^{-1})^3 = \text{const.}$ or, by (9), $\lim_{\kappa \rightarrow \infty} b/\kappa = \text{const.}$ The "contact" potential, therefore, corresponds to $b \rightarrow \infty$ and does not give rise to any stationary state of finite binding energy. A more complete discussion of this question, based on the exact relativistic equations of the two-body problem, is given by KEMMER [37a].

5.132-2, curve (b)). For $b > b_2$ we shall thus have two different states of binding, distinguished by the number (0 or 1) of nodes of the eigenfunction; the stronger binding corresponds to the 0-node eigenfunction. Repeating the argument, we are finally led to the following picture: there is a succession of increasing critical values $b_0 = 0, b_1, b_2, \dots$ of the potential constant b , such that if $b_n < b < b_{n+1}$, there are n S-states of binding of the deuteron, the eigenfunctions of which are characterized by an increasing number of nodes, from 0 to $n-1$, the binding energy $|\eta_k|$ ($k = 1, \dots, n$) decreasing accordingly. As regards the states of positive energy, it is easily seen that their phase constants $\delta^{(0)}$ (in contrast to the case of a repulsive potential) are positive.

5.133. *Attractive potential modified by very short range repulsion* (fig. 2.22). The discussion of this case proceeds on the same lines as that of the preceding ones and clearly leads to general results entirely analogous to those pertaining to a purely attractive potential (5.132). Only the critical values of the potential for which a new state of binding becomes possible will here be determined by the depth of the minimum of the potential function. It is also apparent that the influence of the repulsive potential at small distances will be the more important on the eigenfunction of a state of relative motion, the larger the energy of the system in this state.

5.14. *Eigenfunctions of states of higher orbital momentum.* For the discussion of the stationary states of quantum number $l \neq 0$, we conveniently introduce the *impact parameter* r_l , defined as the closest distance of approach of the two particles. On account of conservation of angular momentum, we may write, with the notation (4.32-11),

$$p r_l = \hbar |\vec{L}|,$$

p being the relative linear momentum at r_l , given by

$$\frac{p^2}{M} = \varepsilon - {}^\tau V^{(l)}(r_l);$$

r_l is therefore determined by the equation

$$\varepsilon - {}^\tau V^{(l)}(r_l) = \frac{\hbar^2}{M} \cdot \frac{l(l+1)}{r_l^2},$$

and the corresponding reduced parameter $\xi_l = \kappa r_l$ by the equation

$$\eta + {}^\tau B^{(l)}(\xi_l) = \frac{l(l+1)}{\xi_l^2}. \quad (28)$$

Comparison with equation (10) shows that at ξ_l the integral curve has an inflexion point.

At small ξ the solution which vanishes for $\xi = 0$ is (5.12) $\sim \xi^{l+1}$; the integral curve thus starts tangentially to the ξ -axis at $\xi = 0$, then (owing to the dominant term $-l(l+1)/\xi^2$) bends upwards till $\xi = \xi_l$, after which point it begins bending downwards (fig. 5.14). If now ξ_l is so

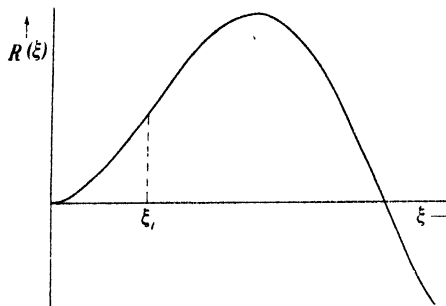


Fig. 5.14. State of orbital momentum l
($\eta > 0$).

large that the wave-function at that point already shows its asymptotic behaviour, we see that only solutions of the periodic type, with positive eigenvalues, can be adjusted at ξ_l to the upward slope of the integral curve. A state of binding, with asymptotic form $\simeq e^{-\sqrt{1-\eta}\xi}$, can therefore only exist if the potential at ξ_l still has a sufficiently large, positive value: physically, this means, according to (28), that there must be an attractive potential sufficiently strong to overcome the repulsion due to centrifugal force. A more quantitative estimate of the required potentials may be obtained as follows: In order to get a lower limit, put the binding energy $\eta = 0$; since ξ_l is to be small, $B(\xi_l)$ is either b or $b\xi_l^{-1}$. In any case, the condition $\xi_l \ll 1$ yields, according to (28), $b \gg l(l+1)$. When this condition is satisfied, there are thus states of binding of orbital momenta $\leq l$ for which the potential is attractive; all states of higher l belong to the continuous spectrum of positive energies.

5.2. Stationary states of binding

5.20. After the qualitative survey of the preceding section, we go over to a more precise treatment of the eigenvalue problem of our equation (5.11–10). We shall first consider some special forms of potential allowing of an exact solution; we shall then briefly recall the general variational method often used to obtain approximate solutions. The examples treated will illustrate the general considerations of the preceding section.

5.21. *The S-states of the potential well.* If the potential is of the type (5.11–2), the eigenfunctions of the S-states of binding are of the form

$$\begin{aligned} &\sim \sin \sqrt{b - |\eta|} \xi & (\xi \leq 1) \\ &\sim e^{-\sqrt{|\eta|} \xi} & (\xi > 1). \end{aligned} \tag{1}$$

The continuity conditions at $\xi = 1$ then yield

$$\sqrt{\eta}|\eta| = -\sqrt{b-|\eta|} \operatorname{ctg} \sqrt{b-|\eta|}, \quad (2)$$

whence for the critical b -values

$$b_{n+1} = (n + \frac{1}{2})^2 \pi^2. \quad (3)$$

From the normalization condition (5.111-11), one deduces, using also the continuity conditions at $\xi = 1$, that the cofactor of $\sin \sqrt{b-|\eta|} \xi$ in the normalized eigenfunction is

$$[2 \sqrt{|\eta|} / (1 + \sqrt{|\eta|})]^{\frac{1}{2}}. \quad (4)$$

In practice, the eigenvalue of the energy may be given by experiment, while the strength of the potential is not known. The relation (2) can then be used to derive b from $|\eta|$. In the vicinity of a critical value, $b - |\eta|$ can be expanded in powers of $\sqrt{|\eta|}$ by means of Bürmann's theorem (W & W, 7.3). So we get for the lowest S -state (BREIT *et al.* [36a, 39a])

$$b = \frac{\pi^2}{4} + 2 \sqrt{|\eta|} + \left(1 - \frac{4}{\pi^2}\right) |\eta| + \left(\frac{32}{\pi^4} - \frac{8}{3\pi^2}\right) |\eta|^{\frac{3}{2}} + \left(\frac{32}{\pi^4} - \frac{320}{\pi^6}\right) |\eta|^2 + \dots \quad (5)$$

5.211. The S -states of the exponential potential. If the potential is of the form (5.11-3) $b e^{-2\xi}$, the substitution

$$y = \sqrt{b} e^{-\xi}$$

reduces the radial equation to a Bessel differential equation of order $\pm \sqrt{b-|\eta|}$. The boundary condition at $\xi \rightarrow \infty$ then determines the eigenfunctions of bound states as Bessel functions of positive order

$$R \sim J_{\sqrt{b-|\eta|}}(\sqrt{b} e^{-\xi}).$$

The eigenvalues are determined by the other boundary condition, $R = 0$ at $\xi = 0$:

$$J_{\sqrt{b-|\eta|}}(\sqrt{b}) = 0.$$

The critical b -values are accordingly the squares of the zeros of $J_0(z)$.

5.22. The S -states of Hulthén's potential. It has been noticed independently by HULTHÉN [42a, b] and by HYLLERAAS and RISBERG [41] that the potential (5.11-7)

$$w = \frac{e^{-\xi}}{1 - e^{-\xi}}, \quad (6)$$

while representing a good approximation of the meson potential $e^{-\xi}/\xi$ (5.11-6), gives rise to a wave-equation which can be solved exactly. Let us make the variable substitution

$$\xi = -\log(1-x), \quad x = 1 - e^{-\xi}, \quad (7)$$

and for an S -state of binding, put

$$R(\xi) = e^{-\sqrt{|\eta|}\xi} \chi(x); \quad (8)$$

the differential equation for χ is

$$x(1-x) \frac{d^2\chi}{dx^2} - (2\sqrt{|\eta|} + 1)x \frac{d\chi}{dx} + b\chi = 0, \quad (9)$$

with the boundary conditions

$$\chi(0) = 0; \quad \lim_{\xi \rightarrow \infty} e^{-\sqrt{|\eta|}\xi} \chi(x) = 0 \text{ if } |\eta| > 0; \quad \chi(1) \text{ finite if } \eta = 0. \quad (10)$$

Now, define n (> 0) by

$$b = n(n + 2\sqrt{|\eta|}). \quad (11)$$

If this expression (11) is substituted for b in equation (9), this equation, for an arbitrary value η_n of η , possesses a complete system of eigenfunctions $\chi_n(x)$, given by the Jacobian polynomials *

$$\begin{aligned} G_n(x) &= \sum_{v=1}^n (-1)^{v-1} \binom{n-1}{v-1} \binom{n+v+2\sqrt{|\eta_n|}-1}{v} x^v \\ &= -\frac{1}{n!} x(1-x)^{-2\sqrt{|\eta_n|}} \frac{d^n}{dx^n} \left[x^{n-1} (1-x)^{n+2\sqrt{|\eta_n|}} \right], \end{aligned} \quad (12)$$

n being an integer ≥ 1 . If then b is considered as a given quantity, the relation (11) will determine in terms of b and n the eigenvalue η_n of η belonging to the eigenfunction χ_n :

$$-\eta_n = \frac{1}{4} \left(\frac{b}{n} - n \right)^2. \quad (13)$$

The condition that the positive square root $\sqrt{|\eta_n|}$ be taken implies, for a given b , the limitation

$$n^2 \leq b \quad (14)$$

on the number of eigenstates; the critical values of b are just

$$b_n = n^2. \quad (15)$$

By a simple calculation, we get for the eigenfunctions normalized according to (5.111-11)

$$\chi_n(x) = \sqrt{\frac{b^2 - n^4}{2bn}} G_n(x). \quad (16)$$

The normalized eigenfunction of the ground state is therefore

$$R_1(\xi) = \sqrt{\frac{b(b^2-1)}{2}} e^{-\frac{b-1}{2}\xi} (1 - e^{-\xi}), \quad (17)$$

with the eigenvalue

$$-\eta_1 = \frac{1}{4}(b-1)^2. \quad (18)$$

* See, e.g., R. COURANT and D. HILBERT, *Methoden der mathematischen Physik I* (1924), p. 75.

5.23. The variational method. As is well-known, the proper solutions of equation (5.11-10) are characterized as the extremal functions of the variation problem

$$\delta \int_0^{\infty} R^* \left[\frac{d^2}{d\xi^2} + \eta + b w(\xi) - \frac{l(l+1)}{\xi^2} \right] R d\xi = 0 \quad (19)$$

which vanish at $\xi = 0$ and $\xi = \infty$. In this problem, the parameter b determining the strength of the potential is usually regarded as given, and the values of the parameter η are sought for which the problem has a solution. An equivalent way of stating the problem is to say that such functions R are to be determined as make the expression

$$\eta = - \frac{\int_0^{\infty} R^* \left[\frac{d^2}{d\xi^2} + b w(\xi) - \frac{l(l+1)}{\xi^2} \right] R d\xi}{\int_0^{\infty} R^* R d\xi} \quad (20)$$

stationary. These extremal values are then just the eigenvalues of the energy parameter η .

But it is clearly equally justified and often more convenient to regard the parameter η as given and to determine the eigenvalues of the parameter b ; one has then to look for the extremal values of

$$b = - \frac{\int_0^{\infty} R^* \left[\frac{d^2}{d\xi^2} + \eta - \frac{l(l+1)}{\xi^2} \right] R d\xi}{\int_0^{\infty} R^* w(\xi) R d\xi} \quad (21)$$

which, for negative η , may also be written, on account of the boundary conditions,

$$b = - \frac{\int_0^{\infty} \left[\left| \frac{dR}{d\xi} \right|^2 + \left(|\eta| + \frac{l(l+1)}{\xi^2} \right) |R|^2 \right] d\xi}{\int_0^{\infty} |R|^2 w(\xi) d\xi} \quad (22)$$

If the potential function $w(\xi)$ has always the same sign, the expression (22) is essentially positive (corresponding, physically, to an attractive potential); this property is peculiar to the form (22) of the variation problem, — the numerator of (21), e.g., is not positive definite.

The above variational principles can be used to find approximate eigen-solutions and eigenvalues. A function R of suitable analytical form is chosen and made to depend on one or more parameters; the latter are then varied and the maximum or minimum problem solved. The solution obtained by inserting in the given form the extremal values of the parameters is the "best" one of the form considered and the corresponding value of η (or b) an approximation to the true eigenvalue of this quantity. By inserting the approximate solutions into the identities (5.121–23, 24), one gets an idea of the degree of accuracy attained.

Another method, which has never been applied to nuclear problems, but which, as suggested by Dr. FRÖHLICH, might be useful, consists in starting, instead of (19), from

$$\delta \int \left[\frac{d^2}{d\xi^2} + \eta + b w(\xi) - \frac{l(l+1)}{\xi^2} \right] R^2 d\xi = 0; \quad (19a)$$

i.e. the problem (similar to that of the "least square" method) would be to find the function R such that the sum of the absolute squares of the values taken by the left-hand side of the wave-equation (5.11–10) be a minimum. See also KRAMERS [38], § 51.

5.231. The S -states of the meson potential. As an example of the variational procedure, we shall briefly discuss the case of the meson potential (HULTHÉN [42a, b, 45a], HYLLERAAS and RISBERG [41]). The similarity with Hulthén's potential (5.22) suggests the variable substitution (7). Introducing again the function $\chi(x)$ by (8), the variational principle (21) takes the form

$$\delta b = 0 \text{ with } b = \frac{\int_0^1 \chi^* \left[(1-x) \frac{d^2}{dx^2} - (2|\eta| + 1) \frac{d}{dx} \right] \chi (1-x)^{2|\eta|} dx}{\int_0^1 |\chi|^2 \frac{(1-x)^{2|\eta|}}{\log(1-x)} dx} \quad (23)$$

For the ground state, the solution $\chi = x$ corresponding to Hulthén's potential suggests the choice of the trial function

$$\chi = x - cx^2, \quad (24)$$

the best value of c being determined by the solution of the variation problem (23). One readily finds that c has to satisfy a quadratic equation, so that for every given $|\eta|$ there are two solutions for χ and b . In particular, for $|\eta| = 0$, these two b -values are the two first critical ones; they happen to be

$$b_1 \approx 1.68, \quad b_2 \approx 4.5.$$

More precisely, the relation between b and the eigenvalue $|\eta|$ can, for $|\eta| \lesssim \frac{1}{2}$, be expressed analytically by the formula

$$b = 1.6798 + 2.2655 |\eta| - 0.2456 |\eta|^2 + 0.1440 |\eta|^3 - \dots \quad (25)$$

Even for $|\eta| = 1$, the terms written down in (25) yield a value of b correct to 1 %. For $|\eta| \leq \frac{1}{2}$, the contribution from the last two terms of (25) does not exceed 2 % of the b -value. More details will be found in Hulthén's papers quoted above.

It is useful to note that the wave-function $R(\xi)$ of the ground state may be represented to a fair approximation by a simple function of the form

$$R(\xi) \sim \xi e^{-w\xi},$$

with a numerical constant w . For this representation, originally proposed by WILSON [38], see also FRÖHLICH, RAMSEY and SNEDDON [46].

5.24. The variation-iteration method. In the case of a central (effective) potential $\mathcal{V} = -Jw(|x|)$, the co-factors $\psi(\vec{x})$ of the spin and charge eigenfunctions ${}^a(o) {}^\tau(\tau)$ in the total eigenfunction (4.33) ${}^3f_1$ satisfy a wave-equation of the same type (4.33-21) as ${}^3f_1$ itself. If we go over to momentum space, by

$$\chi(\vec{k}) = (2\pi)^{-3} \int \psi(\vec{x}) e^{i\vec{k}\vec{x}} d\vec{x} \quad (26)$$

and

$$w_k(\vec{k}) = (2\pi)^{-3} \int w(|x|) e^{i\vec{k}\vec{x}} d\vec{x}, \quad (27)$$

the wave-equation transforms into an integral equation

$$T(\vec{k}) \chi(\vec{k}) - J \int w_k(\vec{k} - \vec{k}') \chi(\vec{k}') d\vec{k}' = 0. \quad (28)$$

in which the coefficient

$$T(\vec{k}) = \frac{\hbar^2 k^2}{M} - \varepsilon \quad (29)$$

is essentially positive for a state of binding ($\varepsilon < 0$).

Upon the assumption that the (symmetrical) "kernel" $w_k(\vec{k} - \vec{k}')$ is positive definite, i.e. that

$$\int f^*(\vec{k}) \cdot w_k(\vec{k} - \vec{k}') \cdot f(\vec{k}') d\vec{k} d\vec{k}' \geq 0 \quad (30)$$

for any $f(\vec{k})$, equation (28) can be solved by a method of successive approximations, starting from an arbitrary trial function $\chi_0(\vec{k})$ and proceeding as follows: Put, for a given $\varepsilon (< 0)$,

$$\chi_{j+1}(\vec{k}) = \frac{1}{T(\vec{k})} \int w_k(\vec{k} - \vec{k}') \chi_j(\vec{k}') d\vec{k}'$$

$$T_j = \int \chi_j^* T(\vec{k}) \chi_j d\vec{k} \quad (31)$$

$$W_j = \int \chi_j^*(\vec{k}) w_k(\vec{k} - \vec{k}') \chi_j(\vec{k}') d\vec{k} d\vec{k}'$$

$$J_j = T_j / W_j, \quad J_{j+1} = W_j / T_{j+1}.$$

It can then be shown that the sequence $J_0, J_1, J_2, \dots$ never increases and

that it converges towards an eigenvalue of J ; this eigenvalue is the lowest one provided the function χ_0 fulfills the condition

$$\int \chi_0^* T(k) \chi dv_k \neq 0, \quad (32)$$

where χ is the exact eigenfunction of the lowest state. In this case, the sequence of functions $\chi_j / \sqrt{T_j}$ converges uniformly towards the eigenfunction χ .

This method has been recently investigated by SVARTHOLM [45] and extended by him to the treatment of the many-nucleon problem, with special application to the 3- and 4-nucleon cases (14.21, 14.22).

5.3. Calculation of the phases

5.30. As we have seen (5.12), the asymptotic behaviour of the states of relative motion is characterized by a phase constant. For the treatment of scattering processes, which depend on just the asymptotic behaviour of the wave-function, it is therefore important to obtain sufficiently accurate estimates of these phases.

5.31. General variational method. The variational method developed for the treatment of the states of binding (5.23) can, as pointed out by HULTHÉN [44d, 46a], also be adapted to the case of the states of positive energy. In our presentation of Hulthén's method, we shall follow a line of argument due to HYLLEAAS [45]. In the case of negative energies, the problem involves two mutually dependent parameters, viz. b and η , and the variation principle consists in expressing that either one of these parameters must be stationary, the other being fixed. In the case of positive energies, we have 3 dependent parameters, viz. b , η and the phase δ , the latter entering the problem through the boundary condition at infinity, which now states that the eigenfunction must have the asymptotic form (5.12-21) $\simeq \sin(\sqrt{\eta}\xi - l\pi/2 + \delta^{(l)})$. Again we may formulate the variation principle by requiring either b , η or δ to be stationary under the boundary condition at infinity just mentioned, the other two parameters being kept fixed.

This may be seen as follows: Let us start from (5.23-20) or (5.23-21), which means that we consider such trial functions $R(\xi; c_1, \dots, c_n)$, depending on certain parameters c_i , as satisfy the equation

$$\mathcal{L}(R) \equiv \int_0^\infty R^* \left[\frac{d^2}{d\xi^2} + \eta + bw(\xi) - \frac{l(l+1)}{\xi^2} \right] R d\xi = 0; \quad (1)$$

this equation being satisfied also by the varied functions, we have, as in (5.23-19) *

$$\delta \mathcal{L} = 0. \quad (2)$$

* In the formulae (2), (3), (4), the letter δ occurs as a symbol of variation, and not in its usual meaning of phase constant.

In calculating $\delta \mathcal{L}$, the expression $\delta \int_0^\infty R^* \frac{d^2 R}{d\xi^2} d\xi$ gives rise to a term

$$\left[R^* \frac{dR}{d\xi} - \frac{dR^*}{d\xi} \delta R \right]_{\xi \rightarrow \infty}, \quad (3)$$

in which also the phase δ occurs. It is easily verified that this term vanishes if δ is treated as a constant. The remaining equation

$$\int_0^\infty \delta R^* [\dots] R d\xi + \int_0^\infty \delta R [\dots] R^* d\xi = 0, \quad (4)$$

with the same operator as in (1) between the brackets, shows that R is a solution of the desired wave-equation, satisfying the given boundary conditions. It can be interpreted again as expressing that either b or η must be stationary, the phase δ being kept constant. Another way of looking at the matter is to say that, starting from (1), we impose on δ the condition of being stationary for given b and η , — for this implies that δ is treated as a constant in computing (3). The latter interpretation was originally put forward by Hulthén, but both he and Hylleraas now favour the former one, which, in fact, has the advantage of reducing both cases of positive and negative energies to entirely analogous forms*.

Anyhow, the practical procedure consists in solving the system of equations

$$\mathcal{L} = 0, \quad \frac{\partial \mathcal{L}}{\partial c_i} = 0 \quad (i = 1, 2, \dots, n) \quad (5)$$

for the parameters c_i and δ . In the case of a short range potential, it will be convenient to take a trial function of the general form

$$R = f(\xi; c_1, \dots, c_n) \sqrt{\xi} J_{l+\frac{1}{2}}(\sqrt{\eta} \xi) + g(\xi; c_1, \dots, c_n) \sqrt{\xi} J_{-(l+\frac{1}{2})}(\sqrt{\eta} \xi), \quad (6)$$

in which f and g must have the properties

$$\begin{aligned} f &\approx \text{const} & , & & g &\sim \xi^{2l+1} & & \text{for small } \xi \\ f &\simeq A \cos \delta & , & & g &\simeq A (-1)^l \sin \delta. \end{aligned} \quad (7)$$

The accuracy of the result can be checked by means of the identities (5.121–25, 26). The method can be improved further by imposing the first of these identities on the trial function as an accessory condition to remain fulfilled when the variation is performed. Further details and examples will be found in Hulthén's papers.

5.32. Estimate of S -phase for nearly critical values of potential strength. In general, the S -phases can only be derived from the explicit integration

* An important difference (HULTHÉN [46a]) is that the *stationary* parameter values, in the continuum case, do not in general correspond to a maximum or a minimum.

of the wave-equation, for instance by the method just outlined. There is, however, a case in which they can be estimated directly. Suppose the positive potential b is very near one of the critical values b_n :

$$b = b_n - \beta, \quad (8)$$

the sign of the small quantity β being arbitrary. Let $\overset{\circ}{\chi}_n$ be the radial eigenfunction of the S -state corresponding to the critical value b_n (and thus to $\eta = 0$); it is a solution of the differential equation

$$\overset{\circ}{\chi}_n'' + b_n w \overset{\circ}{\chi}_n = 0, \quad (9)$$

which we shall take to be real and normalized in such a way that

$$\overset{\circ}{\chi}_n(\infty) = 1. \quad (10)$$

If we write the eigenfunction of an S -state of positive energy η in the form

$$R(\xi) = \chi(\xi) \cdot \frac{\sin(\sqrt{\eta} \xi + \delta)}{\sqrt{\pi} \sqrt{\eta}}, \quad (11)$$

the function χ is a solution of the differential equation

$$\chi'' + 2\sqrt{\eta} \operatorname{ctg}(\sqrt{\eta} \xi + \delta) \chi' + b w \chi = 0, \quad (12)$$

likewise normalized so that $\chi(\infty) = 1$ (5.12–21). Therefore, χ goes over into $\overset{\circ}{\chi}_n$ when $\eta \rightarrow 0$ and $\beta \rightarrow 0$.

Consider now a state of *small* positive energy η . The corresponding function $\chi(\xi)$ can then be approximated by $\overset{\circ}{\chi}_n$. Treating β and $\sqrt{\eta}$ as small quantities of the same order, we derive in the usual way, from the differential equation which must be satisfied by the next term in the expansion $\chi = \overset{\circ}{\chi}_n + \dots$ in powers of $\sqrt{\eta}$, the "orthogonality condition"

$$2\sqrt{\eta} \operatorname{ctg} \delta \int_0^\infty \overset{\circ}{\chi}_n \overset{\circ}{\chi}_n' d\xi - \beta \int_0^\infty w \overset{\circ}{\chi}_n^2 d\xi = 0,$$

or, more simply,

$$\sqrt{\eta} \operatorname{ctg} \delta - \beta_n = 0, \quad (13)$$

since, by (10),

$$2 \int_0^\infty \overset{\circ}{\chi}_n \overset{\circ}{\chi}_n' d\xi = [\overset{\circ}{\chi}_n(\infty)]^2 = 1;$$

in (13), we have put

$$\beta_n = \beta \int_0^\infty w \overset{\circ}{\chi}_n^2 d\xi. \quad (14)$$

The energy η being given, condition (13) yields directly an approximate expression for the phase

$$\operatorname{tg} \delta = \frac{\sqrt{\eta}}{\beta_n}. \quad (15)$$

Now, if β is negative, there is a "last" stationary state of binding with energy η_n very small in absolute value. This energy has a more immediate physical significance than β_n , and is determined by the latter quantity. It will therefore be advantageous to express δ in terms of η_n . To find η_n , exactly the same procedure might be used as has just been applied to the derivation of δ . We would have to put for the eigenfunction

$$R_n(\xi) = \chi_n(\xi) e^{-\sqrt{-\eta_n} \xi} \quad (16)$$

and again approximate χ_n by χ_n^0 . A more elegant method, due to KRAMERS*, consists in regarding the eigenfunction (11) as an analytic function of the phase δ , treated as a complex quantity. For negative η , we get (since $\sqrt{-\eta} = i \sqrt{|\eta|}$)

$$R \sim e^{-\sqrt{|\eta|} \xi} - e^{-2i\delta} e^{\sqrt{|\eta|} \xi},$$

or, using (15) and the well-known formula $e^{2i \arctg z} = \frac{1+iz}{1-iz}$,

$$R \sim e^{-\sqrt{|\eta|} \xi} - \frac{\beta_n + \sqrt{|\eta|}}{\beta_n - \sqrt{|\eta|}} e^{\sqrt{|\eta|} \xi};$$

the eigenvalue η_n is determined by the condition of cancelling the increasing exponential term, i.e.

$$-\eta_n = \beta_n^2. \quad (17)$$

Formula (15) for the phase constant may thus in this case be written in the form

$$\operatorname{tg} \delta = - \sqrt{\frac{\eta}{\eta_n}}. \quad (18)$$

If β is positive, there is no such state of binding; still, we can, for the sake of uniformity, make use of the state of positive energy $\eta_n = \beta_n^2$, and write in a similar way

$$\operatorname{tg} \delta = + \sqrt{\frac{\eta}{\eta_n}}; \quad (19)$$

to distinguish it from the actual level $-\eta_n$, the state of positive energy η_n will be called a "virtual" level. The phase constant is then generally

given, in terms of the energy $|\eta_n|$ of the actual or virtual level concerned, by

$$\operatorname{tg} \delta = \pm \sqrt{\frac{\eta}{|\eta_n|}}, \quad (20)$$

the $\pm$ sign being that of β , i.e. $+$ or $-$ according as the level is virtual or actual.

To the approximation here considered, a physical meaning can also be attached, as pointed out by HULTHÉN [42b], to the concept of virtual level: it is a state in which the mean absolute value of the potential energy $\int_0^\infty R^2 w d\xi$ (the wave-function being normalized in the energy scale, 5.12) is a maximum, thus corresponding to a maximum concentration of the particle density within the potential trough. This property is readily verified by inserting for R the value (11) with $\chi \approx \chi_n$ and $\sin^2(\sqrt{\eta}\xi + \delta) \approx \frac{1}{2}(1 - \cos 2\delta)$; according to (14) and (15), we have thus to find the maximum of $(1 - \cos 2\delta)/\operatorname{tg} \delta$, and this occurs for $\delta = 45^\circ$, i.e. $\eta = \eta_n$. Hulthén suggested taking this extremal property as an *exact* definition of the virtual level; but we shall see presently that such a definition is not necessarily the most adequate when dealing with higher approximations.

A simple geometrical interpretation of $\operatorname{tg} \delta$ for S -states of very small energies is obtained by noticing that the tangent at the point ξ' where the eigenfunction $R(\xi)$ practically assumes its asymptotic form (5.12–21) cuts the ξ -axis at a point of abscissa

$$\xi' - \left(R \frac{dR}{d\xi} \right)_{\xi=\xi'} = \xi' - \frac{1}{\sqrt{\eta}} \operatorname{tg} (\sqrt{\eta} \xi' + \delta),$$

which, for small η , reduces to

$$\xi_a \equiv \kappa a = -\operatorname{tg} \delta / \sqrt{\eta}. \quad (21)$$

According to (20), we have in first approximation,

$$\xi_a = \mp 1/\sqrt{|\eta_n|}. \quad (22)$$

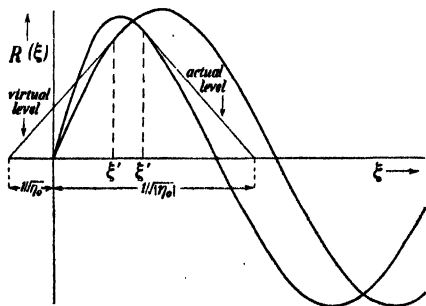


Fig. 5.32. Actual and virtual levels of small energy. The curves represent in both cases the eigenfunction of a state of small positive energy.

the point in question thus falls on the positive or negative side of the ξ -axis according as the level η_n is actual or virtual (fig. 5.32). In ordinary

units, the characteristic length a_0 corresponding to this approximation is given by

$$a_0 = \mp \frac{\hbar}{\sqrt{M|\epsilon_n|}}. \quad (22a)$$

5.321. Calculation of S-phases for a potential well. In the case of a potential well (5.11-2; 5.21), the calculation of the S-phases can be carried out explicitly. The eigenfunction of an S-state of positive energy is

$$\begin{aligned} &\sim \sin \sqrt{b + \eta} \xi \quad (\xi < 1) \\ &\sim \sin (\sqrt{\eta} \xi + \delta) \quad (\xi > 1); \end{aligned} \quad (23)$$

the continuity conditions at $\xi = 1$ yield

$$\sqrt{b + \eta} \operatorname{ctg} \sqrt{b + \eta} = \sqrt{\eta} \operatorname{ctg} (\sqrt{\eta} + \delta), \quad (24)$$

whence

$$\operatorname{ctg} \delta = \frac{\sqrt{\eta} \sin \sqrt{\eta} + \sqrt{b + \eta} \operatorname{ctg} \sqrt{b + \eta} \cos \sqrt{\eta}}{\sqrt{\eta} \cos \sqrt{\eta} - \sqrt{b + \eta} \operatorname{ctg} \sqrt{b + \eta} \sin \sqrt{\eta}}. \quad (25)$$

Suppose, now, that b is such that there is an actual level of small binding energy $|\eta^{(0)}|$. We may then, using the series for b given in (5.21-5), expand in powers of $\sqrt{|\eta^{(0)}|}$. For the quantity of physical interest $\sin^2 \delta = (1 + \operatorname{ctg}^2 \delta)^{-1}$, the result is (KITTEL and BREIT [39c])

$$\sin^2 \delta = \frac{\eta}{\eta + |\eta^{(0)}|} [1 + \sqrt{|\eta^{(0)}|} + G_2 |\eta^{(0)}| + G_3 |\eta^{(0)}|^{\frac{3}{2}} + \dots] \quad (26)$$

with

$$\begin{aligned} G_2 &= \left(1 - \frac{4}{\pi^2}\right) - \frac{1}{4} \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} = 0.5947 - 0.25 \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ G_3 &= \left(\frac{1}{3} - \frac{4}{\pi^2} + \frac{32}{\pi^4}\right) + \left(\frac{1}{\pi^2} - \frac{1}{3}\right) \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ &= 0.2556 - 0.2320 \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ G_4 &= \left(\frac{1}{3} - \frac{4}{\pi^2} + \frac{48}{\pi^4} - \frac{320}{\pi^6}\right) + \left(-\frac{1}{4} + \frac{2}{\pi^2} - \frac{8}{\pi^4}\right) \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ &\quad + \left(-\frac{1}{48} + \frac{1}{2\pi^2}\right) \left(\frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|}\right)^2 \\ &= 0.0880 - 0.1295 \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} + 0.0298 \left(\frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|}\right)^2, \dots \end{aligned} \quad (27)$$

In the case of a virtual level, $b = \pi^2/4 - \beta$, we readily find, in first approximation, $\eta^{(0)} = \beta^2/4$. If we took Hulthén's extremal property as an

exact definition of the virtual state, we should get in the next approximation (HULTHÉN [42b])

$$\eta^{(0)} = \frac{\beta^2}{4} \left[1 - \frac{\beta}{2} \left(1 - \frac{4}{\pi^2} \right) \right].$$

But in order to obtain an expansion closely parallel to the above, it is more convenient to start from a slightly different definition (KITTEL and BREIT [39c]), analogous to the relation (5.21-2) valid for an actual level:

$$\sqrt{b - \eta^{(0)}} \operatorname{ctg} \sqrt{b - \eta^{(0)}} = \sqrt{\eta^{(0)}}, \quad (28)$$

which gives instead

$$\eta^{(0)} = \frac{\beta^2}{4} \left[1 + \frac{\beta}{2} \left(1 - \frac{4}{\pi^2} \right) + \dots \right]. \quad (29)$$

The expansion for $\sin^2 \delta$ is then

$$\sin^2 \delta = \frac{\eta}{\eta + \eta^{(0)}} [1 - \sqrt{\eta^{(0)}} + G_2 \eta^{(0)} - G_3 (\eta^{(0)})^{\frac{3}{2}} + \dots] \quad (30)$$

with the same expressions as above for the G 's.

5.3211. Calculation of S -phases for an exponential potential (RARITA and PRESENT [37]). Treating the problem as in 5.211, we take as eigenfunction of a state of relative motion the linear combination of Bessel functions

$$c_1 J_{i\sqrt{\eta}}(y) + c_2 J_{-i\sqrt{\eta}}(y) \quad (y = \sqrt{b} e^{-\xi}).$$

The ratio of the coefficients c_1, c_2 is fixed by the boundary condition at $\xi = 0$:

$$c_1 J_{i\sqrt{\eta}}(\sqrt{b}) + c_2 J_{-i\sqrt{\eta}}(\sqrt{b}) = 0,$$

while the phase δ is defined by the asymptotic form

$$\lim_{y \rightarrow 0} [c_1 J_{i\sqrt{\eta}}(y) + c_2 J_{-i\sqrt{\eta}}(y)] \simeq \sin(\sqrt{\eta} \xi + \delta).$$

This means that δ is the argument of the complex quantity

$$J_{i\sqrt{\eta}}(\sqrt{b}) \frac{e^{-i\sqrt{\eta} \log \sqrt{b}/2}}{I'(1+i\sqrt{\eta})}.$$

Hence, in the limit of very small $\sqrt{\eta}$, we readily derive for $\operatorname{tg} \delta$ an expression of the general form (20) or (21), with

$$\xi_a = -\frac{1}{J_0(\sqrt{b})} \left[\frac{\pi}{2} N_0(\sqrt{b}) - J_0(\sqrt{b}) \left(\log \frac{\sqrt{b}}{2} + 0.5772 \dots \right) \right].$$

N_0 denoting the Neumann function of order zero.

5.322. Calculation of S -phases for Hulthén's potential. The theory of 5.32 can also be illustrated by the case of Hulthén's potential (5.11-7; 5.22). For χ_n we have, in virtue of (5.22-8,12),

$$\chi_n = \frac{(-1)^n}{n!} x \frac{d^n}{dx^n} [x^{n-1} (1-x)^n], \quad (31)$$

and therefore, by (14), (5.22-7),

$$\beta_n = \beta \int_0^1 \chi_n^2 \frac{dx}{x} = \beta \int_0^1 \frac{1}{n!} \frac{d^n \chi_n^0}{dx^n} x^{n-1} (1-x)^n dx = \frac{\beta}{2n}. \quad (32)$$

For small values of η , we then get the phase by applying formula (15).

5.323. Calculation of S -phases for the meson potential. In this case (5.231), we write the wave-function of a state of small positive energy in the form (11) with

$$\chi(x) = \frac{x - cx^2}{1 - c}, \quad (33)$$

the variable x being again defined by (5.22-7). If now

$$b = 1.68 - \beta \quad (|\beta| \ll 1), \quad (34)$$

the variational method outlined in 5.31 leads (HULTHÉN [42b]) to

$$\text{tg } \delta = \frac{2.266 \sqrt{\eta}}{\beta} \left(1 - 0.125 \beta - 3.885 \frac{\eta}{\beta} + \dots \right). \quad (35)$$

Suppose that $\beta > 0$, so that we have a virtual level of low energy η_1 . By putting $\text{tg } \delta = 1$ and $\eta = \eta_1$ in (35), solving for β and introducing the expression obtained in (34), we get a relation, entirely analogous to (5.231-25), between the potential strength b and the energy η_1 of the virtual level

$$b = 1.6798 - 2.2655 \sqrt{\eta_1} + 0.6416 \eta_1 - \dots;$$

in particular, the identity (apart from the sign) of the coefficients of $\sqrt{\eta}$ in the two formulae is in accordance with the theory of 5.32.

5.33. Estimate of phases of states with higher orbital momentum. In actual cases, the potential ${}^\sigma B^{(l)}(\xi)$ will, at any rate for $l \geq 2$, be negligible for the value of ξ given by the impact parameter ξ_l (5.14-28). If this is the case, we can directly use the identity (5.121-25) to obtain an approximate value of ${}^\sigma \delta^{(l)}$. In fact, we may then, in the integral, replace ${}^\sigma R^{(l)}(\xi)$

throughout by $\sqrt{\frac{\xi}{2}} J_{l+1/2}(\sqrt{\eta} \xi)$: for $\xi < \xi_l$, because both functions behave approximately as $\sim \xi^{l+1/2}$; for $\xi > \xi_l$, because ${}^\sigma B^{(l)}$ is negligible, so that ${}^\sigma R^{(l)} \approx \sqrt{\frac{\xi}{2}} J_{l+1/2}(\sqrt{\eta} \xi)$. We get accordingly

$$\sin {}^\sigma \delta^{(l)} \approx \frac{\pi}{2} \int_0^\infty [\sqrt{\xi} J_{l+1/2}(\sqrt{\eta} \xi)]^2 \cdot {}^\sigma B^{(l)} d\xi, \quad (36)$$

or, with the ordinary (not dimensionless) variables,

$$\sin {}^\sigma \delta^{(l)} \approx -\frac{\pi}{2} \cdot \frac{M}{\hbar^2} \int_0^\infty [J_{l+1/2}(kr)]^2 \cdot {}^\sigma V^{(l)} r dr, \quad (37)$$

on account of (5.11-1,9), (5.111-13).

Formula (36) or (37) corresponds to the so-called *Born approximation*. According to the assumption made about ${}^{\circ}B^{(l)}(\xi)$, its validity is restricted to nuclear potentials insensible at a distance

$$\xi_l \approx \frac{\sqrt{l(l+1)}}{\sqrt{\eta}}, \quad \text{or} \quad r_l \approx \frac{\sqrt{l(l+1)}}{k}, \quad (38)$$

i.e. for sufficiently large l or sufficiently small k . The phase ${}^{\circ}\delta^{(l)}$ given by (36) is always a small quantity; for the integrand in (36) is small for all ξ : for $\xi < \xi_l$ on account of $\sqrt{\xi} J_{l+\frac{1}{2}}(\sqrt{\eta}\xi)$ being small; for $\xi > \xi_l$ because ${}^{\circ}B^{(l)}$ is then negligible.

5.331. More accurate estimate. A more accurate estimate can be obtained by a variant of Hulthén's method (5.31), proposed by PAIS [46]. It consists in taking as trial function

$$R \approx \sqrt{\frac{\xi}{2}} J_{l+\frac{1}{2}+\Delta}(\sqrt{\eta}\xi); \quad (39)$$

according to (5.12-20,21), the parameter Δ is directly related to the phase δ by

$$\delta = -\frac{\pi}{2} \Delta. \quad (40)$$

It therefore represents only a small change in the order of the Bessel function, and in conformity with the general procedure (5.31-5), it can be determined from the equation (5.31-1), in which the value (39) for R is inserted. Using a property of Bessel functions*, this gives

$$\frac{1}{2} \frac{\Delta(\Delta + 2l + 1)}{2\Delta + 2l + 1} + \int_0^{\infty} B R^2 d\xi = 0. \quad (41)$$

The function (39) can be expected to be a good approximation when the influence of the nuclear forces is smaller throughout than that of the centrifugal forces. If Δ is very small, equations (40), (41) reduce to a form practically equivalent to the Born approximation (36), on account of $\sin \delta \approx \delta$.

For the meson potential $B = be^{-\xi/\xi}$, e.g., the integral in (41) becomes

$$\frac{1}{2} b \int_0^{\infty} J_{l+\frac{1}{2}+\Delta}^2(\sqrt{\eta}\xi) e^{-\xi} d\xi = \frac{b}{2\pi\sqrt{\eta}} Q_{l+\Delta} \left(1 + \frac{1}{2\eta} \right), \quad (42)$$

$Q_n(x)$ denoting a Legendre function of the second kind**. For $x > 1$, one has (W & W, 15.31, p. 317)

$$Q_n(x) = \frac{\sqrt{\pi}}{(2x)^{n+1}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \left[1 + \frac{(n+1)(n+2)}{2(2n+3)} \frac{1}{x^2} + \dots \right]. \quad (43)$$

We must insert the value (42) of the integral, expanded according to (43), into equation (41) and solve for Δ . The Born approximation in this case is

$$\delta \approx \frac{b}{2\sqrt{\eta}} Q_l \left(1 + \frac{1}{2\eta} \right). \quad (44)$$

* Cf. G. WATSON, *Treatise on the theory of Bessel functions* (1922), p. 403.

** Cf. G. WATSON, *l.c.*, p. 389.

CHAPTER VI

PHYSICAL PROPERTIES OF THE PROTON-NEUTRON SYSTEM

6.0. In the present Chapter, we shall discuss the available empirical material about the proton-neutron system on the assumption of a static central interaction between proton and neutron, with a view to deriving as much information as possible concerning the form of this interaction.

6.1. The ground state of the deuteron

6.11. Energy and angular momentum. The most precise determination of the binding energy of the deuteron in its normal state is afforded by the study of the disintegration of this nucleus by γ -rays: the threshold energy of the latter gives directly the *binding energy* (MYERS and VAN ATTA [42], WIEDENBECK and MARHOEFER [45])

$$|\varepsilon_0| = 2,185 \pm 0,006 \text{ MeV.} \quad (1)$$

The value 1 found (1.22) for the angular momentum of the ground state is compatible, according to the parity of the state, either with a (triplet or singlet) P -type or, for the most general static interaction, with a mixture ${}^3S + {}^3D_1$ (4.331–22). The former eventuality would present itself if the interaction were described by a Majorana potential with a repulsive distance dependence (4.341); but it is ruled out *a posteriori* by the general consistency of an interpretation of the empirical data based on the assumption that we are dealing with an *even state*. In particular, a P state would give rise to an electric quadrupole moment much larger than observed.

6.12. Electromagnetic properties. The next question is whether one has actually to deal with a mixture of S and D components, or with a pure S or D state, i.e. whether or not the interaction essentially involves a non-central coupling. Definite conclusions have been reached in this respect as a result of Rabi's investigations on the electromagnetic behaviour of hydrogen and deuterium molecules. His powerful magnetic resonance method of analysis of molecular beams deflected by an inhomogeneous magnetic field (RABI *et al.* [39, 40]) yields very accurate information on the energies of the stationary states of the molecules investigated, including not only the interaction with the external magnetic field, but also the mutual interaction of the constituent nuclei and electrons due to their magnetic dipole and electric quadrupole moments (A2.1). The discussion of the results obtained with H_2 , HD and D_2 molecules led not only to a precise determination of the *magnetic moment* of the ground state of the deuteron,

but also to the disclosure of a small *electric quadrupole moment* in this state.

The latter feature is of fundamental importance, in as much as it implies the existence of a non-central interaction between proton and neutron. In fact, it is in the first place obvious that the electric quadrupole moment, defined (A2.13) as the expectation value of the operator

$$Q = \frac{1}{4} (3z^2 - r^2) = \frac{r^2}{4} (3 \cos^2 \vartheta - 1) = \frac{\sqrt{4\pi}}{2\sqrt{5}} r^2 Y_2^0(\vartheta) \quad (2)$$

in the substate of maximum magnetic quantum number, would vanish if the ground state were a pure ${}^3S$ state; this remains true when the relativistic correction of the ${}^3S$ eigenfunction, which involves P components (4.34–29b), is taken into account (MØLLER and ROSENFELD [40]) *.

Let us therefore consider a general ${}^3S + {}^3D_1$ mixture, represented by the (unrelativistic) eigenfunction

$$\psi = \psi_S \cos \omega + \psi_D \sin \omega; \quad (3)$$

in this formula, ψ_S and ψ_D are the wave-functions of the ${}^3S$ and ${}^3D_1$ states which combine to form the ground state, both normalized to unity; $\sin^2 \omega$ denotes the amount of D state admixture. For ψ_S and ψ_D we write, according to (4.34–28), (4.32–17),

$$\begin{aligned} \psi_S &= {}^1(t)_0 \cdot \frac{R_0(r)}{r} \cdot {}^3Z_1^{(0)m} = {}^1(t)_0 \cdot \frac{R_0(r)}{r} \frac{(-1)^m}{\sqrt{4\pi}} \cdot {}^3(o)_m \\ \psi_D &= {}^1(t)_0 \cdot \frac{R_2(r)}{r} \cdot {}^3Z_1^{(2)m} \end{aligned} \quad (4)$$

* The exact eigenfunction of a ${}^3S$ state is of the form (4.33–18)

$$\psi = {}^3(o)_1 \cdot {}^3f_1 + {}^3(o)_0 \cdot {}^3f_0 + {}^1(o)_0 \cdot {}^1f_0 + {}^3(o)_{-1} \cdot {}^3f_{-1};$$

for the substate of magnetic quantum number $m = 1$, we have, according to (4.34–29b), (4.32–14, 17),

$$\begin{aligned} {}^3f_0 &= \frac{1}{\sqrt{3}} {}^1Z_1^{(1)1} \cdot {}^1(t)_0 \cdot \mathcal{D}_1 {}^3R^{(0)} = \frac{1}{\sqrt{3}} {}^1(o)_0 \cdot Y_1^1 \cdot {}^1(t)_0 \cdot \mathcal{D}_1 {}^3R^{(0)} \\ {}^1f_0 &= \sqrt{\frac{2}{3}} {}^3Z_1^{(1)1} \cdot {}^1(t)_0 \cdot \mathcal{D}_1 {}^3R^{(0)} = \frac{1}{\sqrt{3}} [{}^3(o)_0 Y_1^1 + {}^3(o)_1 Y_1^0] {}^1(t)_0 \cdot \mathcal{D}_1 {}^3R^{(0)}. \end{aligned}$$

The expression for the quadrupole moment, exact up to the second order in the velocities inclusively (the first order correction vanishes) involves as a factor the integral

$$\sum_{\sigma_z} \int d\Omega Y_2^0 |{}^3f_0|^2 + |{}^1f_0|^2;$$

and since

$$\sum_{\sigma_z} |{}^3f_0|^2 + |{}^1f_0|^2 \sim 2 |Y_1^1|^2 + |Y_1^0|^2 \sim 2 \cdot \frac{1}{2} \sin^2 \vartheta + \cos^2 \vartheta$$

is independent of ϑ , this integral vanishes.

and, by a calculation which will be developed in 6.121 below, get for the quadrupole moment $Q = av Q$:

$$Q = q_1 \sin \omega \cos \omega - q_2 \sin^2 \omega,$$

$$q_1 = -\frac{1}{10} \Re \int_0^\infty R_0^* R_2 r^2 dr \quad (5)$$

$$q_2 = \frac{1}{20} \int_0^\infty |R_2|^2 r^2 dr.$$

Putting

$$\frac{q_1}{q_2} = \operatorname{tg} \alpha, \quad \frac{2Q}{q_1} = \operatorname{tg} \xi,$$

we may transform relation (5) into

$$\cos(2\omega - \alpha) = \frac{\cos(\xi - \alpha)}{\cos \xi}. \quad (6)$$

Regarded as an equation in ω , equation (6) has two essentially distinct solutions ω_1, ω_2 , connected by the relation *

$$\omega_1 + \omega_2 = \alpha.$$

Now, we may expect the quantities q_1, q_2 to be comparable in order of magnitude with $d^2 \approx 10^{-25} \text{ cm}^2$, while the value of Q deduced from the measurements of RABI and his collaborators **, viz.

$$Q = 2.73 \cdot 10^{-27} \text{ cm}^2, \quad (7)$$

* Taking into account this relation, one derives from (6)

$$\cos(\omega_1 - \omega_2) = \frac{\cos(\xi - \alpha)}{\cos \xi}$$

and further

$$\sin^2 \omega_1 + \sin^2 \omega_2 = 1 - \cos \alpha \cos(\omega_1 - \omega_2)$$

$$\sin^2 \omega_1 - \sin^2 \omega_2 = \sin \alpha \sin(\omega_1 - \omega_2),$$

so that the exact solutions of equation (6) may be expressed in the form

$$\sin^2 \omega = \frac{\sin^2 \alpha}{2} [1 - \operatorname{tg} \xi \operatorname{ctg} \alpha \pm \sqrt{1 - 2 \operatorname{tg} \xi \operatorname{ctg} \alpha - \operatorname{tg}^2 \xi}].$$

** The experiments yield with great accuracy the product of Q with the mean gradient of the electric field exerted on one of the nuclei of the deuterium molecule by the other nucleus and the electrons. The calculation of this factor is a delicate matter, because it requires an accurate knowledge of the electronic wave-function; it has been carried out by NORDSIECK [40]. The possible error in the estimation of this factor is the main source of uncertainty in the value of the quadrupole moment; it amounts to about 2%.

is of a much smaller order of magnitude. Accordingly, we may write, in a more or less rough approximation,

$$\omega_1 \approx \frac{1}{2} \xi \quad , \quad \omega_2 \approx a - \frac{1}{2} \xi$$

i.e.

$$\sin^2 \omega_1 \approx \left(\frac{Q}{q_1} \right)^2 \quad (8a)$$

$$\sin^2 \omega_2 \approx \frac{q_1^2 - 2 Q q_2}{q_1^2 + q_2^2} \quad (8b)$$

These solutions are of very different characters: the former represents a very small, the latter a very large modification of the ${}^3S$ eigenfunction by admixture of a ${}^3D_1$ wave-function.

A decision between these two possibilities can be reached by means of the empirical value of the magnetic moment. Indeed, this value (RABI *et al.* [39, 40], ARNOLD and ROBERTS [46]),

$$\mu_d = 0,8565 \pm 0,0004 \quad (9)$$

nuclear magnetons, is found to differ markedly from that of the magnetic moment of a pure ${}^3S$ state, viz.

$$\mu_d^0 = \mu_n + \mu_p = 0,8793 \pm 0,0020 \quad (10)$$

according to (1.21-3), (1.22-6): and the difference is directly interpretable in terms of an admixture of D state. In fact, for the $S + D$ mixture (3), one finds (A2.13-9) a magnetic moment

$$\mu_d = \mu_d^0 - \frac{3}{2} (\mu_d^0 - \frac{1}{2}) \sin^2 \omega. \quad (11)$$

Inserting in this formula the values of μ_d and μ_d^0 given by (9) and (10), one gets

$$\sin^2 \omega = 0,04. \quad (12)$$

This result is compatible only with the first solution, (8a). We are therefore led to the conclusion that *the ground state of the deuteron is predominantly a ${}^3S$ state, with a small admixture of ${}^3D_1$ state*. It may further be observed (SCHWINGER [41]) that the fact that the D admixture cannot be *larger* than (12), together with (8a) and the numerical value (7) of the quadrupole moment, leads to an inequality

$$q_1 \gtrsim 0,5 \cdot 10^{-27} \text{ cm}^2, \quad (13)$$

implying a *lower* limitation of the range of nuclear interaction*.

* Some implications of the large mixing ratio (8b) were discussed at an early stage (when this eventuality could not yet be excluded) by INGLIS [39b] and (not quite correctly) by FLÜGGE [39].

It must be emphasized, however, that the smallness of D -admixture by no means implies that the non-central coupling necessarily contributes only a relatively small correction to the binding energy of the ground state. On the contrary, several types of non-central potentials have been studied, in which the non-central terms in general play a considerable part, while nevertheless giving rise to a small D -admixture of the ${}^3S$ ground state. The discussion of such cases will, however, be postponed until we have examined the various implications of the assumption that the main part of nuclear interaction is describable by a central potential. From now on, in this and the following Part, we shall thus limit ourselves to *central interactions*; the ground state of the deuteron will accordingly be treated as a pure ${}^3S$ state.

6.121. Calculation of the deuteron quadrupole moment. We must still briefly indicate how the explicit calculation of the quadrupole moment can be carried out. Let us define

$$\mathcal{D}^{(12)} = (\vec{\sigma}^{(1)} \cdot \vec{x}_0) (\vec{\sigma}^{(2)} \cdot \vec{x}_0) - \frac{1}{3} (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}), \quad (14)$$

where $\vec{x}_0$ is the unit vector $\vec{x}/|\vec{x}|$. The easily established identity (RARITA and SCHWINGER [41b])

$$\Delta (r^2 \mathcal{D}^{(12)}) = 0 \quad (15)$$

expresses that $\mathcal{D}^{(12)}$ depends on the angle variables through a linear combination of tesseral harmonics of order $l = 2$. On the other hand, $\mathcal{D}^{(12)}$ commutes with the total angular momentum and (since it is symmetrical in $\sigma^{(1)}, \sigma^{(2)}$) with the square of the total spin (4.32-8), so that it conserves the multiplicity. If, therefore, $\mathcal{D}^{(12)}$ operates on ${}^3Z_1^{(0)m}$, the resulting combination of the $Y_2^{m'}$ will just be (apart from a constant factor) ${}^3Z_1^{(2)m}$ (with the same $j = 1$ and m). The co-factor can be ascertained by explicit calculation of the coefficient of one particular ${}^3(\sigma)_m$ (for $m = 1$, say) in $\mathcal{D}^{(12)} \cdot {}^3Z_1^{(0)m}$. One finds

$${}^3Z_1^{(2)m} = -\frac{3}{2\sqrt{2}} \mathcal{D}^{(12)} \cdot {}^3Z_1^{(0)m}. \quad (16)$$

By direct calculation, one may further verify the relation

$$[\mathcal{D}^{(12)}]^2 = \frac{4}{3} (1 + P_\sigma) - \frac{2}{3} (2 P_\sigma - 1) \mathcal{D}^{(12)}. \quad (17)$$

According to (2), (3) and (4), the numerical coefficients occurring in the expressions (5) for q_1 and q_2 are, respectively,

$$\begin{aligned} & \frac{\sqrt{4\pi}}{\sqrt{5}} \Re \sum_{\sigma'} \int {}^3Z_1^{(0)1*} \cdot Y_2^0 \cdot {}^3Z_1^{(2)1} d\Omega \\ & - \frac{\sqrt{4\pi}}{2\sqrt{5}} \sum_{\sigma'} \int {}^3Z_1^{(2)1*} \cdot Y_2^0 \cdot {}^3Z_1^{(2)1} d\Omega; \end{aligned}$$

by (16), the first quantity becomes successively

$$\begin{aligned}
 & -\frac{\sqrt{4\pi}}{\sqrt{5}} \cdot \frac{3}{2\sqrt{2}} \sum_{\sigma'} \int {}^3Z_1^{(0)1*} \cdot \mathcal{D}^{(12)} \cdot {}^3Z_1^{(0)1} \cdot Y_2^0 d\Omega \\
 & = -\frac{\sqrt{4\pi}}{\sqrt{5}} \cdot \frac{3}{2\sqrt{2}} \cdot \frac{1}{4\pi} \cdot \frac{2}{3} \frac{\sqrt{4\pi}}{\sqrt{5}} \int (Y_2^0)^2 d\Omega = -\frac{\sqrt{2}}{10}.
 \end{aligned}$$

while, by (16) and (17), the second reduces to

$$\begin{aligned}
 & -\frac{\sqrt{4\pi}}{2\sqrt{5}} \cdot \frac{9}{8} \cdot \sum_{\sigma'} \int {}^3Z_1^{(0)1*} \cdot [\mathcal{D}^{(12)}]^2 \cdot {}^3Z_1^{(0)1} \cdot Y_2^0 d\Omega \\
 & = \frac{\sqrt{4\pi}}{2\sqrt{5}} \cdot \frac{3}{4} \sum_{\sigma'} \int {}^3Z_1^{(0)1*} \cdot \mathcal{D}^{(12)} \cdot {}^3Z_1^{(0)1} \cdot Y_2^0 d\Omega = \frac{1}{20}.
 \end{aligned}$$

6.122. Para-ortho conversion. For the sake of completeness, we recall here that an alternative, though less direct and much less precise method of measuring the magnetic moment of the deuteron, or rather its ratio to the proton magnetic moment, consists in studying the rate of conversion of para-hydrogen or ortho-deuterium into their respective ortho- or para-states under the influence of paramagnetic gases such as O_2 or NO (WIGNER [33b], KALCKAR and TELLER [35], FARKAS *et al.* [35, 38]). The ortho-para-transitions will be brought about by the inhomogeneous magnetic field of the paramagnetic molecules; the probability of their occurrence will evidently be proportional to the square of the magnetic moment of the H or D nucleus, and a comparison of the effect for H_2 and D_2 under suitably chosen conditions yields directly the ratio $|\mu_p/\mu_d|$. Owing to the approximate character of the theoretical formulae, however, the values calculated for this ratio from the empirical data are not quite definite: according to the form of theory adopted, one obtains either 3.8 or some value between 3.2 and 4; these may be compared with the figure 3.28 computed from the values of μ_d and μ_p measured in Rabi's laboratory. On the other hand, if one compares the ortho-para conversion of hydrogen and deuterium when dissolved in water or other diamagnetic liquids (FARKAS and SANDLER [39], CASIMIR [40]), it appears that besides the magnetic interaction another mechanism must be active to accelerate the deuterium conversion; a closer examination shows that the electric interaction between the dipole field of the water molecules and the deuteron quadrupole moment can account even quantitatively for the corresponding empirical result.

6.13. Magnetic interaction of proton and neutron. There are two *a priori* possible configurations of the deuteron which have no orbital momentum: they correspond to a triplet and a singlet S state. If the nuclear forces did not depend on the spins of the nucleons, the only energy difference between these states would be due to the different magnetic interactions of the constituent nucleons; this effect has been calculated by CASIMIR [36a] under the assumption that the magnetic moment of the nucleon results from a current density distribution of the form $\text{rot } \vec{M}$, $\vec{M}$ being the density of magnetization. There can of course

be no certainty about such an assumption as long as the theory of the anomalous magnetic moment is not better founded, but it is in any case in accordance with the charged field theory outlined in 1.21. The interaction operator of the current distributions of proton and neutron* will then be, if $r_{PP'}$ denotes the distance of the points P, P' (1.31-7),

$$\mathcal{V}_{\text{magn}} = - \int \text{rot}_P \vec{M}_P^{(1)} \cdot \text{rot}_{P'} \vec{M}_{P'}^{(2)} \frac{dv_P dv_{P'}}{r_{PP'}}, \quad (18)$$

the operators $\vec{M}^{(i)}$, in the configuration space $(\vec{x}^{(1)}, \vec{x}^{(2)})$ of the nucleons, having the form

$$\vec{M}^{(i)}(\vec{\xi}) = \vec{\mu}^{(i)} \delta(\vec{\xi} - \vec{x}^{(i)}) \quad (19)$$

with $\vec{\mu}^{(i)}$ given by (4.42-9). We have to average this over the density distribution defined by the eigenfunctions (large components) of the ^1S and ^3S states (4.34-28).

For the calculation of the expectation value in such spherically symmetrical states, the operator (18) is readily found, by partial integrations, to be equivalent to

$$\begin{aligned} \mathcal{V}_{\text{magn}} &= -\frac{8\pi}{3} \int \vec{M}_P^{(1)} \vec{M}_P^{(2)} dv_P + \int \vec{M}_P^{(1)} \text{grad}_P (\vec{M}_{P'}^{(2)} \text{grad}_{P'}) \frac{dv_P dv_{P'}}{r_{PP'}} \\ &= -\frac{8\pi}{3} \vec{\mu}^{(1)} \vec{\mu}^{(2)} \delta(\vec{x}) - (\vec{\mu}^{(1)} \text{grad}) (\vec{\mu}^{(2)} \text{grad}) \frac{1}{r} \\ &= -\frac{8\pi}{3} \vec{\mu}^{(1)} \vec{\mu}^{(2)} \delta(\vec{x}) - \frac{1}{r^3} \left[3 \left(\vec{\mu}^{(1)} \frac{\vec{x}}{r} \right) \left(\vec{\mu}^{(2)} \frac{\vec{x}}{r} \right) - \vec{\mu}^{(1)} \vec{\mu}^{(2)} \right]. \end{aligned} \quad (20)$$

The first term arises from a surface integral over an infinitesimal sphere with centre P , which is accordingly excluded from the domain of integration of the second term (as indicated by the sign $^{\circ}$); as it stands, the first term is correct only for the calculation of $\langle S | \dots | S \rangle$ matrix elements, but its more general expression would not offer any interest, since it does not yield any contribution to other matrix-elements, on account of the vanishing of the radial part of the wave-functions at the origin. For S states, however, it is just this first term which gives the whole expectation value, the average of the second one over all angles being zero**. The expectation

* If we assume that there is no spin-dependent interaction, there is no contribution of the exchange magnetic moment to the magnetic interaction energy (4.43).

** At first sight, the contribution from the second term of (20) would seem to be indeterminate, owing to the divergence of the mean value of r^{-3} in an S state. But this singularity arises only through the neglect of a relativistic correction factor $\left(1 + \frac{e - \mathcal{V}_{\text{total}}}{M}\right)^{-2}$, $\mathcal{V}_{\text{total}}$ being the total interaction energy. The latter quantity becoming infinite (as r^{-3}) at the origin, this factor indeed secures the convergence of the mean value of r^{-3} .

value of the factor containing the isotopic variables is, of course, $\mu_n \mu_p$; as to the spin factor $\vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$, or $2P_\sigma - 1$ (4.15-26), its average σ_s is -3 for the singlet state and $+1$ for the triplet state. We finally get for the magnetic interaction energy in those states

$$\sigma \bar{V}_{\text{magn}}^{(0)} = -\mu_0^2 \mu_n \mu_p \cdot \frac{2}{3} \sigma_s \left| \frac{R^{(0)}(r)}{r} \right|_{r=0}^2 \quad (\sigma = 1, 3). \quad (21)$$

If we suppose the nuclear potential to be the same for both states, we should take ${}^1R^{(0)} = {}^3R^{(0)}$. Since $\mu_n < 0$, $\mu_p > 0$, this would mean, however, that the singlet state would be lower than the triplet state, — a conclusion in contradiction with the spectroscopic evidence indicating that the ground state has angular momentum 1 (6.11). This is a strong argument pointing to a significant *spin dependence of the proton-neutron potential*, large enough to depress the ${}^3S$ below the ${}^1S$ state. The order of magnitude of singlet-triplet separation due to magnetic interaction is easily estimated from (21). One may write in general *

$$\left| \frac{R^{(0)}(r)}{r} \right|_{r=0}^2 = \frac{2\kappa^2}{|a_0|} \alpha, \quad (22)$$

a_0 being the characteristic length (5.32-22a) associated with the ground state and α a numerical factor of the order of 2 or 3. The level separation is therefore, by (21), (22),

$$\begin{aligned} {}^3\bar{V}_{\text{magn}}^{(0)} - {}^1\bar{V}_{\text{magn}}^{(0)} &\approx 4 \cdot \frac{2}{3} \mu_0^2 |\mu_n| |\mu_p| \frac{2 \sqrt{M|\epsilon_0|} \kappa^2}{\hbar} \alpha \\ &\approx \frac{4}{3} \frac{\hbar}{e^2} \left(\frac{m}{M} \right) |\mu_n| |\mu_p| \sqrt{m|\epsilon_0|} \alpha' \\ &\approx 0.013 \alpha' \text{ MeV}, \end{aligned} \quad (23)$$

if the values (1.21-3), (1.22-6) and (1) are inserted, and κ^{-1} is taken to be of the order of d (we have written $\kappa^2 \alpha = d^{-2} \alpha'$; α' is again of the

* Formula (22) is easily obtained, in a rough approximation corresponding to the limiting case of a force of zero range, from the normalization and continuity conditions for a potential well of vanishing width. For a well of finite width, we get, according to (5.21-1, 4, 5)

$$\left[\frac{R(\xi)}{\xi} \right]_{\xi=0}^2 = \frac{2 \sqrt{|\eta|} (b - |\eta|)}{1 + \sqrt{|\eta|}} \approx 2 \sqrt{|\eta|} \frac{\pi^2}{4} (1 - 0.19 \sqrt{|\eta|}),$$

which, by (5.32-22), is of the form (22), with $\alpha \approx 2.2$ (taking $|\epsilon_0| \approx 2.18 \text{ MeV}$ and $\kappa^{-1} \approx d$). For Hulthén's potential, formulae (5.22-17, 18) yield

$$[R(\xi)/\xi]_{\xi=0}^2 = b(b+1) \sqrt{|\eta|} = 2 \sqrt{|\eta|} (1 + 3 \sqrt{|\eta|} + 2 |\eta|);$$

taking here for κ the value corresponding to a meson mass $M_m \approx 240 m$ (1.32-12; 1.33-13), i.e. $\kappa \approx 1.75 d^{-1}$, we get $\alpha \approx 2.4$.

order of 2 to 6). This is a quite small interval compared with the separation due to the spin dependent nuclear interaction: in fact, as we shall see, this interaction is so large that there is no ^1S state of binding of the deuteron.

6.2. Scattering of slow neutrons by protons

6.20. The evidence for the existence of a strong spin dependence of the proton-neutron interaction, discussed in the preceding section (6.13), is corroborated by the study of the scattering of slow neutrons by protons. In this section, we shall first develop the theory of the scattering process (in a form valid also for larger neutron velocities); we shall then summarize the relevant empirical data and discuss their implications as regards the nuclear potential.

6.21. *Theory of proton-neutron scattering.* In order to describe the process of proton-neutron scattering, we look for a solution of the two-body problem, with given total energy, asymptotically representing an incident plane wave Ψ_{inc} and a scattered spherical wave Ψ_{scatt} :

$$\Psi \simeq \Psi_{\text{inc}} + \Psi_{\text{scatt}}. \quad (1)$$

Such a solution will be a linear combination of the eigenstates of the given (positive) energy η , of the form (4.33–18), (4.34–28). To the first order in the nucleon velocities, the differential cross-section for the scattering of a given incident wave will therefore be

$$dS = \sum_{\varrho'_3, \sigma'_z} |\Psi_{\text{scatt}}|^2 r^2 d\Omega = \sum_{\varrho'_3, \sigma'_z} \{ |\Psi^{(0)}|^2 + (\Psi^{(0)} \Psi^{(1)*} + \Psi^{(0)*} \Psi^{(1)}) \}_{\text{scatt}} r^2 d\Omega,$$

the incident wave being normalized to unity. From (4.33–18) it appears that the summation over ϱ'_3 will in any case make the first order contribution vanish, so that we have only to compute the large component $\Psi^{(0)}_{\text{scatt}}$; in the following calculations, we shall accordingly omit the index $^{(0)}$ and the factor $^3(\varrho)_1$ in Ψ , and the summation over ϱ'_3 (which just gives 1) in dS , so that we simply write

$$dS = \sum_{\sigma'_z} |\Psi_{\text{scatt}}|^2 r^2 d\Omega. \quad (2)$$

The incident wave representing a proton and a neutron moving with relative momentum p in the z -direction has the asymptotic form

$${}^\sigma \Psi_{\text{inc}} \simeq \frac{1}{\sqrt{2}} \{ u_+ (1) u_- (2) e^{ikz} \mp u_- (1) u_+ (2) e^{-ikz} \}, \quad (3)$$

u_+ , u_- being the eigenfunctions of τ_3 , and the double sign being — or + according as the spins of the two particles are parallel ($\sigma = 3$, triplet system) or antiparallel ($\sigma = 1$, singlet system); k is $p/\hbar$ as given by

(5.111–13). With the help of the well-known expansion

$$\begin{aligned} e^{ikr} &= \sum_{l=0}^{\infty} (2l+1) i^l P_l(\cos \vartheta) \cdot \sqrt{\frac{\pi}{2kr}} J_{l+\frac{1}{2}}(kr) \\ &\simeq \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)} Y_l^0 \cdot \frac{e^{ikr} - (-1)^l e^{-ikr}}{2ikr} \end{aligned} \quad (4)$$

(the normalized spherical harmonic $Y_l^0 = \sqrt{(2l+1)/4\pi} P_l(\cos \vartheta)$, P_l being the corresponding Legendre polynomial; see N.14), we may write (3) in the form

$${}^{\sigma}\Psi_{\text{inc}} \simeq \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)} Y_l^0 \cdot {}^{\tau}(\tau)_0 \cdot \frac{e^{ikr} - (-1)^l e^{-ikr}}{2ikr}, \quad (5)$$

where

$${}^{\tau}(\tau)_0 = \frac{1}{\sqrt{2}} \{u_+(1) u_-(2) \mp (-1)^l u_-(1) u_+(2)\} \quad (6)$$

is just the eigenfunction defined by (4.331–23). For the incident wave we may thus take as the most general combination

$$\Psi_{\text{inc}} \simeq \sum_{\sigma=1,3} \sum_{m_s} {}^{\sigma}J_{m_s} \cdot {}^{\sigma}(\sigma)_{m_s} \cdot {}^{\sigma}\Psi_{\text{inc}} \quad (7)$$

with arbitrary amplitudes ${}^{\sigma}J_{m_s}$ (for $\sigma = 1$, $m_s = 0$; for $\sigma = 3$, $m_s = -1, 0, 1$).

The total wave is, according to (4.34–28) and (5.12–21), of the general asymptotic form *

$$\Psi \simeq \sum {}^{\sigma}A_j^{(l)m} \cdot {}^{\sigma}Z_j^{(l)m} \cdot {}^{\tau}(\tau)_0 \cdot \frac{e^{ikr+i^{\sigma}\delta^{(l)}} - (-1)^l e^{-ikr-i^{\sigma}\delta^{(l)}}}{2ir \cdot i^l}. \quad (8)$$

The condition that $\Psi - \Psi_{\text{inc}}$ represents only an outgoing spherical wave readily yields

$$\Psi_{\text{scatt}} \simeq \sum_{\sigma, m_s, l} {}^{\sigma}J_{m_s} \cdot \sqrt{2l+1} \cdot {}^{\sigma}(\sigma)_{m_s} \cdot Y_l^0 \cdot {}^{\tau}(\tau)_0 \cdot \frac{e^{ikr}}{ikr} \cdot \sqrt{\pi} (e^{2i^{\sigma}\delta^{(l)}} - 1). \quad (9)$$

According to (6), this may be written as

$$\Psi_{\text{scatt}} \simeq \frac{1}{\sqrt{2}} u_+(1) u_-(2) \sum_{\sigma, m_s, l} \dots + \frac{1}{\sqrt{2}} u_-(1) u_+(2) \sum_{\sigma, m_s, l} (\mp) (-1)^l \dots$$

Now, in a scattering experiment, the identity of the particles is fixed by the production and detection arrangements, so that e.g. particle (1) is in the neutron state, particle (2) in the proton state. We then have

* The normalization factors occurring in (5.12–21) have been included in the coefficients A .

$u_+(1) = u_-(2) = 1$, $u_-(1) = u_+(2) = 0$ and only the first sum in the expression of Ψ_{scatt} subsists. If we further assume the incident wave to be unpolarized*, all $|\sigma J_{ms}|^2$ are equal; normalizing to unity, we get from (3), (7) for their absolute value $\frac{1}{2} |\sigma J_{ms}|^2 = \frac{1}{4}$ and the differential cross-section (2) becomes**

$$dS = \frac{\pi}{k^2} d\Omega \sum_{l,l'} \sqrt{(2l+1)(2l'+1)} Y_l^0 Y_{l'}^0 \sum_{\sigma} \sigma \sin^{\sigma} \delta^{(l)} \sin^{\sigma} \delta^{(l')} \cos [\sigma \delta^{(l)} - \sigma \delta^{(l')}]. \quad (10)$$

For the total cross-section we get, on account of the orthogonality of the harmonics

$$S = \frac{\pi}{k^2} \sum_l (2l+1) \sum_{\sigma} \sigma \sin^2 \delta^{(l)}. \quad (11)$$

Each state of the given energy contributes to the total cross-section a term proportional to the square of the sine of its phase and to its "weight" $(2l+1)\sigma$.

The differential cross-section, giving the angular distribution of the scattered particles, consists of an expansion in terms of the successive powers of $\cos \vartheta$. Since the phases decrease with increasing l (5.12), the main contribution to the scattering will in any case be that of the S -waves, yielding a spherically symmetrical distribution (in the barycentric system of reference). The P -waves will mainly contribute, by interference with the S -waves, a term in $\cos \vartheta$ in the angular distribution, and so on. It is important to observe, however, that the coefficient of any power of $\cos \vartheta$ in the differential cross-section strictly speaking depends on *all* waves of different orbital momenta.

The total cross-section, on the other hand, is less sensitive to the influence of the waves of higher orbital momenta, as formula (11) shows. In the case, treated in (5.32), that there exists an actual or virtual S -level of *sufficiently small energy* $|\eta^{(0)}|$, we get from (5.32-20), (11), (5.11-9) and (5.111-13), for the corresponding contribution of the S -wave to the scattering cross-section for not too fast neutrons

$$S^{(0)} = \frac{\pi}{k^2} \cdot \frac{\eta}{\eta + |\eta^{(0)}|} = \frac{\pi \hbar^2}{M} \frac{1}{\varepsilon + |\varepsilon^{(0)}|}, \quad (12)$$

or 3 times this quantity if the level belongs to the triplet system. In the limit of *zero energy of the scattered particles*, the scattering cross-section becomes, by (5.32-21) and (5.111-13)

$$S^{(0)} = \pi a^2, \quad (13)$$

where a has the geometrical meaning explained in 5.32; it also appears

* For the scattering of polarized neutrons, see 6.35.

** Of course, the same result would have been obtained from the second sum of Ψ_{scatt} , corresponding to the other possible numeration of proton and neutron.

here as the radius of the "sphere of action" of a proton with respect to a very slow neutron. A more or less rough approximation to a is the quantity a_0 defined by (5.32-22a); for the deuteron ground state, one has

$$^3a_0 = \frac{\hbar}{\sqrt{M|\epsilon_0|}} = 0.44 \cdot 10^{-12} \text{ cm.} \quad (14)$$

The binding energy of the deuteron is so large, however, that the formula (12) or the approximation $a \approx a_0$ in (13) are insufficient as regards the ^3S contribution to the scattering. The necessary correction cannot be given independently of the form of the potential; we shall now calculate it in the case of the well potential and we shall see that it is quite considerable.

6.211. Case of the well potential. According to the explicit calculation of the S -phase, carried out in 5.321, we see that the right-hand side of formula (12) must be multiplied by a correction factor (5.321-26, 30)

$$f = 1 \pm \sqrt{|\eta^{(0)}|} + G_2 |\eta^{(0)}| \pm G_3 |\eta^{(0)}|^2 + \dots, \quad (15)$$

the $\pm$ sign corresponding to an actual or virtual level, respectively, and the G 's being given in terms of $\epsilon/|\epsilon^{(0)}|$ by (5.321-27). If the width a_0 of the well is denoted by D , the parameter $\pm \sqrt{|\eta^{(0)}|}$ is, by (5.32-22), just D/a_0 (with its sign); we may thus re-write

$$f = 1 + \frac{D}{a_0} + G_2 \left(\frac{D}{a_0}\right)^2 + G_3 \left(\frac{D}{a_0}\right)^3 + \dots \quad (16)$$

Since the range D does not occur in (12), but only in the co-factor f , the expression given by (12) is often called the *zero range cross-section*, and f the *correction for finite range*.

Likewise, one has

$$a = a_0 f_0^{\frac{1}{2}}, \quad (17)$$

where f_0 represents the series (16) for $\epsilon = 0$; the coefficients G_i are then

$$G_2 = 0.345, \quad G_3 = 0.025, \quad G_4 = -0.012 \quad (\epsilon = 0). \quad (18)$$

For the ^3S scattering, the general expressions of the G 's may be written, on account of (6.11-1),

$$\begin{aligned} G_2 &= 0.345 - 0.057 E \\ G_3 &= 0.025 - 0.053 E \\ G_4 &= -0.012 - 0.016 E + 0.002 E^2 \end{aligned} \quad (19)$$

(E expressed in MeV);

in these formulae, $E (= 2\epsilon)$ denotes the kinetic energy of the neutron with respect to the proton at rest (6.212). The range D being of the order d , the ratio $D/^3a_0$ is, by (14), of the order 0.64, which shows the importance of the finite range correction in this case.

6.212. Transformation to laboratory system. The preceding treatment of the scattering process utilizes the barycentric system of reference, in

which the behaviours of proton and neutron are entirely equivalent. For practical purposes, however, we must go over to a "laboratory" system of reference, in which the bulk of the scatterer is at rest. When we have to do with the scattering of neutrons of large kinetic energy (i.e. large compared to the binding energy of protons in substances like paraffin, ≈ 0.4 eV), the protons may be considered as free and we may take as laboratory system that in which they are at rest. The above theory is then immediately applicable and the transition from the barycentric to the laboratory system, as appears from the self-explanatory figure 6.212, is effected in the following way*:

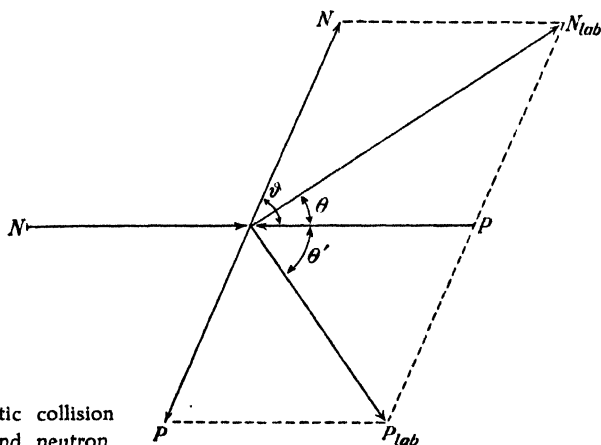


Fig. 6.212. Elastic collision between proton and neutron.

	Laboratory	Barycentric	
Energy of scattered particle	E	$= 2\varepsilon$	
Deflection angle	Θ	$= \frac{1}{2}\vartheta$	
Azimuth	Φ	$= \varphi$	
Element of solid angle	$4 \cos \Theta d\Omega_{\text{lab}}$	$= d\Omega_{\text{bar}}$	
If recoil proton is observed	Deflection angle of recoil particle of scatterer	$\Theta' = \frac{\pi}{2} - \frac{\vartheta}{2}$	(20)
	Element of solid angle for recoil particle	$4 \cos \Theta' d\Omega'_{\text{lab}} = d\Omega_{\text{bar}}$	

The relation between the elements of solid angle expresses the fact that an angular distribution with spherical symmetry in the barycentric system corresponds to a "cos Θ -law" in the laboratory system.

In the case of very slow neutrons, however, the binding of the protons and their thermal motion becomes of considerable influence on the scattering process. This effect was first pointed out by FERMI [36]** and further discussed by BETHE [37] (§ 59C, p. 122—127), ARLEY [38], and SACHS

* It has been assumed that particle (1) is the neutron, particle (2) the proton.

** A somewhat more accurate method than FERMI's has been developed by BREIT [47a] and BREIT and ZILSEL [47b].

and TELLER [41] *. We cannot enter into this question here, we shall only indicate, in the next sub-section, the principle of the method by which it can be treated. Empirical results obtained with very slow neutrons should thus always be corrected for the binding effect ** before being compared with the free proton formulae ***.

6.213. Scattering of slow neutrons by bound protons. In order to account for the binding of the proton, one has to add to the Hamiltonian of the proton-neutron system a term of potential energy depending only on the proton coordinates. At first sight, this would seem greatly to complicate the solution of the corresponding wave-equation. But an ingenious argument, due to FERMI [36], leads to a very simple treatment of the scattering problem in this case also. This argument is based on the fact that, on the one hand, the neutron wave lengths involved and, on the other hand, the distances over which the binding potential changes appreciably are both much larger than the range of the nuclear potential.

The asymptotic behaviour of the wave-function $\dagger \Psi(\vec{x}^{(n)}, \vec{x}^{(p)})$, responsible for the scattering, will thus be the same as that of a function $\Phi(\vec{x}^{(n)}, \vec{x}^{(p)})$, smoothed out by taking the mean value of Ψ with respect to the neutron coordinates over a sphere of radius intermediate between neutron wave-length and range of nuclear force. A wave-equation can then be set up for Φ , in which the influence of the nuclear potential appears only as a small perturbation.

In order to avoid unessential complications, we shall first carry out this procedure under the simplifying assumption that the nuclear potential is spin independent; we have then to do with only one scattering radius a for neutrons of zero velocity (6.21), the same in triplet and singlet configurations. Let us then consider some length r_0 much smaller than the neutron wave-length, but at the same time large compared both with the range κ^{-1} of the nuclear force and with the scattering radius $|a|$. Let Σ be the volume of a sphere of radius r_0 , and

$$\Phi(\vec{x}^{(n)}, \vec{x}^{(p)}) = \frac{1}{\Sigma} \int_{\Sigma} \Psi(\vec{x}^{(n)} + \vec{x}', \vec{x}^{(p)}) dv'. \quad (21)$$

Since Φ differs appreciably from Ψ only for such values of $\vec{x}^{(n)}, \vec{x}^{(p)}$ that the distance $r \equiv |\vec{x}^{(n)} - \vec{x}^{(p)}|$ is smaller than r_0 , we may put

$$\Psi = \Phi \chi(r), \quad (22)$$

* The main formulae are not written down correctly in this last paper; their correct expression is given by GIBERT and ROSSEL [46].

** ARLEY's paper [38] contains numerical evaluations useful for the reduction of experimental data.

*** Conversely, an interesting attempt has recently been made by ROSSEL [47] to develop the study of the scattering of slow neutrons by chemical compounds in definite states of aggregation into a method of analysis of molecular and intermolecular forces.

† For convenience, we here treat proton and neutron as different kinds of particles.

the function $\chi(r)$, which expresses so to speak the distortion of the wave-function due to the nuclear potential, tending asymptotically to unity; we have further neglected the angular dependence of χ , since we know that for slow neutrons we have mainly to do with S -wave scattering. Calling $\mathcal{H}'$ the total Hamiltonian without the nuclear potential $\mathcal{V}$, we may write the wave-equation for Φ as

$$\mathcal{H}' \Phi + \frac{1}{\Sigma} \int_{\Sigma} [\mathcal{V} \Psi]_{\vec{x}^{(n)} + \vec{x}'} d\nu' = E \Phi,$$

or approximately, by (22),

$$\mathcal{H}' \Phi + \mathcal{U} \Phi \approx E \Phi, \quad (23)$$

with

$$\mathcal{U} = \frac{1}{\Sigma} \int_{\Sigma} [\mathcal{V} \chi]_{\vec{x}^{(n)} + \vec{x}'} d\nu'. \quad (24)$$

Combining (23) with the equation $\mathcal{H}' \Psi + \mathcal{V} \Psi = E \Psi$ for Ψ and with (22), we get for χ , to the same approximation,

$$\left[\frac{1}{M} p^2 + \mathcal{V} - \mathcal{U} \right] \chi = 0,$$

p being the relative momentum operator. Now, we may solve this equation by disregarding the last term, since, with the solution so obtained, this term turns out to be negligible in comparison with the first. We thus have

$$\chi = \frac{R}{r}, \quad R'' - \frac{M}{\hbar^2} \mathcal{V} R = 0, \quad (25)$$

i.e. just the radial equation for an S -wave of zero energy. Therefore, remembering the geometrical meaning of the quantity a introduced in (5.32–21) and normalizing so that $\chi(\infty) = 1$,

$$R(0) = 0, \quad R(r) \simeq -a + r. \quad (26)$$

We may now evaluate $\mathcal{U}$ in the following way; for $r > r_0$, $\mathcal{U} \approx 0$; for $r < r_0$, we approximate it by putting $r = 0$, so that

$$[\text{by (25)}] \quad \mathcal{U} \approx \frac{4\pi}{\Sigma} \int_0^{r_0} \mathcal{V}(r') \chi(r') r'^2 dr' = \frac{4\pi}{\Sigma} \cdot \frac{\hbar^2}{M} \int_0^{r_0} R' r' dr$$

$$[\text{by (26)}] \quad = \frac{4\pi}{\Sigma} \cdot \frac{\hbar^2}{M} [R' r - R]_0^{r_0} = \frac{4\pi \hbar^2 a}{M} \cdot \frac{1}{\Sigma}.$$

Denoting by $\delta_{r_0}(r)$ the function

$$\begin{aligned}\delta_{r_0}(r) &= \frac{1}{\Sigma} \text{ for } r < r_0 \\ &= 0 \text{ for } r > r_0.\end{aligned}\quad (27)$$

so that $\int_{-\infty}^{\infty} \delta_{r_0}(r) dv = 1$, we may thus write

$$\mathcal{U} \approx \frac{4\pi\hbar^2 a}{M} \delta_{r_0}(r); \quad (28)$$

this is the potential which in the equation (23) for Φ represents the influence of the nuclear forces. It is easily seen that in comparison with the other terms of the equation, it can be regarded as a small perturbation throughout. We may therefore apply to the scattering problem the Born method of approximation. Consider the process by which a neutron of wave-vector $\vec{k}_0$ ($\equiv \vec{p}_0/\hbar$) is scattered with wave-vector $\vec{k}_1$, the proton going over from the state 0 of binding energy $-E_0$, represented by the eigenfunction $\varphi_0(\vec{x}^{(p)})$, to the state 1 of energy $-E_1$ and eigenfunction $\varphi_1(\vec{x}^{(p)})$; the partial cross-section for this process is given by

$$\begin{aligned}dS(\vec{k}_0; 0 \rightarrow 1) &= \frac{k_1}{k_0} |f(\vec{k}_0; 0 \rightarrow 1)|^2 d\Omega^* \\ f(\vec{k}_0; 0 \rightarrow 1) &= \frac{M^*}{2\pi\hbar^2} \int e^{i(\vec{k}_0 - \vec{k}_1)\vec{x}^{(n)}} \varphi_1^*(\vec{x}^{(p)}) \mathcal{U}(|\vec{x}^{(n)} - \vec{x}^{(p)}|) \varphi_0(\vec{x}^{(p)}) dv^{(n)} dv^{(p)} \quad (29) \\ \frac{\hbar^2}{2M^*} (k_0^2 - k_1^2) &= E_0 - E_1.\end{aligned}$$

In this formula, $d\Omega^*$ refers to the system of reference in which the centre of gravity of the total system (neutron + proton + system to which proton is bound) is at rest, and M^* denotes the reduced mass of the neutron in this system of reference, which may be taken as "laboratory system". Inserting the value (28) for $\mathcal{U}$ and replacing $\delta_{r_0}(r)$ by the δ -function (which corresponds to $r_0 \rightarrow 0$), we get

$$f(\vec{k}_0; 0 \rightarrow 1) = 2a \frac{M^*}{M} \int e^{i(\vec{k}_0 - \vec{k}_1)\vec{x}} \varphi_1^*(\vec{x}) \varphi_0(\vec{x}) dv. \quad (30)$$

This theory is easily generalized to the case of the scattering of a slow neutron by a bound nucleus of mass A : we have then simply to replace, in formulae (28) and (30), the reduced mass $\frac{1}{2}M$ by $\frac{A}{A+1}M$; the general meaning of the scattering radius a is that the expression $S_{\text{free}} = 4\pi a^2$ represents the total cross-section for scattering of slow neutrons by free nuclei of mass A . In particular, in the limiting case of infinitely tight binding

of the scattering nuclei, we have only elastic scattering, and $M^* = M$; the differential cross-section (29), (30) reduces to the form

$$dS_{\infty} = \left(\frac{A+1}{A} \right)^2 a^2 d\Omega^* \quad (29a)$$

the scattering is spherically symmetrical in the *laboratory* system and the total scattering cross-section becomes

$$S_{\infty} = \left(\frac{A+1}{A} \right)^2 S_{\text{free}}. \quad (29b)$$

For tightly bound protons, for instance, it is 4 times that for free protons; for tightly bound deuterons, the corresponding factor is 2.25.

We have still to indicate how the influence of spin dependent nuclear forces is taken into account. In the case of scattering by protons, this means that the scattering radius has different values ${}^{\sigma}a$ for the singlet and triplet configurations. For a given spin transition, we have then to replace a in (30) by the corresponding matrix-element of the operator

$$a = {}^3a \cdot \frac{1}{2} (1 + P_{\sigma}) + {}^1a \cdot \frac{1}{2} (1 - P_{\sigma}),$$

P_{σ} being the exchange operator for the spin variables; by (4.13-26), this may be written

$$a = \frac{1}{4} (3 \cdot {}^3a + {}^1a) + \frac{1}{4} ({}^3a - {}^1a) \frac{\vec{\sigma}^{(p)} \cdot \vec{\sigma}^{(n)}}{\sigma^{(p)} \sigma^{(n)}}. \quad (31)$$

This formula is readily extended to the case that one of the interacting particles has an intrinsic angular momentum $\vec{S}$, with quantum number S different from $\frac{1}{2}$; one finds

$$a = \frac{S+1}{2S+1} ({}^{2S+2}a) + \frac{S}{2S+1} ({}^{2S}a) + \frac{({}^{2S+2}a - {}^{2S}a)}{2S+1} \frac{\vec{\sigma} \cdot \vec{S}}{\sigma S}. \quad (31a)$$

The mean value of a^2 for all possible spin states of the colliding particles is

$$\overline{a^2} = \frac{S+1}{2S+1} \cdot ({}^{2S+2}a)^2 + \frac{S}{2S+1} \cdot ({}^{2S}a)^2. \quad (31b)$$

For scattering of slow neutrons by bound deuterons, e.g., we have to take

$$a = \frac{1}{3} (2 \cdot {}^4a + {}^2a) + \frac{1}{3} ({}^4a - {}^2a) \frac{\vec{\sigma}^{(n)} \cdot \vec{S}}{\sigma^{(n)} S} \quad (31c)$$

and, instead of (28),

$$U \approx \frac{2}{3} \cdot \frac{2\pi \hbar^2 a}{M} \delta(\vec{x}^{(n)} - \vec{x}^{(d)}). \quad (31d)$$

6.22. Experiments on the scattering of slow neutrons by protons. Numerous determinations of the scattering cross-section of protons for neutrons of various energies have been carried out. The arrangement of

all such experiments is quite straightforward: between the neutron source and the detector, a sample of the material to be tested is introduced and the resulting decrease of the neutron intensity is measured; from this, the scattering cross-section is easily deduced. In our case, the scatterer will consist of some hydrogenous substance, such as paraffin or water and the

6.22-1. Scattering of thermal neutrons by protons			
Reference	Scatterer	S' 10^{-24} cm^2	S 10^{-24} cm^2
GOLDHABER and BRIGGS [37]	paraffin	$47,5 \pm 5$	$17,7 \pm 2$
FRISCH <i>et al.</i> [37]	paraffin	48 ± 3	$18 \pm 1,5$
POWERS <i>et al.</i> [38]	$\approx \text{C}_{16}\text{H}_{34}$	$42,5 \pm 0,9$	$15,8 \pm 0,4$
CAROLL and DUNNING [38]	C_4H_{10}	50,0	18,6
HANSTEIN [40b]	$\approx \text{C}_{22}\text{H}_{46}$	50,7	18,8
	$\text{C}_{16}\text{H}_{34}^*$	49,0	18,2
BRICKWEDDE <i>et al.</i> [38]	H_2O	$42,2 \pm 1,8$	—
LIBBY and LONG [39]	CH_4	36 ± 4	—
	C_2H_4	46 ± 3	$17,1 \pm 1,1$
	C_2H_6	46 ± 3	$17,1 \pm 1,1$
GIBERT and ROSSEL [46]	H_2	$31,6 \pm 0,6$	—
ROSSEL [47]	H_2O	$40,7 \pm 1,5$	—
CARROLL [41]	H_2	$31,8 \pm 0,5$	—
	H_2O	$44,6 \pm 0,5$	—
	CH_4	$45,4 \pm 0,3$	—
	C_2H_6	$46,4 \pm 0,5$	$17,4 \pm 0,2$
	C_3H_8	$46,9 \pm 0,6$	$17,6 \pm 0,2$
	C_4H_{10}	$48,7 \pm 0,6$	$18,3 \pm 0,2$
	$n \text{ C}_{16}\text{H}_{34} \left\{ \begin{array}{l} \text{solid} \\ \text{gas} \end{array} \right.$	$44,9 \pm 0,4$	$16,8 \pm 0,15$
		$50,1 \pm 0,4$	$18,8 \pm 0,15$
	$\approx \text{C}_{22}\text{H}_{46}$	$49,8 \pm 0,2$	$18,7 \pm 0,1$
	$n \text{ C}_{32}\text{H}_{66}$	$50,2 \pm 0,2$	$18,8 \pm 0,1$
MARSHALL [46] **	H_2O	44,4	—
* Thick In detectors covered with In foils to minimize resonance activation were used.			
** Neutrons from uranium-graphite pile.			

cross-section for H is calculated from the molecular cross-section by assuming additivity of the atomic cross-sections (incoherent scattering) *.

* This additivity may break down for thermal neutrons, owing to interference effects, especially if the scatterer is in a crystalline or polycrystalline state. See on this point H. BEYER and M. WHITAKER, *P.R.* 57, 976, 1940; F. RASETTI, *P.R.* 58, 321, 1940; O. HALPERN, M. HAMERMESH and H. JOHNSON, *P.R.* 59, 981, 1941; R. SEEGER and E. TELLER, *P.R.* 62, 37, 1942; R. WEINSTOCK, *P.R.* 65, 1, 1944; E. FERMI, W. STURM and R. SACHS, *P.R.* 71, 589, 1947.

It is therefore advisable to measure also the cross-section of C or O, as the case may be, under the same conditions. (In the experiments reported below, this has always been done, unless otherwise indicated.) The experiments cover an extended range of energies of the impinging neutrons; the energy regions explored are actually determined by the available neutron sources. At the moment, we shall be concerned with the region of *slow neutrons*, up to about 50 eV, within which the scattering cross-section will not be expected to vary appreciably. The value obtained can therefore be regarded as the limiting cross-section for zero energy.

Numerous measurements have been carried out with "*thermal*" neutrons, i.e. neutrons in thermal equilibrium with the material from which they emerge; if T denotes the absolute temperature of this material, their mean kinetic energy is of the order of kT , i.e. $\approx 0,026$ eV at room temperature. Until recently, the usual source of thermal neutrons, or neutron "howitzer", has been a Rn-Be tube surrounded with paraffin. But some experiments (MARSHALL [46]) have already been performed with the thermal neutrons issuing from a uranium-graphite pile. The thermal neutrons are selected by comparing the measurements with and without interposition of a Cd filter. The neutron beam is collimated by suitable walls and diaphragms lined with B and Cd, and detected by means of a BF_3 ionization chamber connected with a linear amplifier. The cross-sections obtained must be corrected for the effect of proton binding (6.212, 6.213). We shall first give results of measurements in which the paraffin of the source as well as the scatterer were at room temperature. In the table 6.22-1, we have collected — together with some cross-section values obtained with lighter molecules — chiefly those pertaining to hydrocarbons of high molecular weight; only in the latter case has the reduction factor from the cross-section S' per bound proton to the free proton cross-section S been worked out with any accuracy (ARLEY [38], p. 31, 32).

If the source is cooled (by liquid air, say) a marked increase of the cross-section S is observed, in agreement with the theoretical expectation. For the ratio of the new cross-section to that at room temperature, it is found:

6.22-2. Scattering of cooled neutrons by paraffin			
Reference	Neutron temperature	Ratio observed	Ratio theoretical
FRISCH <i>et al.</i> [37]	$\approx 88^\circ\text{K}$	$1,09 \pm 0,05$	1,34
POWERS <i>et al.</i> [38]	$\approx 120^\circ\text{K}$	$1,3 \pm 0,06$	1,25

No better agreement can be expected in view of the far-reaching schematization of Arley's theory.

More satisfactory results, however, have been obtained by investigating the scattering of neutrons of ordinary thermal velocity by hydrogen gas cooled to very low temperatures; for the reduction formula of SACHS and

TELLER [41] may then be applied to derive from the observed cross-section S' the free proton value S . The following table summarizes the relevant data:

6.22-2a. Scattering of thermal neutrons by cooled hydrogen gas			
Reference	Gas temperature	S' 10^{-24} cm^2	S 10^{-24} cm^2
LIBBY and LONG [39]	90 °K	22 ± 4	$14,7 \pm 2,7$
GIBERT and ROSSEL [46] }	77 °K	$27,2 \pm 1,6$	$19,2 \pm 1,1$
	20 °K	$26,2 \pm 2,9$	$20,3 \pm 2,3$

The large and rather uncertain correction necessitated by the binding effect makes it desirable to carry out measurements with neutrons of energy $> \approx 1 \text{ eV}$, when this effect becomes negligible. As already stated, the cross-section for neutrons of a few volts may, of course, equally well as that for thermal neutrons, be identified with its limiting value for zero energy: for the other energy quantities which enter into the description of the scattering process are of "nuclear" order of magnitude (0,1 ... 1 MeV). Now, the scattering of *resonance neutrons*, the energies of which are just in the region from 1 to about perhaps 50 eV, may be

6.22-3. Scattering of resonance neutrons by protons			
Reference	Source	Filter	S 10^{-24} cm^2
SIMONS [40]	Be (γ_{Ra}, n) in paraffin	Ag, I	$14,4 \pm 0,8^*$
COHEN <i>et al.</i> [39]	id.	Rh	} $20 \pm 2^{**}$
COHEN <i>et al.</i> [40]	id.	Ag, In	
HANSTEIN [40b]	Be (p, n), beam collimated	In	} 21 ± 1
HANSTEIN [41]	id.	In, I	

* In deriving this value, large corrections had to be applied owing to the unfavourable "geometry" of the arrangement. Moreover, the value here given differs from that published by Simons, because a better value of the oxygen cross-section has been used in reducing the experimental results. According to CARROLL [41], the slow neutron cross-section of oxygen is

$$S_{\text{O}} = (4,12 \pm 0,10) \cdot 10^{-24} \text{ cm}^2;$$

BRICKWEDDE *et al.* [38] give essentially the same value.

** In the reduction of the data, the slow neutron scattering cross-section of carbon

$$S_{\text{C}} = 4,83 \cdot 10^{-24} \text{ cm}^2.$$

as given by GOLDBABER and BRIGGS [37], has been adopted; it agrees with the more recent measurements of HANSTEIN and DUNNING [40a] and MARSHALL [46].

studied thanks to a "resonance filter" method, devised independently by COHEN, GOLDSMITH and SCHWINGER [39] and by L. SIMONS [40]. A filter of a material possessing a resonance level for slow neutrons is placed before the scatterer, and the detector is made of the same material (shielded by Cd to eliminate thermal neutrons); measurements are carried out with and without the filter. The same source as for thermal neutrons can be used; but larger intensities have been obtained by means of the reaction Be (p, n), with a cyclotron as the source of the necessary protons of high energy (threshold of the reaction at 2.01 MeV) (HANSTEIN and DUNNING [40a]).

From table 6.22-3 it may safely be concluded that SIMONS' [40] value is definitely too low; in particular, it is not (as was for some time believed owing to an overestimation of the proton binding effect) in agreement with the thermal neutron measurements. It must also be observed, in this connexion, that systematic errors in such experiments will tend to lower the observed value of the scattering cross-section. There is no doubt that the most reliable figure is that given by HANSTEIN [40b, 41]; we shall therefore adopt, for the scattering cross-section for neutrons of zero energy, the value

$$S = (21 \pm 1) \cdot 10^{-24} \text{ cm}^2. \quad (32)$$

6.23. The ^1S level of small energy of the deuteron. Let us now compare the value (32) of the scattering cross-section for zero energy neutrons with the theoretical result. If there were no spin-dependent term of nuclear interaction, the singlet and triplet S levels would practically coincide and the quantity in question would be just four times the limiting value $S^{(0)}$ given by (13); i.e., disregarding the finite range correction (which does not alter the order of magnitude) and taking account of (14), $4\pi ({}^3a_0)^2 = \frac{4}{3} 2.39 \cdot 10^{-24} \text{ cm}^2$. This figure is in complete disagreement with the experimental result (32), which is about 9 times larger. For larger neutron energies, on the other hand, the discrepancy would soon become very much smaller: according to (12) and (20), one would have (again neglecting the finite range correction)

$$S_{\text{no spin force}} = \frac{4\pi\hbar^2}{M} \cdot \frac{1}{\frac{1}{2}E + |\epsilon_0|};$$

for $E = 4.1 \text{ MeV}$, say, this gives $1.24 \cdot 10^{-24} \text{ cm}^2$, indeed of the same order of magnitude as the measured value $1.73 \cdot 10^{-24} \text{ cm}^2$ (table 6.413).

The way out of this difficulty, as first suggested by WIGNER (quoted by B & B, p. 117), is just to assume that owing to a *strong spin dependence of the nuclear potential*, there is some (actual or virtual) ^1S level of much smaller energy (in absolute value) than the ground state. In fact, the cross-section (for zero range) then becomes (writing, for symmetry, $|{}^3\epsilon^{(0)}|$

instead of $|\varepsilon_0|$ for the energy of the ground state)

$$S(E) = \frac{\pi \hbar^2}{M} \left[\frac{3}{\frac{1}{2}E + |{}^3\varepsilon^{(0)}|} + \frac{1}{\frac{1}{2}E + |{}^1\varepsilon^{(0)}|} \right] \quad (\text{zero range}), \quad (33)$$

and for $E = 0$,

$$S(0) = 3\pi ({}^3a_0)^2 + \pi ({}^1a_0)^2 \quad (\text{zero range})$$

$${}^3a_0 = \frac{\hbar}{\sqrt{M|{}^3\varepsilon^{(0)}|}} = 0.44 \cdot 10^{-12} \text{ cm}, \quad {}^1a_0 = \pm \frac{\hbar}{\sqrt{M|{}^1\varepsilon^{(0)}|}}. \quad (34)$$

While this modification has but small effect for relatively large values of E , it gives much larger results for very small energies if $|{}^1\varepsilon^{(0)}| \ll |{}^3\varepsilon^{(0)}|$, the contribution to the scattering from the singlet term then becoming preponderant. Adjusting the parameter $|{}^1\varepsilon^{(0)}|$ so as to get for $S(0)$ Hanstein's value (32), we get from (34)

$${}^1a_0 = (2.47 \pm 0.07) \cdot 10^{-12} \text{ cm}, \quad |{}^1\varepsilon_0| = 0.067 \pm 0.004 \text{ MeV (zero range); } (35)$$

we shall see presently that the correction for finite range does not appreciably modify these values. The analysis of the scattering data yields only the absolute value of the quantities ${}^1a_0$ and ${}^1\varepsilon_0$, so that the question whether the ${}^1S$ level is actual or virtual cannot be decided on this evidence.

6.231. Radiative capture of slow neutrons. The large spin dependence of the proton-neutron force revealed by the strong scattering of slow neutrons by protons is further confirmed by the discussion of the radiative capture of such neutrons by protons, leading to the formation of deuterons with emission of γ -rays. This point will be treated together with the other radiative processes in a later section (6.52). It must be pointed out that in contrast to the elastic scattering, the radiative capture effect is sensitive to the actual or virtual character of the ${}^1S$ level; however, as we shall see, the evidence in this respect is not entirely conclusive. A definitive settlement of the question is provided by experiments on neutron scattering by para- and ortho-hydrogen, to the study of which we shall now proceed.

6.3. Slow neutron scattering by hydrogen molecules

6.30. The scattering of very slow neutrons by molecular hydrogen requires special consideration when the energy of the neutrons becomes of the order of magnitude of, or lower than, the excitation energy of the rotation levels of the molecule; such a kinetic energy ($\approx 0.01 \text{ eV}$) corresponds, of course, to a de Broglie wave-length of the order of magnitude of the mean distance between the two H atoms. We must then take account of two peculiarities, which, as emphasized by TELLER [36], have a considerable bearing on the problem of the spin dependence of the proton-neutron interaction and in particular on the question whether the ${}^1S$ level of low energy of the deuteron is an actual or virtual one. In the first place, hydrogen molecules provide us with two different arrangements

of the proton spins, viz. the para-configuration (antiparallel spins) and the ortho-configuration (parallel spins). If the nuclear forces did not depend on spin, this would of course make no difference to the scattering of neutrons*; but if they are spin-dependent, the two configurations will react differently. Assume, in fact, (to consider a simple case) that the temperature of the hydrogen is so low that practically all molecules are in the lowest para- or ortho-configuration (in some given proportion). If the neutron energy is too small to excite a para-molecule into the ortho-state, the scattering by para-hydrogen will only involve para-para transitions in the molecule, while the scattering by ortho-hydrogen will comprise both ortho-ortho and ortho-para transitions, the latter being entirely brought about by the spin dependent interaction between neutron and proton. Further, since the wave-length of the neutrons is of the same order of magnitude as the distance between the protons, we shall expect strong interference of the neutron waves scattered from the two protons. The effect of such an interference will depend very sensitively on the relative sign of the triplet and singlet phase constants, i.e. of the constants ^3a and ^1a ; we thus see that, while the scattering by protons only yields information on the absolute values of these constants, the phenomenon under discussion may reveal whether the ^1S -level connected with ^1a is an actual or a virtual one.

6.31. Experiments. The effects just described have actually been found when looked for. The chief difficulty in these experiments is to get rid of any admixture of neutrons of higher velocity which tend to blur the characteristic difference sought. If one uses as a neutron source the usual "howitzer" (6.22), the paraffin being cooled, say, to liquid air temperature, the Cd filter procedure becomes insufficient, since it only suppresses neutrons of energy above 0.4 eV. Still, the first experiments, carried out with this source (HALPERN *et al.* [37], BRICKWEDDE *et al.* [38], LIBBY and LONG [39]), were already able to establish the essential qualitative conclusions that

- (1) $o\text{ H}_2$ scatters neutrons of ≈ 0.01 eV mean energy (≈ 120 °K temperature) much more than $p\text{ H}_2$;
- (2) the cross-section of $p\text{ H}_2$ is markedly smaller for neutrons of ≈ 0.01 eV mean energy than for ordinary (≈ 300 °K thermal neutrons).

The quantitative results for liquid hydrogen (BRICKWEDDE *et al.* [38]) and for hydrogen gas cooled down to ≈ 90 °K (LIBBY and LONG [39]) were as follows:

* Except in so far as the spacing of the rotational levels is different in ortho- and para-hydrogen. This effect would only be noticeable for a critical energy of the impinging neutrons. Cf. SCHWINGER and TELLER [37a].

6.31. Scattering of neutrons by molecular hydrogen				
Neutron temperature		$\approx 300^\circ\text{K}$	$\approx 120^\circ\text{K}$	"p-filtered" 120°K
Molecular cross-section * (10^{-24} cm^2)	$\circ \text{H}_2$	55,8 (44 ± 8)	78,7 (78 ± 10)	100,1
	$p \text{H}_2$	29,0 (44 ± 8)	17,6 (38 ± 8)	12,6
* The figures between brackets refer to hydrogen gas at $\approx 90^\circ\text{K}$.				

The figures given in this table for liquid hydrogen correspond to a molecular cross-section of $48,7 \cdot 10^{-24} \text{ cm}^2$ for scattering of ordinary thermal neutrons by liquid hydrogen of normal composition (75 % $\circ \text{H}_2$); the comparison of this value with that for gaseous hydrogen at the same temperature of 20°K , viz. $(52,4 \pm 5,8) \cdot 10^{-24} \text{ cm}^2$ (table 6.22-2a), shows that there is no appreciable effect of intermolecular forces. Libby and Long's values for hydrogen gas at 90°K , quoted in the above table, point, though less convincingly, to the same conclusion, when allowance is made for the thermal motion of the gas molecules. The last column in table 6.31 concerns 120°K neutrons which have been filtered by a layer of liquid parahydrogen; owing to the relative transparency of $p\text{H}_2$ to "cold" neutrons, the effect of this filtering is to attenuate the high energy "tail" of the neutron distribution. The large differences caused by such filtering strikingly illustrate the disturbing influence of the inhomogeneity of the neutrons.

A considerable improvement in this respect could be obtained with the help of the neutron monochromator devised by ALVAREZ [38]. The principle of this apparatus is very simple: it consists in modulating the ion beam which produces neutrons by some suitable nuclear reaction, and arranging the linear amplifier of the detecting BF_3 ionization chamber to be sensitive only when the ion source is off. This arrangement will only record neutrons of velocity d/t , d being the distance of the chamber from the source and t the phase difference in time between the modulations of the beam and the amplifier. In the actual apparatus (ALVAREZ and PITZER [40b]), the deuteron beam from a cyclotron activated a Be target and the $\text{Be}(d, n)$ neutrons were slowed down by a paraffin block placed near the target*. The distance d was about 7 m; the modulation frequency being 60 sec^{-1} , the time t could be chosen so as to select neutrons of $\approx 20^\circ\text{K}$ (the corresponding effective velocity of these neutrons** being $\sqrt{\pi kT/2M} \approx 509 \text{ m/sec}$, this gives $t \approx 1/73 \text{ sec}$). The scatterer was hydrogen gas cooled by boiling liquid hydrogen at $20,4^\circ\text{K}$. After correction for capture of the neutrons by the protons, the scattering cross-sections obtained were

$$\text{for } \left\{ \begin{array}{l} \circ \text{H}_2 : 100 \pm 3 \\ p \text{H}_2 : 5,2 \pm 0,6 \end{array} \right\} \cdot 10^{-24} \text{ cm}^2, \quad (1)$$

* The time t must then be sufficiently long compared with the life-time of the neutrons in paraffin, to avoid any disturbance from the interposition of the paraffin block.

** This is the effective velocity for B absorption, cf. BETHE [37], § 60A.

thus qualitatively agreeing with the former measurements, but yielding a much larger value of the ratio ortho/para.

6.32. Theory. The quantitative treatment of the effect under discussion (SCHWINGER and TELLER [37a]) requires only a slight generalization of the procedure sketched in 6.213. We have just to replace in (6.213–29) the functions φ_0, φ_1 by the required eigenfunctions of the hydrogen molecule (including the motion of its centre of gravity) and the operator $\bar{U}$ by

$$\bar{U} = U(|\vec{x}^{(n)} - \vec{x}^{(p_1)}|) + U(|\vec{x}^{(n)} - \vec{x}^{(p_2)}|). \quad (2)$$

Separating $\bar{U}$ into two parts, symmetrical and antisymmetrical, respectively, in the proton spins, we get, by (6.213–28) [with $\delta_{r_0}(r) \rightarrow \delta(\vec{x})$] and (6.213–31),

$$\begin{aligned} \bar{U} = \frac{\pi \hbar^2}{M} \left\{ (3 \cdot {}^3a + {}^1a) + ({}^3a - {}^1a) \vec{\sigma}^{(n)} \cdot \frac{\vec{\sigma}^{(p_1)} + \vec{\sigma}^{(p_2)}}{2} \right\} \cdot [\delta(\vec{x}^{(n p_1)}) + \delta(\vec{x}^{(n p_2)})] \\ + \frac{\pi \hbar^2}{M} ({}^3a - {}^1a) \vec{\sigma}^{(n)} \cdot \frac{\vec{\sigma}^{(p_1)} - \vec{\sigma}^{(p_2)}}{2} [\delta(\vec{x}^{(n p_1)}) - \delta(\vec{x}^{(n p_2)})], \end{aligned} \quad (3)$$

with $\vec{x}^{(n p_i)} \equiv \vec{x}^{(n)} - \vec{x}^{(p_i)}$. From this expression, it is easily deduced that the scattering cross-section for transitions involving no change of the total spin $\frac{1}{2} [\vec{\sigma}^{(p_1)} + \vec{\sigma}^{(p_2)}]$ of the molecule (i.e. para-para and ortho-ortho transitions) contains the factor

$$(3 \cdot {}^3a + {}^1a)^2 + S(S+1)({}^3a - {}^1a)^2, \quad (4)$$

S being the spin quantum-number of the molecular state ($S = 0$ for para-states, $S = 1$ for ortho-states). For transitions involving a change of spin (ortho-para and para-ortho transitions), the cross-section is proportional to

$$({}^3a - {}^1a)^2 \frac{3}{2S+1}, \quad (5)$$

S denoting the spin quantum-number of the initial state (that of the final state being $1-S$). These expressions exhibit very clearly the interference features described above (6.30).

For the cross-sections of para- and ortho-hydrogen in their lowest states, we can write accordingly

$$\begin{aligned} \mathcal{S}_{\text{para}} &= \mathcal{S}_{p \rightarrow p} (3 \cdot {}^3a + {}^1a)^2 + \mathcal{S}_{p \rightarrow o} ({}^3a - {}^1a)^2 \\ \mathcal{S}_{\text{ortho}} &= \mathcal{S}_{o \rightarrow o} (3 \cdot {}^3a + {}^1a)^2 + (2\mathcal{S}_{o \rightarrow o} + \mathcal{S}_{o \rightarrow p}) ({}^3a - {}^1a)^2, \end{aligned} \quad (6)$$

the coefficients $\mathcal{S}$ being certain functions of the neutron energy, represented by * fig. 6.32; the indices correspond to the transition involved: $\mathcal{S}_{p \rightarrow p}$ and

* Analytical expressions for the $\mathcal{S}$'s, which, besides the neutron energy, involve only parameters of the hydrogen molecule, are given by SCHWINGER and TELLER [37a]. A

$S_{o \rightarrow o}$ pertain to elastic scattering from the lowest para- or ortho-state, respectively, while $S_{o \rightarrow p}$ and $S_{p \rightarrow o}$ refer to inelastic scattering processes. The latter ($S_{p \rightarrow o}$) is, of course, only different from zero for neutron energies larger than the excitation energy of the lowest ortho-level, viz. 0.023 eV. As to $S_{o \rightarrow p}$, it varies as $1/v$ at low velocities and remains practically constant for energies higher than about 0.01 eV.

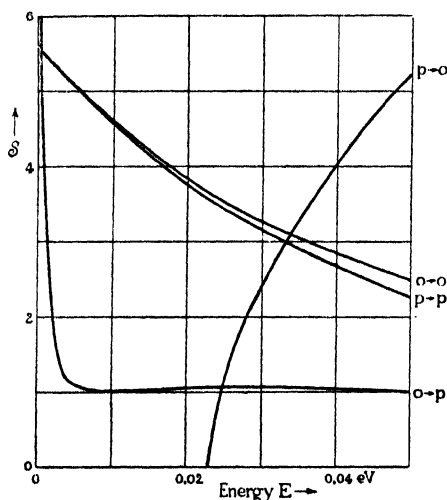


Fig. 6.32. Coefficients of the neutron scattering cross-sections of para- and ortho-hydrogen, in terms of the neutron energy E .

The states of the deuterium molecule D_2 can also be divided into para- and ortho-configurations according to their parity; in this case, the para-states correspond to a molecular spin $S = 1$ and odd values of the total angular momentum J , the ortho-states to $S = 0$ or 2 and even values of J . In view of possible experiments with deuterium, the theory of slow neutron scattering has been worked out for this case by HAMERMESH and SCHWINGER [46]. It proceeds on exactly the same lines as above, starting from the expression (6.213–31c, 31d) for the equivalent scattering potential of the deuteron.

6.33. Analysis of experiments. Let us now, by way of illustration, take $^1a/^3a = \pm 4$, — an order of magnitude suggested by (6.23–34,35). From formula (6), together with figure 6.32, we see that, according as the ^1S level of low energy of the deuteron is actual or virtual, the ratio

$$(3 \cdot ^3a + ^1a)^2 / (^3a - ^1a)^2$$

will change from 5.44 to 0.11. In the first case, the elastic scattering would predominate, so that (considering the properties of the coefficients S) the ortho and para cross-sections would always be of the same order of

more precise calculation of effective cross-sections, taking account of the Maxwellian contribution of the velocities of the molecules, is carried out by SCHWINGER [40]; in this last paper, the expression for $S_{o \rightarrow p}$ actually utilized differs from that given by SCHWINGER and TELLER by a factor $\frac{1}{2}$, which had been omitted by these authors.

magnitude, and the values for cold neutrons would be only moderately larger than those for ordinary thermal neutrons. In the case of a virtual ^1S level, however, in which the inelastic processes are enhanced, the para cross-section will be very much smaller for cold than for thermal neutrons, owing to the vanishing in the former energy range of the coefficient $\mathcal{S}_{p \rightarrow o}$; again, for cold neutrons, the para cross-section will for the same reason be very much smaller than the ortho cross-section. It has been stressed above (6.31) that both these predictions are fully confirmed by all experiments, which conclusively establishes that the ^1S level is a virtual one.

The measurements by Alvarez, given by (1), allow of a more quantitative analysis (SCHWINGER [40]). For the neutron energy $E \approx kT \approx 0.00173$ eV, one gets after correcting for the thermal motion of the molecules

$$\mathcal{S}_{p \rightarrow p} = 6,473, \quad \mathcal{S}_{o \rightarrow o} = 6,291, \quad \mathcal{S}_{o \rightarrow p} = 1,447,$$

while $\mathcal{S}_{p \rightarrow o} = 0$. According to (6), the para cross-section gives directly

$$(3 \cdot ^3a + ^1a)^2 = (0,803 \pm 0,092) \cdot 10^{-24} \text{ cm}^2; \quad (7)$$

the ortho cross-section then yields

$$(^3a - ^1a)^2 = (6,77 \pm 0,26) \cdot 10^{-24} \text{ cm}^2. \quad (8)$$

These numbers are not in very good agreement with the proton scattering cross-section for neutrons of zero energy, given by (6.22–32); in fact, the latter quantity may be written

$$\mathcal{S}(0) = 3\pi(^3a)^2 + \pi(^1a)^2 = \frac{\pi}{4} [(3 \cdot ^3a + ^1a)^2 + 3(^3a - ^1a)^2], \quad (9)$$

which would give only $\mathcal{S}(0) = (16,57 \pm 0,69) \cdot 10^{-24} \text{ cm}^2$. It is therefore desirable to repeat the measurements: especially the quantity $(3 \cdot ^3a + ^1a)^2$, immediately furnished by the para cross-section, is very sensitive to the value of ^3a , i.e. — as appears from (6.211–17) — to the range of the nuclear force*.

6.34. The neutron spin. The general evidence of scattering of slow neutrons incidentally provides us, as pointed out by SCHWINGER [37b], with an indirect proof that the neutron spin is $\frac{1}{2}$ rather than, say, $\frac{3}{2}$. If the neutron has an arbitrary intrinsic momentum $\vec{S}^{(n)}$, and if S_n denotes the

* If we try to calculate the a 's from (7) and (8), we get — taking account of the fact that ^3a and ^1a must be of opposite signs — two systems of solutions:

	^3a (10^{-12} cm)	^1a (10^{-12} cm)
I	$0,874 \pm 0,025$	$1,728 \pm 0,025$
II	$0,426 \pm 0,025$	$2,176 \pm 0,025$

According to (6.211–16, 17, 18) and (6.21–14), the corresponding widths of a potential well yielding such values of ^3a are $\approx 0,8 \cdot 10^{-12}$ cm and ≈ 0 , respectively. Neither of these range values is of a reasonable order of magnitude. Cf. also HAMERMESH and SCHWINGER [47].

corresponding quantum number, we must (6.213–31a, b) introduce two amplitudes $^{(2S_n)}a$, $^{(2S_n+2)}a$, in terms of which the scattering cross-section of protons for very slow neutrons may be written,

$$\mathcal{S} = 4\pi \left[\frac{S_n + 1}{2S_n + 1} \cdot ^{(2S_n+2)}a^2 + \frac{S_n}{2S_n + 1} \cdot ^{(2S_n)}a^2 \right], \quad (10)$$

while the scattering radius takes the form

$$a = \left[\frac{S_n + 1}{2S_n + 1} \cdot ^{(2S_n+2)}a + \frac{S_n}{2S_n + 1} \cdot ^{(2S_n)}a \right] + \frac{1}{2S_n + 1} [(^{(2S_n+2)}a - ^{(2S_n)}a) \vec{S}^{(n)} \cdot \vec{\sigma}^{(p)}], \quad (11)$$

The factors (4), (5) then become

$$(S \rightarrow S') \quad \left[\frac{S_n + 1}{2S_n + 1} \cdot ^{(2S_n+2)}a + \frac{S_n}{2S_n + 1} \cdot ^{(2S_n)}a \right]^2 + \frac{S(S+1) S_n(S_n+1)}{3(2S_n+1)^2} [(^{(2S_n+2)}a - ^{(2S_n)}a)^2] \quad (12)$$

$$(S \rightarrow 1 - S) \quad [(^{(2S_n+2)}a - ^{(2S_n)}a)^2] \cdot \frac{S_n(S_n+1)}{3(2S_n+1)^2} \cdot \frac{3}{2S+1}.$$

If, e.g., the neutron spin were $\frac{3}{2}$, the excited state of the deuteron would be a quintet state, and (10) would give

$$\mathcal{S} = 4\pi \left[\frac{3}{8} (^3a)^2 + \frac{5}{8} (^5a)^2 \right]; \quad (13)$$

from this formula, we can derive $|^5a|$ and the energy $|^5\epsilon^{(0)}|$ of the corresponding actual or virtual quintet level. If we apply the finite range correction (6.211) on the assumption of a potential well of given width, we find, by a procedure explained in detail in the next section (6.431), the following results:

6.34. Scattering radii and excited deuteron level for neutron spin $\frac{3}{2}$					
D	^3a	$ ^5a $	$ ^5a/3a $	$ ^5\epsilon^{(0)} $ (actual level)	$^5\epsilon^{(0)}$ (virtual level)
10^{-13} cm	10^{-12} cm	10^{-12} cm		MeV	MeV
1.94	0.538	1.58	2.94	0.19	0.14
2.8	0.585	1.57	2.68	0.20	0.14

On the other hand, according to (12) and (6), the scattering cross-sections of para- and ortho-hydrogen for cold neutrons may be written

$$\begin{aligned} \mathcal{S}_{\text{para}} &= \frac{1}{4} \mathcal{S}_{p \rightarrow p} (3 \cdot ^3a + 5 \cdot ^5a)^2 \\ \mathcal{S}_{\text{ortho}} &= \frac{1}{4} \mathcal{S}_{o \rightarrow o} (3 \cdot ^3a + 5 \cdot ^5a)^2 + \frac{1}{4} (2 \mathcal{S}_{o \rightarrow o} + \mathcal{S}_{o \rightarrow p}) (^3a - ^5a)^2, \end{aligned} \quad (14)$$

the coefficients $\mathcal{S}$ being the same as in the case of spin $\frac{1}{2}$. With the above values of the ratio $|^5a/3a|$, it will be seen that the ratio $\mathcal{S}_{\text{ortho}}/\mathcal{S}_{\text{para}}$ is of the

order of magnitude 1 if the ${}^5S$ level is actual, or 2 ... 3 if it is virtual; in either case in sharp contradiction with experiment*.

6.35. Scattering of polarized neutrons by protons. In view of the possibility of producing partially polarized neutron beams**, it is interesting to note that, as pointed out by SCHWINGER and RABI [37c], the effect of the scattering of such a beam by (unpolarized) protons on its polarization affords a quite sensitive means of testing the spin dependence of the nuclear force and the nature of the excited ${}^1S$ level of the deuteron. No experiments on these lines, however, have as yet been attempted.

In a general way, the spin dependence of the wave-function representing the incident neutron (1) and proton (2) may be expressed as follows:

$$(\alpha^{(1)} v_+(1) + \beta^{(1)} v_-(1)) (\alpha^{(2)} v_+(2) + \beta^{(2)} v_-(2)),$$

$v_{\pm}$ being the eigenfunctions of σ_z . The polarization p of the incident neutron is defined by

$$p = \frac{|\alpha^{(1)}|^2 - |\beta^{(1)}|^2}{|\alpha^{(1)}|^2 + |\beta^{(1)}|^2}, \quad (15)$$

while for the amplitudes of the unpolarized protons $|\alpha^{(2)}|^2 = |\beta^{(2)}|^2$. It is then easily seen from (6.21-9) that the absolute squares of the amplitudes corresponding to the two spin orientations of the scattered neutron wave (restricted to its S -component) are proportional to

$$|\alpha^{(1)}|^2 \cdot [|\delta^3|^2 + \frac{1}{4}|\delta^3\delta + \delta^1\delta|^2] + |\beta^{(1)}|^2 \cdot \frac{1}{4}|\delta^3\delta - \delta^1\delta|^2$$

and

$$|\beta^{(1)}|^2 \cdot [|\delta^3\delta|^2 + \frac{1}{4}|\delta^3\delta + \delta^1\delta|^2] + |\alpha^{(1)}|^2 \cdot \frac{1}{4}|\delta^3\delta - \delta^1\delta|^2$$

respectively; we have put $\delta^i\delta = e^{2i\sigma_i(0)} - 1$. Hence the polarization of the scattered neutron wave is found to be

$$p_{\text{scatt}} = p \left[1 - \frac{\sin^2(3\delta(0) - \delta(0))}{\sin^2 \delta(0) + 3 \sin^2 \delta(0)} \right]. \quad (16)$$

The scattering thus causes a change of polarization of the incident neutron beam whenever the potential is spin dependent. The sensitiveness of the effect can be judged from the case of very small neutron energies. According to (5.32-21), formula (16) then becomes

$$p_{\text{scatt}} = p \left[1 - \frac{(3a - \delta^1 a)^2}{(\delta^1 a)^2 + 3(\delta^3 a)^2} \right]; \quad (17)$$

this result follows also directly from (6.213-29, 30, 31) and is therefore

* On the contrary, no clear-cut evidence against the spin $\frac{1}{2}$ can be drawn from the proton scattering cross-section for neutrons of 2.5 MeV, as was wrongly asserted by H. GOLDSMITH and L. MOTZ, *P.R.* 53, 947. 1938. For this energy, assuming the ${}^5S$ -level to be virtual, we get in fact a cross-section

$$2.68 \cdot 10^{-24} \text{ cm}^2 \text{ for } D = 1.94 \cdot 10^{-13} \text{ cm} \\ \text{or } 2.80 \cdot 10^{-24} \text{ cm}^2 \text{ for } D = 2.8 \cdot 10^{-13} \text{ cm}$$

of the same order of magnitude as the observed one (table 6.412).

** According to a recent remark by SCHWINGER [46], the resonance scattering of neutrons by Helium (17.42) should have a strong polarizing effect.

valid also for bound protons. For $|1a/3a| = 4$ one gets

$$\begin{aligned} {}^1S \text{ level actual: } b_{\text{scatt}}/b &\approx 0.53 \\ {}^1S \text{ level virtual: } b_{\text{scatt}}/b &\approx -0.32. \end{aligned} \quad (18)$$

6.4. Scattering of fast neutrons by protons

6.40. We may now proceed to the study of neutrons of higher energy (from, say, 0.1 MeV), which, for brevity, may be termed "fast". In this section, we shall first collect all the relevant experimental evidence. This comprises, on the one hand, numerous measurements of the total scattering cross-section of protons for fast neutrons of various energies, and, on the other hand, some results concerning the angular distribution of the scattered neutrons (or of the recoil protons), likewise for different energies of the incident particles. The theoretical discussion will be limited, however, to an analysis of the dependence of the total cross-section on the energy of the incident neutrons, in so far as it can be accounted for by the *S*-phase only, i.e. in so far as it depends on the effective potentials for even states. The discussion of the contributions from higher phases both to the total cross-section and to the departure of the angular distribution of the scattered particles from isotropy (in the barycentric system) necessitates some knowledge of the effective potentials for odd states as well, and will conveniently be postponed to a later section (8.33).

6.41. *Total scattering cross-section of protons for fast neutrons.* When experimenting with "fast" neutrons, the chief difficulty is to get sufficiently homogeneous neutron beams of different energies *. In successive intervals of increasing energy, various sources and methods of detection have to be used for that purpose. We shall begin with a survey of the determinations of the total cross-section, treating these different energy regions separately.

6.411. *Scattering of neutrons of energies 0.1 ... 1 MeV.* In the interval 0.1 ... 1 MeV, one may use photoneutrons from Be or D activated by the γ -rays of ThC'' (2.62 MeV) or RaC (mainly 1.76 MeV); the thresholds of the reactions $D(\gamma, n)$, $\text{Be}(\gamma, n)$ are in fact 2.18 and 1.63 MeV, respectively. The reaction ${}^{12}\text{C}(d, n)$, with deuterons from a high voltage tube, also yields low energy neutrons, the energy balance of the (endothermic) reaction being -0.27 MeV. Another convenient neutron source in the energy interval considered is provided by the ${}^7\text{Li}(p, n)$ reaction. The neutrons can be detected either in an ionization chamber (I.C.) filled with hydrogen under pressure, or by the activation of a suitable element surrounded with paraffin or water. In this way, the values of the cross-section listed in table 6.411 have been obtained; the older ones are not very accurate **.

* In the following survey, we shall omit a few measurements in which the neutron energy was not sufficiently well-defined, or which for some reason do not seem quite reliable. They are listed in the very complete tables of DIEBNER, HERRMANN and GRASSMANN [42].

** In tables 6.411 and 6.412, the values of the neutron energies have been recalculated from the data about the nuclear reactions mentioned in the text.

6.411. Scattering of neutrons of 0.1 . . . 1 MeV energy by protons				
Reference	Source	Detector	Energy (approxim.) MeV	$\mathcal{S}$ 10^{-24} cm^2
LEIPUNSKI <i>et al.</i> [36 b]	$\text{Be}(\gamma_{\text{Ra}}, n)$	Ag in water	0.12	$8 \pm 2^*$
GOLOBORODKO and LEIPUNSKI [39]	$\text{D}(\gamma_{\text{RdTh}}, n)$	Dy in paraffin	0.22	5.0 ± 1.0
AMALDI <i>et al.</i> [40]	$^{12}\text{C}(d, n)$	I.C. with 20 atm. H_2 ; electrometer	0.4	8.7
GOOD and SCHARFF-GOLDHABER [40]	$\text{Be}(\gamma_{\text{RdTh}}, n)$	I.C. with 5 atm. H_2 ; linear amplifier	0.9	3.70 ± 0.35
FRISCH [46]	$^7\text{Li}(p, n)$	I.C. with 0.55 to 2.65 atm. H_2 ; linear amplifier	0.035	16.74 ± 0.41
			0.095	13.46 ± 0.39
			0.265	9.12 ± 0.24
			0.490	6.33 ± 0.21
BAILEY <i>et al.</i> [46]	$^7\text{Li}(p, n)$	I.C. with 4 atm. methane; linear amplifier	0.35	7.15 ± 0.24
			0.46	6.52 ± 0.15
			0.72	5.22 ± 0.12
			0.97	4.45 ± 0.08
	$^{12}\text{C}(d, n)$	argon filled I.C. with paraffin radiator	1.0 ± 0.1	4.16 ± 0.15
			1.6	3.36 ± 0.08
			2.0	2.96 ± 0.07

* Value calculated with the help of the C cross-section determined by FEDOROV and PERFILIEVA [37] with the same apparatus.

6.412. Scattering of neutrons produced by the $\text{D}(d, n)$ reaction. A very suitable homogeneous neutron source, which has been extensively used, is provided by the reaction $\text{D}(d, n)$, which gives rise to a single neutron group. The energy set free in this exothermic reaction is 3.26 MeV, so that even with (say) 0.2 MeV deuterons from a high voltage tube, it yields homogeneous neutron beams with energies from 2 to 3 MeV according to the direction of the beam with respect to the impinging deuterons (cf. M & F, p. 52—54). Mostly, the experiments are carried out with the neutron beam parallel or perpendicular to the deuteron beam; but the apparatus used by AOKI [39] permitted a variation of the angle of the two beams and thus a controllable variation of the neutron energy in the interval 2 ... 3 MeV. With the same source, BAILEY *et al.* [46] could extend the range of neutron energies up to 6 MeV.

6.413. Scattering of very fast neutrons by protons. In order to obtain neutrons of still larger energies, we must at present have recourse to such (d, n) reactions as can be produced with Li, Be, B as targets; but owing to the occurrence of disintegrations into more than two particles or of excited states of the product nucleus, the energy spectra of the neutrons produced in these reactions present partly a continuous distribution, partly numerous homogeneous groups which are not easily separated. An energy selecting device has therefore to be introduced in the apparatus; no very

6.412. Scattering of $D(d,n)$ neutrons by protons			
Reference	Detector	Energy MeV	S 10^{-24} cm
BOOTH and HURST [37]	In in paraffin	3,1 *	$1,7 \pm 0,4^{**}$
ALLEN and HURST [40]	P	3,1 *	$2,1 \pm 0,2$
LADENBURG and KANNER [37]	paraffin-lined I.C.; linear amplifier	2,5 *	$2,11 \pm 0,21$
ZINN <i>et al.</i> [39]	He-filled I.C.; linear amplifier	$2,85 \pm 0,04$	$2,36 \pm 0,12$
KIKUCHI and AOKI [39]	I.C. with 20 atm. methane; electrometer	$2,45 \pm 0,05$	$2,28 \pm 0,09$
AOKI [39]	id.	2,21 ***	$2,76 \pm 0,10$
		2,33	$2,71 \pm 0,08$
		2,48	$2,53 \pm 0,10$
		2,63	$2,51 \pm 0,08$
		2,76	$2,42 \pm 0,10$
BAILEY <i>et al.</i> [46]	argon filled I.C. with paraffin radiator	2,6	$2,60 \pm 0,05$
		3,0	$2,33 \pm 0,13$
		3,5	$2,09 \pm 0,09$
		4,0	$1,85 \pm 0,09$
		4,5	$1,83 \pm 0,10$
		5,0	$1,63 \pm 0,05$
		5,5	$1,48 \pm 0,06$
		6,0	$1,32 \pm 0,12$
* Maximum energy. — ** Value corrected according to estimate of C cross-section from data of ZINN <i>et al.</i> [39], AOKI [39] and AMALDI <i>et al.</i> [43]. — *** The absolute error may be $\pm 0,15$ MeV, but the relative values are more accurate.			

great accuracy is needed, however, since the cross-section is expected to vary slowly with energy in this region. SALANT [40] obtains a certain amount of selection by using for the detection of the neutrons the activity induced in Cu by the reactions $^{63}\text{Cu}(n, 2n)^{62}\text{Cu}$, $^{62}\text{Cu} \rightarrow ^{62}\text{Ni} + \beta^+$; in fact, the first reaction has a threshold at about 12 MeV, so that the Cu-detector selects from the $\text{Li}(d, n)$ continuum, for instance, the part extending from 12 to the upper end at about 15 MeV; if we make the natural assumption that the yield of the reaction increases very rapidly with increasing energy, we may locate the mean effective energy nearer the upper extremity of the interval; this effective energy can further be varied to some extent by varying the direction of the neutron beam. The same method has been used by SHERR [45] for still higher energy neutrons, produced by bombarding Li with 10,2 MeV D^+ ions: the maximum neutron energy is then 25,4 MeV and the reaction $^{12}\text{C}(n, 2n)^{11}\text{C}$ provides an energy-sensitive detector with (calculated) threshold at 20,6 MeV.

AMALDI *et al.* [43] detect the neutrons by letting them fall on a paraffin layer and counting the recoil protons with a coincidence system of three counters (fig. 6.413); by interposing between paraffin and counters an Al

absorber of suitable thickness, they only record protons above a certain energy (which, for head-on collisions, is also the energy of the impinging

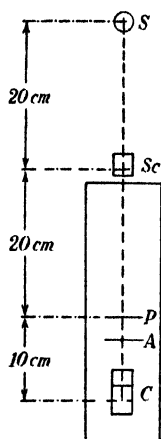


Fig. 6.413. Fast neutron scattering according to AMALDI *et al.* [43]. S source, Sc scatterer, P paraffin layer, A absorber, C counters.

neutrons) and in this way they contrive to select an interval of neutron energies between the lower limit defined by the absorber and the maximum energy of the spectrum. The advantage of this device is clearly the control over the lower limit of the energy interval investigated. Essentially the same method has quite recently been applied by SLEATOR [47], whose preliminary results are also quoted in the table 6.413.

6.413. Scattering of very fast neutrons by protons					
Reference	Source	Selector	Energy interval MeV	Mean energy MeV	S 10^{-24} cm^2
SALANT and RAMSAY [40]	Li (<i>d, n</i>)	Cu	12 ... 15	14	$0,70 \pm 0,06$
				15	$0,66 \pm 0,07$
AMALDI <i>et al.</i> [43]	Be (<i>d, n</i>)	Al absorber	3,2 ... 4,6	4,1	$1,73 \pm 0,06$
	B (<i>d, n</i>)	id.	9,4 ... 13,2	12,5	$0,69 \pm 0,11$
	Li (<i>d, n</i>)	id.	11,15 ... 15	13,5	$0,71 \pm 0,04$
	Li (<i>d, n</i>)	Cu	12 ... 15	14	$0,69 \pm 0,021$
SHERR [45]	Li (<i>d, n</i>)	C	20,6 ... 25,4	25	$0,39 \pm 0,04$
SLEATOR [47]	Be (<i>d, n</i>)	Al absorber	5,68 ... 8,16	6,5	$1,40 \pm 0,11$
			8,24 ... 11,70	9,3	$0,92 \pm 0,08$
	Li (<i>d, n</i>)	id.	8,94 ... 13,9	10,6	$0,78 \pm 0,09$
			11,2 ... 15,9	12,8	$0,83 \pm 0,09$
			13,2 ... 18,1	14,8	$0,61 \pm 0,09$
			15,0 ... 19,6	16,5	$0,66 \pm 0,10$
			16,5 ... 21,2	18,1	$0,55 \pm 0,08$
			17,9 ... 22,9	19,6	$0,52 \pm 0,09$
			19,4 ... 24,6	21,1	$0,41 \pm 0,09$

6.42. Angular distribution of proton-neutron scattering. The investigation of the angular distribution of the scattered neutrons or recoil protons

presents considerable difficulties. The first experiments gave widely contradictory results, chiefly due, — as a critical discussion (DEE and GILBERT [37]) shows —, to the inhomogeneity of the impinging neutrons. DEE and GILBERT [37] were first able to obtain reliable results, making a very careful study of the question for $D(d, n)$ neutrons. They observed the recoil proton tracks in a large expansion chamber filled with a mixture of methane and argon at 3,5 atm. A special apparatus was designed to measure the scattering angle Θ' (6.212–20) from the stereoscopic pair of photographs of each track. Thanks to the high pressure, the tracks were sufficiently short to terminate in the chamber, so that the range, and consequently the energy, of each recoil proton could also be ascertained. Since for elastic scattering of neutrons of energy E (in the laboratory system), the energy E' of the recoil protons is uniquely related to the scattering angle Θ' by the formula (see fig. 6.212)

$$E' = E \cos^2 \Theta', \quad (1)$$

it was possible in this way to eliminate the tracks due to neutrons not coming directly from the source. The result was that, at this energy, the scattering is *isotropic in the barycentric system*; the experimental uncertainty, however, would still allow a deviation from isotropy up to about 10 %. Other experiments (KRUGER *et al.* [37], BONNER [37]) on essentially the same lines, but under less favourable conditions, led to the same conclusion. A further confirmation was given by BARSCHALL and KANNER [40], who utilized a specially constructed ionization chamber filled with a mixture of H_2 at 4 atm. and Kr at 5 atm. and derived the angular distribution of the recoil protons with the help of relation (1) from a measurement, by the linear amplifier, of their energy distribution*. Particulars of all these experiments** with $D(d, n)$ neutrons are collected in the following table:

6.42-1. Investigation of recoil proton tracks			
Reference	Energy of neutrons MeV	Range of scattering angle Θ'	Number of tracks
DEE and GILBERT [37]	2.44 ... 2.52	0 ... 51°	1534
KRUGER <i>et al.</i> [37]	2.44 ... 2.7	0 ... 69°	635
BONNER [37]	2.44 ... 2.47	0 ... 65°	207
BARSCHALL and KANNER [40]	2.44 ... 2.5	0 ... 55°	—

* Further experiments on the same lines have recently been announced (COON, DAVIS and BARSCHALL [46a]).

** Quite at variance with the measurements just mentioned, are those of the Japanese physicists (KIKUCHI *et al.* [39b], AOKI [39]). They compare the neutron intensity coming directly from the source with that scattered by a flat paraffin ring (the direct beam being then stopped by a block of Fe placed in the hole of the ring). From this intensity ratio they derive in a rather indirect way the value of the differential cross-section per unit solid angle $S(\theta)$ for the mean scattering angle defined by the dimensions and position of the ring. By using rings of different sizes they get 4 values of $S(\theta)$ in the angular

Even for much higher neutron energies, only small (if any) deviations from isotropy could be perceived. This result is well-established for energies up to about 11 MeV. Neutrons of this energy have been studied by TATEL [42]. He selects them from the continuous neutron spectrum of the $\text{Be}(d, n)$ reaction by the same device as the Italian physicists (6.413), the recoil proton detector consisting of a coincidence system of two ionization chambers. He investigates their scattering by hydrogen gas in two directions, determined by the orientation of the axis joining the hydrogen cell to the detector with respect to the neutron beam (collimated by a hole in a thick wall of water). The mean scattering angles are $\Theta' = 16^\circ$ and $\Theta' = 45^\circ$ (i.e. $\vartheta = 148^\circ$ and 90° , respectively); the recoil proton yields correspond, within 10 % experimental uncertainty, to isotropic distribution of the scattered neutrons.

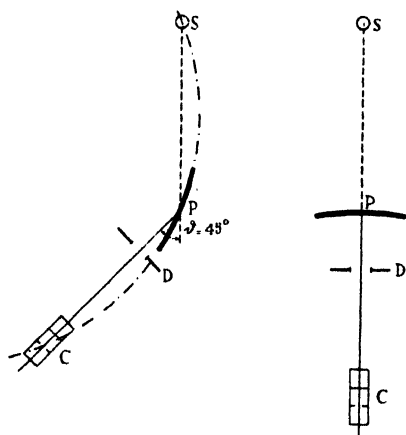


Fig. 6.42-1. Angular distribution of fast neutron scattering, according to AMALDI *et al.* [42a]. S source, P paraffin scatterer, D diaphragm, C counters.

In the case of neutrons of 12 ... 14 MeV, the question is not definitely settled, as different experiments lead to conflicting results. According to AMALDI *et al.* [42a, b], a considerable anisotropy of scattering angle distribution would occur in this energy region. The Italian physicists compare the intensity of the recoil protons from a paraffin scatterer in the direction of the incident neutron beam ($\Theta' = 0$) with that of the protons scattered at $\Theta' = 45^\circ$ under the same geometrical conditions. For this purpose (fig. 6.42-1), the detector (consisting of a coincidence system of three proportional counters) is rotated through an angle of 45° about the middle P of the scatterer, while the latter (which is bent into the required shape) is disposed along the circle passing through the source S, the point P and the center C of the counter system. The sources used were

range $\Theta = 23^\circ \pm 5^\circ \dots 55^\circ \pm 6^\circ$ (thus $\Theta' = 35^\circ \pm 6^\circ \dots 67^\circ \pm 5^\circ$) for D (d, n) neutrons of about 3 MeV. While $\mathcal{S}(\Theta)/\cos \Theta$ turns out to be practically independent of Θ , it differs by a factor $\approx \frac{1}{2}$ from the value $\mathcal{S}/\pi$ expected in the case of S-wave scattering ($\mathcal{S}$ being the total cross-section). The authors attribute this discrepancy to a strong anisotropy of the angular distribution, but in view of the other results, this conclusion has very little plausibility.

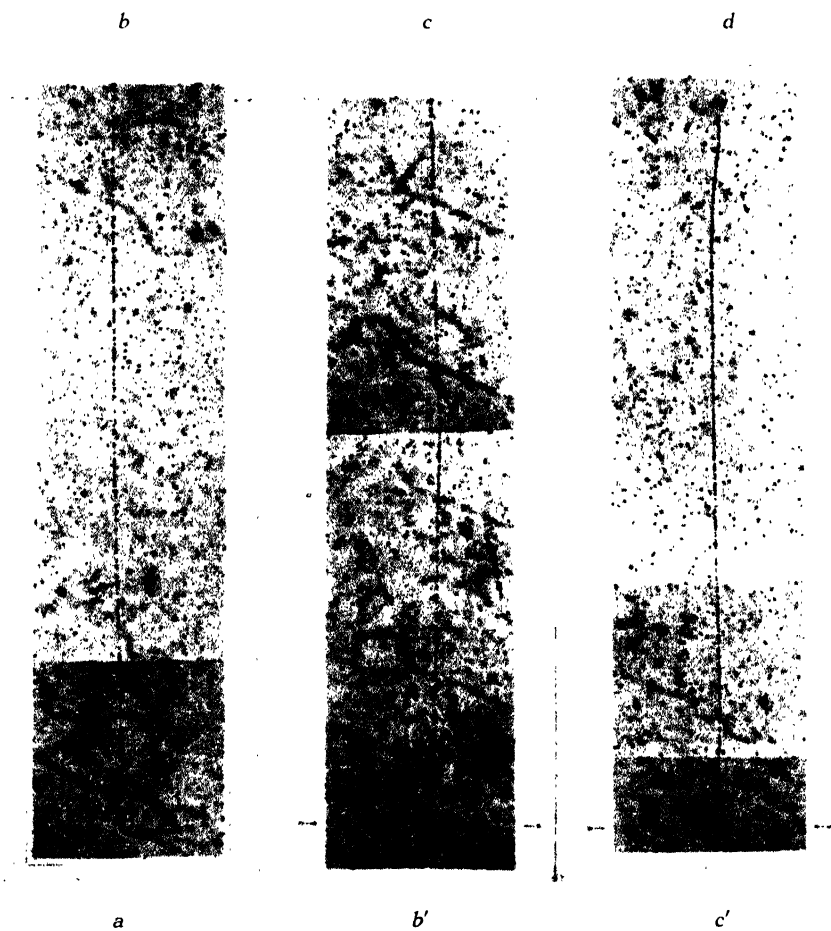


Fig. 6.42-2. Microphotograph of a 13 MeV proton track in a photographic emulsion layer (POWELL *et al.* [46a]). The track begins at *a* and ends at *d*; the three successive segments *a b*, *b' c*, *c' d* should be pieced together at the places indicated by the arrows.

the reactions $B(d, n)$ and $Li(d, n)$, the lower limit of the neutron spectrum being defined by Al absorbers (as above, 6.413). The ratio $\mathcal{S}(\pi)/\mathcal{S}(\pi/2)$ of the values of the differential cross-section $\mathcal{S}(\vartheta)$ per unit solid angle (in the *barycentric* system) for backward ($\vartheta = \pi$) and sideways ($\vartheta = \pi/2$) scattering, as derived from these measurements, would be:

6.42-2. Scattering anisotropy of fast neutrons according to the Italian physicists			
Source	Energy interval MeV	Mean energy MeV	$\mathcal{S}(\pi)/\mathcal{S}(\pi/2)$
$B(d, n)$	10 ... 13,5	12,5	$0,71 \pm 0,04$
$Li(d, n)$	11 ... 15	13,3	$0,53 \pm 0,03^*$
	12,5 ... 15	14	$0,52 \pm 0,03^*$
* These values may be too low.			

In contrast with these results, measurements performed by POWELL and his collaborators [40, 44, 46a, b] on the two fastest neutron groups from $B(d, n)$ with mean energies 8,8 and 13 MeV, point for both groups to an angular distribution which, if it deviates at all from isotropy, would rather exhibit a small excess of scattered neutrons in the backward direction. Powell's method is more straightforward and much more powerful than the ingenious apparatus of Amaldi, and its results are very probably more reliable; but it is desirable that the Italian experiments be repeated with a view to tracing the cause of this puzzling discrepancy*. The method developed at Bristol consists in observing tracks of recoil protons in the emulsion layer of a photographic plate exposed to an approximately longitudinal neutron beam. The direction of the impinging neutrons can be defined with an accuracy of $\pm 0^\circ,5$, and a procedure of microscopic focussing permits the measurement of the dip of a proton track with an error as small as $\pm 1^\circ$, so that the uncertainty in the scattering angle Θ' does not exceed 2° . The measurement of the range of each track together with its scattering angle allows its assignment to a specific neutron group; special emulsions with very large grain densities have been prepared (POWELL *et al.* [46a]) in order to ensure an accurate definition of the tracks (fig. 6.42-2, plate III). Account must be taken, however, of the tracks not ending in the emulsion, the thickness of which is 120 or 150 μ : for each kind of emulsion, a careful empirical determination has been made of the fraction of tracks remaining in the emulsion for different ranges and angles of dip, and before being compared with the actual results of counts of tracks, theoretical distribution curves must be corrected for this "loss" at each scattering angle (corresponding, by (1), to a different range). This circumstance practically restricts the utilization of the material

* A recent investigation by LAUGHLIN and KRUGER [47] of the angular distribution of the recoil protons produced in a high pressure cloud chamber filled with methane by neutrons of energy 12 ... 13 MeV supports the Bristol results.

to tracks with dips smaller than 10° . The method has the obvious advantage of surveying the whole angular range $\theta' = 0 \dots 45^\circ$ and yielding a complete angular distribution function, which allows of a much more stringent theoretical test than the mere ratio $S(\pi)/S(\pi/2)$. The latest results on the two fast neutron groups are summarized in fig. 6.42-3, 4; owing to

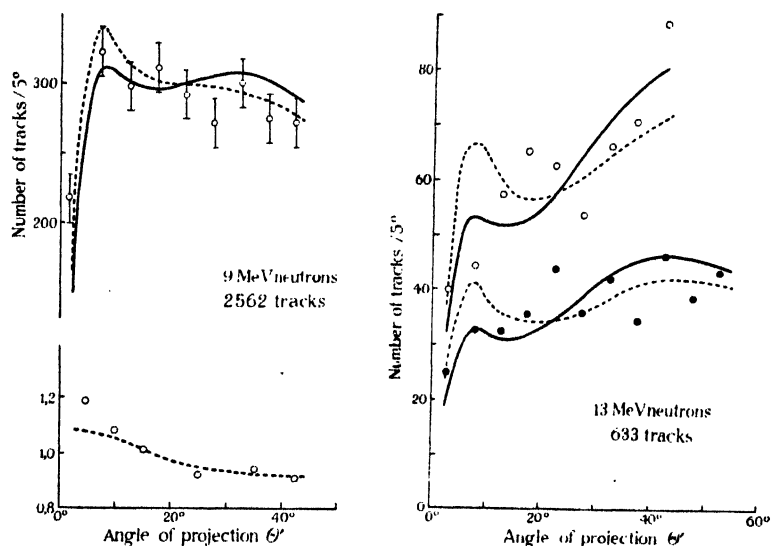


Fig. 6.42-3, 4. Angular distribution of fast neutron scattering.

• Measurements of tracks in a 150μ emulsion layer.

○ Same, combined with tracks in a 120μ emulsion layer.

—— Isotropic distribution (corrected for "loss").

..... Distribution (corrected for "loss") according to central meson potential of symmetrical type (8.33) with $M_m = 225 m$.

In the lower part of fig. 6.42-3, the data are shown as referred to the isotropic distribution.

the statistical character of the method, an increase in accuracy may be expected from the accumulation of further data, especially on the 13 MeV group. Moreover, there is hope of applying the same analysis to neutrons of still higher energies; which would be of great importance from a theoretical point of view (8.33).

6.43. Comparison with theory. Leaving for later discussion the possible significance of the small deviations from isotropic distribution of the scattered particles indicated by Powell's experiments with 9 and 13 MeV neutrons, we may draw from the results of the preceding sub-section (6.42) the general conclusion that, at any rate for smaller neutron energies, we have mainly to do with S-wave scattering, in conformity with elementary theoretical considerations (6.21). In this energy region, we may thus compare the empirical results concerning the total scattering cross-section at various energies with the predictions of a theory of S-wave scattering, utilizing only the effective nuclear potentials for even states and adopting a definite form for these potentials. From such a detailed comparison we

may hope to derive in each case quantitative information on the ranges and strengths of the effective potentials.

6.431. Scattering by potential well. The calculations can be carried out in analytical form (without numerical integration of the wave-equation) only in the case of the potential well. The theory developed in 6.21, 6.211, 6.212, when combined with the existence of the low energy ^1S level (6.23), leads to the general formula

$$S(E) = \frac{\pi \hbar^2}{M} \left[\frac{3 \cdot {}^3f(E)}{\frac{1}{2}E + |\varepsilon_0|} + \frac{{}^1f(E)}{\frac{1}{2}E + |{}^1\varepsilon^{(0)}|} \right], \quad (2)$$

valid for neutron energies E satisfying the condition $\eta < 1$, i.e. (5.11–9; 6.212–20) $E \lesssim 10$ MeV. If the energy is expressed in MeV, we have to use the numerical value

$$\frac{\pi \hbar^2}{M} = 1.3 \cdot 10^{-24} \text{ cm}^2 \text{ MeV}; \quad (3)$$

the finite range corrections are given by (6.211–16) in terms of the potential width D (assumed to be the same for singlet and triplet states) and the scattering constants ${}^3a_0$, ${}^1a_0$.

The triplet contribution can immediately be calculated for any energy, since the parameters occurring in (2), viz. $|\varepsilon_0|$, ${}^3a_0$ and the G 's in ${}^3f(E)$ are all determined by the deuteron binding energy $|\varepsilon_0|$ (6.11–1; 6.21–14; 6.211–19). But the analogous parameters pertaining to the ^1S level have first to be derived from the experimental data, more specifically from the zero energy cross-section (6.23–34; 6.211–17)

$$S(0) = 3\pi ({}^3a)^2 + \pi ({}^1a)^2, \quad (4)$$

$${}^3a = {}^3a_0 \sqrt{{}^3f(0)} \quad (5)$$

$${}^1a = {}^1a_0 \sqrt{{}^1f(0)}. \quad (6)$$

Having adopted some value for D , we begin by calculating ${}^3a$ with the help of (5), using the expression (6.211–16; 18) for ${}^3f(0)$ (with the terms in $(D/{}^3a_0)^n$ up to $n = 4$). From (4) we then obtain ${}^1a$ and solve (6) for $|{}^1a_0|$ (${}^1a_0$ and ${}^1a$ are negative); it is sufficient to write (6) in the form

$$\left| \frac{{}^1a}{{}^1a_0} \right|^2 = 1 - \frac{D}{|{}^1a_0|} + 0.345 \left(\frac{D}{|{}^1a_0|} \right)^2. \quad (7)$$

Finally $|{}^1a_0|$ yields the energy $|{}^1\varepsilon_0|$. The following table gives an idea of the variation of the various parameters with the range D :

6.431-1. Scattering parameters in terms of width of potential well					
D 10^{-13} cm	${}^3a$ 10^{-13} cm	$-{}^1a$ 10^{-13} cm	$-{}^1a_0$ 10^{-13} cm	${}^1\varepsilon^{(0)}$ MeV	$ {}^1a/{}^3a $
0	0.437	2.47 ± 0.07	2.47 ± 0.07	0.067 ± 0.004	5.66 ± 0.16
1.94	0.538	2.41 ± 0.07	2.50 ± 0.07	0.066 ± 0.004	4.48 ± 0.13
2.8	0.585	2.38 ± 0.07	2.51 ± 0.07	0.065 ± 0.004	4.07 ± 0.12

The corresponding corrections for finite range are:

6.431-2. Finite range corrections		
D 10^{-13} cm	${}^3f(E)$	${}^1f(E)$
1,94	$1,51 - 0,02 E + 10^{-4} E^2$	$0,92 - 0,01 E + 10^{-4} E^2$
2,8	$1,79 - 0,04 E + 4 \cdot 10^{-4} E^2$	$0,89 - 0,02 E + 4 \cdot 10^{-4} E^2$
E in MeV		

Graphs of the cross-section (2) for the range values 1,94 and $2,8 \cdot 10^{-13}$ cm have been drawn in fig. 6.431, in which the empirical data have also been plotted. The value $D = 1,94 \cdot 10^{-13}$ cm has been chosen so as to fit the cross-section $2,53 \cdot 10^{-24}$ cm² as measured by AOKI [39] for $E \approx 2,5$ MeV. It will be observed that not only are AOKI's measurements in the range 2,21 ... 2,76 MeV beautifully consistent with the theoretical curve, but also those by ZINN *et al.* [39] at 2,85 MeV and by the Italian physicists (6.413) at 4,1 MeV fall on or very near the same curve. However, it must be noted that the energy values at which Aoki's measurements have been performed could unfortunately (owing to uncertain corrections) not be fixed with a smaller margin of error than $\pm 0,15$ MeV; and further, — as also exhibited by the older measurements (points B, K, L in fig. 6.431) —, that systematic errors tend to give too low a value of the scattering cross-section. One cannot, therefore, as shown by the figure, exclude the possibility that the true value of the range would even be as high as $2,8 \cdot 10^{-13}$ cm ($\approx d$): the importance of this observation will appear on a later occasion (8.1) *.

As regards the experimental points at higher neutron energies (12 ... 15 MeV), we cannot directly relate them to the calculated curve, since ** we ought here to take account of the waves of higher orbital momentum. The fact that also these points practically fall on the S-wave scattering curve indicates that *the contribution of waves of higher orbital momentum to the total scattering cross-section in the region of 12 ... 15 MeV neutron energies is very small*. This property may serve as a test for different kinds of nuclear interaction (16.22).

Summing up, we see that for any assumed width of the potential well, the "depths" or strengths of the effective wells for ${}^3S$ and ${}^1S$ states are determined by the binding energy of the ground state of the deuteron, and the scattering cross-section of the proton for neutrons of zero energy; the variation of the scattering cross-section with neutron energy helps to a certain

* BOHM and RICHMAN [47] show that the adjunction of a broad and shallow "tail" to the potential well tends to lower the cross-section at higher energies and thus to improve the fit with the empirical curve.

** This energy region is already outside the domain of validity of our approximate treatment of the S-wave scattering; but the deviation, for such energies, should not amount to more than a few percent.

extent to fix the value of the width, but the accuracy of this determination is as yet rather poor. The explicit calculation of the potential strengths ${}^3j^{(0)}$, ${}^1j^{(0)}$ from ${}^3\varepsilon^{(0)}$, ${}^1\varepsilon^{(0)}$ is easily performed by means of formula (5.21-5) for an actual level and the analogous one for a virtual level; on account of the definition (5.321-28) adopted for such a level, one has simply to replace, in (5.21-5), $\sqrt{|\eta|}$ by $-\sqrt{\eta}$. The result is:

6.431-3. Widths and depths of potential wells		
D 10^{-13} cm	${}^3j^{(0)}$ MeV	${}^1j^{(0)}$ MeV
1,94	38,1	25,4
2,8	20,9	11,8

It may be observed that, owing to the smallness of the energy of the ${}^1S$ level, the value of the strength, ${}^1j^{(0)}$ of the singlet potential is quite insensitive to any experimental uncertainty affecting the scattering cross-section for zero energy neutrons. This remark is valid for any simple type of central potential.

6.432. Scattering by other types of central potential. In principle, the procedure followed in the case of the potential well may just as well be applied to the other simple forms of potential quoted in 5.11, but it will be much more laborious, since for every value of the energy which is considered, the S -phase has to be computed numerically. For simplicity, we shall always assume that all effective potentials have the same range; but even so the cross-section (as exemplified by the case of the well potential) is so insensitive with respect to this range that one cannot hope to determine this parameter by a comparison with the available data. One must therefore adopt some value of the range, drawn from other evidence, such as the proton-proton scattering experiments (7.13). The triplet potential strength ${}^3j^{(0)}$ is then derived from ${}^3\varepsilon^{(0)}$ and the triplet contribution to the scattering cross-section for zero energy neutrons can be calculated; no great accuracy is here needed, since this contribution is small. From the empirical value of the total cross-section, the singlet contribution is thus obtained and the singlet potential strength ${}^1j^{(0)}$ necessary to account for it can finally be found. One should then check the constants determined in this way by calculating the scattering cross-section for different values of the neutron energy; but apart from the case of the well potential, treated above, this has only been done (to some extent) for the meson potential.

Let us take this last case as an illustration. From the value of $|{}^3\eta^{(0)}|$, that of ${}^3b^{(0)}$ can be calculated by formula (5.231-25), while ${}^1b^{(0)}$ is given in terms of ${}^1\eta^{(0)}$ by the analogous formula (5.323-34a). In order to determine ${}^1\varepsilon^{(0)}$, we need not compute accurately the triplet contribution to the zero energy cross-section, but (following the example of BREIT *et al.* [39b, d]) we may simply put it equal to the "well" value (for a width

$D = 2,8 \cdot 10^{-13}$ cm, say); this amounts to adopting for $1\epsilon^{(0)}$ the corresponding "well" value, given in table 6.431-1. Inserting, in the formulae just quoted, the numerical values of $3\epsilon^{(0)}$ and $1\epsilon^{(0)}$, going over to ordinary units by means of (5.11-9) and introducing the meson mass $M_m = \hbar\kappa$ instead of κ , we get for the effective potentials

$$\begin{aligned}
 {}^3J^{(0)} &= 4,67 \cdot 10^{-4} \left(\frac{M_m}{m} \right)^2 + 5,57 \cdot 10^{-2} \frac{M_m}{m} - 0,535 \quad (\text{MeV}) \\
 {}^1J^{(0)} &= 4,67 \cdot 10^{-4} \left(\frac{M_m}{m} \right)^2 - 0,965 \cdot 10^{-2} \frac{M_m}{m} \quad (\text{MeV}).
 \end{aligned}
 \tag{8}$$

Calculations of scattering cross-sections using these potentials will more suitably be dealt with in a later Chapter (8.33).

We shall here bring together in tabular form results obtained by the procedure outlined above for different types of potentials. The assumed range values are those suggested by the analysis of the proton-proton scattering data, but in the case of the meson potential figures pertaining to a whole range of values of the meson mass are quoted for future reference.

6.432. Data on proton-neutron potentials					
Type of potential	Assumed range κ^{-1}	${}^3J^{(0)}$	Triplet wave contribution to zero energy cross-section	${}^1J^{(0)}$	${}^1J^{(0)}/{}^3J^{(0)}$
	10^{-13} cm	MeV	10^{-24} cm ²	MeV	
Well (5.11-2)	2,8	20,9	3,225	11,8	0,56
Exponential (5.11-3) *	1,73	123,3	—	≈ 74	0,6
MORSE (5.11-4) **	1,4	99,06	4,47	≈ 63	$\approx 0,63$
Gauss (5.11-5) ***	1,9	43,7	3,15	27,2	0,62
Meson (5.11-6) $M_m/m =$	200	1,9	29,3	16,7	0,57
	225	1,69	—	21,4	0,60
	248	1,57	—	26,3	0,62
	278	1,37	—	33,4	0,65
	326	1,17	—	46,5	0,69
HULTHÉN (5.11-7) †	1	60	—	38	0,63
HYLLERAAS (5.11-8) ††	1	112	—	77	0,69

* The value of ${}^3J^{(0)}$ is that given by RARITA and PRESENT [37] for the range indicated; that of ${}^1J^{(0)}$, which was derived by these authors from a superseded value of $\mathcal{S}$, has not been recalculated, but — following BUCKINGHAM and MASSEY [41] — the ratio q has just been put = 0,6.
 ** From the formula and graphs in the paper of MORSE *et al.* [36].
 *** According to numerical calculations of BREIT *et al.* [39a]; the value of ${}^1J^{(0)}$ is obtained by interpolation from their results.
 † MOSELEY and ROSEN [47]
 †† EISNER and ROSEN [47]

6.5. Radiative processes

6.50. The present section is devoted to a theoretical treatment of the photo-disintegration of the deuteron and a discussion of the inverse process of radiative capture of neutrons by protons. A note is added on the theory of the disintegration of the deuteron by electron impact.

6.51. *The photo-disintegration of the deuteron.* The way in which a deuteron in its ground state of binding can be dissociated by sufficiently hard γ -radiation is apt to give valuable information on the forces holding proton and neutron together. Thus the threshold frequency of the γ -rays immediately yields the binding energy; it is in fact this method which at the moment provides us with the best value of this constant (6.11). But, as we shall see later (8.34), the angular distribution of the ejected protons also gives rise to important theoretical inferences. In order to discuss this last phenomenon, especially for relatively small frequencies, it will be necessary to take into consideration not only the disintegrations due to the electric field of the γ -ray (*photo-electric effect*) but also those produced by its magnetic field (*photo-magnetic effect*). The first depend on electric dipole transitions of the deuteron*, the second on magnetic dipole transitions. In fact, if we assume the wave-length of the γ -ray to be large compared with the dimensions of the deuteron, the interaction energy with the electromagnetic field ($\vec{E}, \vec{H}$) of the radiation (treated as constant in the spatial region of interest) may to a sufficient approximation (4.42-5) be written as $-\vec{E}\vec{P} - \vec{H}\vec{M}$, $\vec{P}$ and $\vec{M}$ being the electric and magnetic dipole moments.

The differential cross-section of the photo-effect for a given polarization of the incident radiation is given by

$$d\Phi = \nu \overline{\sum_{\tau_2, e_2, \sigma_2} |\Psi(\infty)|^2} r^2 d\Omega; \quad (1)$$

here ν denotes the relative velocity of the ejected nucleons, determined, as well as the total kinetic energy ε , from the binding energy ε_0 of the ground state and the frequency $c\nu/2\pi$ of the incident radiation by the equation $\hbar\nu = \varepsilon + |\varepsilon_0|$; $\Psi(\infty)$ is the asymptotic value of the wave-function of the ground state, perturbed by the electromagnetic field; the bar indicates the average corresponding to the degeneracy of the ground state. The wave-function $\Psi(\infty)$ must be normalized to unity, and the magnitude of the field must be chosen so as to correspond to an intensity of one photon per cm^2 and per sec. Writing the interaction energy operator in the form

$$\mathcal{V}_{\text{el. magn.}} = \mathcal{W} e^{-i\nu t} + \mathcal{W}^\dagger e^{i\nu t}, \quad \text{with } \mathcal{W} = -\vec{E}_0 \vec{P} - \vec{H}_0 \vec{M}, \quad (2)$$

* It can be verified that electric quadrupole transitions play only a negligible role, compared to the other (PAIS [43]).

we must take

$$|\vec{E}_0|^2 = |\vec{H}_0|^2 = 4\pi \cdot \frac{\hbar\nu}{2c}. \quad (3)$$

It will appear that only the large components of the wave-function come into play; for these we may write *, on account of (4.34-28), (5.12-21), (5.111-13),

$$\Psi(\infty) \simeq {}^3(\varrho)_1 \cdot \sum_{\sigma, l, j, m} \int d\varepsilon \cdot a_0(\varepsilon; \sigma, l, j, m; t) \cdot {}^\sigma Z_j^{(l)m} \cdot \sqrt{\frac{2c}{\pi \hbar \nu}} \cdot \frac{\sin(kr - \frac{1}{2}l\pi + {}^\sigma\delta^{(l)})}{r};$$

evaluating the coefficients a and performing the integration over ε , we find **

$$\Psi(\infty) \simeq - {}^3(\varrho)_1 \cdot \sqrt{\frac{2\pi c}{\hbar \nu}} \cdot \frac{e^{ikr}}{r} \cdot e^{-iEct/\hbar} \sum_{\sigma, l, j, m} \tau(\tau)_0 \cdot {}^\sigma Z_j^{(l)m} \cdot \frac{e^{i\tau_\delta^{(l)}}}{i^l} \cdot (\varepsilon; \sigma, l, j, m | \mathcal{W} | 0), \quad (4)$$

where $(\varepsilon; \sigma, l, j, m | \mathcal{W} | 0)$ is a matrix-element of the operator $\mathcal{W}$, the ground state being denoted by the symbol 0.

According to the considerations of 4.43, we may take as electric dipole moment the operator given by (4.43-11), which clearly gives rise to transitions from the ${}^3S$ ground state to ${}^3P$ states of the continuum. As regards the magnetic moment operator, since it must operate on the ${}^3S$ state eigenfunction, it reduces to the form (4.42-6, 7, 8, 9); the operator $\vec{M}_{\text{nuc}}$ may be written

$$\begin{aligned} \vec{M}_{\text{nuc}} = & \mu_0 (\mu_n + \mu_p) \frac{\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}}{2} - \mu_0 (\mu_p - \mu_n) \frac{\vec{\tau}_3^{(1)} + \vec{\tau}_3^{(2)}}{2} \frac{\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}}{2} \\ & - \mu_0 (\mu_p - \mu_n) \frac{\vec{\tau}_3^{(1)} - \vec{\tau}_3^{(2)}}{2} \frac{\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}}{2}. \end{aligned} \quad (5)$$

From this expression, it is immediately apparent that the photomagnetic transitions due to $\vec{M}_{\text{nuc}}$ lead (through the last term of (5)) to ${}^1S$ -waves. The exchange moment, as a detailed investigation (PAIS [43]) shows, likewise gives rise to transitions to ${}^1S$ states, as well as (but to a negligible extent) to ${}^1D$ states. Summarizing, we thus get a superposition of a ${}^3P$, i.e. $\sin^2\vartheta$ -distribution and a spherically symmetrical ${}^1S$ distribution of ejected nucleons (in the barycentric system).

* The normalization must be referred to the ordinary (not reduced) energy scale, and to r instead of ξ (5.111).

** This method is due to BETHE, who applied it to the treatment of the atomic photo-effect. Cf. e.g. A. SOMMERFELD [39], Chapt. VI, § 7.

The total cross-section is the sum of the two effects:

$$d\Phi = d\Phi_{el} + d\Phi_{magn}. \quad (6)$$

The relevant matrix-elements are calculated without difficulty, by means of (4.32-17), from (1), (4), (2), (3); averaging further $d\Phi_{el}$ for an unpolarized incident radiation, one gets

$$d\Phi_{el} = \frac{e^2}{\hbar} \cdot \hbar\nu \cdot \frac{\pi}{8} |K_{el}|^2 \sin^2 \vartheta d\Omega$$

$$d\Phi_{magn} = \frac{\mu_0^2}{\hbar} \cdot \hbar\nu \cdot \frac{\pi}{3} |K_{magn}|^2 d\Omega, \quad (7)$$

where

$$K_{el} = \int_0^\infty R_0 \cdot {}^3R_t^{(1)} \cdot r dr$$

$$K_{magn} = (\mu_p - \mu_n) \int_0^\infty R_0 \cdot {}^1R_t^{(0)} dr - \int_0^\infty R_0 \cdot \mu(r) \cdot {}^1R_t^{(0)} dr, \quad (8)$$

R_0 designating the radial part of the ground state eigenfunction, and $\mu(r)$ the radial dependence of the exchange magnetic moment. The total cross-sections are given by

$$\Phi_{el} = \frac{e^2}{\hbar} \cdot \hbar\nu \cdot \frac{\pi^2}{3} |K_{el}|^2$$

$$\Phi_{magn} = \frac{\mu_0^2}{\hbar} \cdot \hbar\nu \cdot \frac{4\pi^2}{3} |K_{magn}|^2. \quad (9)$$

For frequencies just above the threshold, i.e. small velocities of the ejected nucleons, it is easy to see that the photomagnetic effect is predominant. In fact, for the eigenfunction ${}^1R_t^{(0)}$ we may take the form (5.32-11), which for small η is proportional to

$$\chi(\xi) \cdot \frac{\sin \delta}{\eta^{\frac{1}{2}}} \cdot (\sqrt{\eta} \xi \operatorname{ctg} \delta + 1),$$

i.e., by (5.32-20),

$${}^1R_t^{(0)} \sim \chi(\xi) \cdot \frac{\eta^{\frac{1}{2}}}{\sqrt{\eta + |{}^1\eta^{(0)}|}} \cdot (\sqrt{|{}^1\eta^{(0)}|} \xi \pm 1), \quad (10)$$

the sign of the last term being + or — according as the low energy ${}^1S$ level defining the phase is virtual or actual. On the other hand, we may in (8), replace ${}^3R_t^{(1)}$ by the corresponding function W obtained by neglecting the nuclear potential in comparison with the centrifugal forces (5.12):

$${}^3R_t^{(1)} \sim \frac{1}{\eta^{\frac{1}{2}}} \sqrt{z} J_{\frac{3}{2}}(z) \quad (z \equiv \sqrt{\eta} \xi)$$

$$\sim \frac{1}{\eta^{\frac{1}{2}}} \left(\frac{\sin z}{z} - \cos z \right)$$

$$\sim \frac{1}{\eta^{\frac{1}{2}}} \cdot \frac{z^2}{3} \sim \eta^{\frac{1}{2}} \xi^2. \quad (11)$$

From (8) it then follows, according to (10) and (11), that we may quite generally write, for small energies ε of the ejected nucleons,

$$\begin{aligned} |K_{el}|^2 &\approx \varepsilon^{\frac{1}{2}} F_{el}(\varepsilon) \\ |K_{magn}|^2 &\approx \frac{\sqrt{\varepsilon}}{\varepsilon + |\varepsilon^{(0)}|} F_{magn}(\varepsilon), \end{aligned} \quad (12)$$

the F 's being analytic functions of ε . The formulae (9), together with (12), finally show that for small ε the photomagnetic cross-section increases with ε much more rapidly than the photo-electric one. The photomagnetic effect will chiefly affect in a sensitive way the angular distribution of the ejected nucleons (8.34).

To get an approximate idea of the order of magnitude of the photo-disintegration cross-sections, we may in (8) replace R_0 by its asymptotic form $e^{-\sqrt{|\varepsilon_0|}} \frac{1}{2}$, with the normalization factor $\sqrt{2\pi} \sqrt{|\varepsilon_0|}$. Neglecting further the magnetic exchange effect, we may then easily evaluate the K 's and we get for the corresponding functions $F(\varepsilon)$ introduced in (12)

$$F_{el}(\varepsilon) \approx \frac{8}{\pi} \frac{\hbar^2}{M} \frac{\sqrt{|\varepsilon_0|}}{(\varepsilon + |\varepsilon_0|)^{\frac{1}{2}}} \quad (13)$$

$$F_{magn}(\varepsilon) \approx \frac{2}{\pi} (\mu_p - \mu_n)^2 \frac{\sqrt{|\varepsilon_0|}}{(\varepsilon + |\varepsilon_0|)^2} (\sqrt{|\varepsilon_0|} \pm \sqrt{|\varepsilon^{(0)}|})^2,$$

the $\pm$ sign pertaining to the virtual or actual character of the $1S$ level. Taking account of the energy relation $\hbar\nu = \varepsilon + |\varepsilon_0|$ and of the formulae (9), (12) and (5.32-22a), we get for the cross-sections

$$\begin{aligned} \Phi_{el} &= A \left(\frac{\hbar\nu}{|\varepsilon_0|} - 1 \right)^{\frac{1}{2}} \left(\frac{\hbar\nu}{|\varepsilon_0|} \right)^{-3} \\ \Phi_{magn} &= B \left(\frac{\hbar\nu}{|\varepsilon_0|} - 1 \right)^{\frac{1}{2}} \frac{\left(1 \pm \sqrt{\frac{|\varepsilon^{(0)}|}{|\varepsilon_0|}} \right)^2}{\frac{\hbar\nu}{|\varepsilon_0|} \left(\frac{\hbar\nu}{|\varepsilon_0|} - 1 + \sqrt{\frac{|\varepsilon^{(0)}|}{|\varepsilon_0|}} \right)}, \\ A &\equiv \frac{8}{3} \pi \frac{e^2}{\hbar} |\varepsilon_0^{(0)}|^2, \quad B \equiv \frac{8}{3} \pi \frac{\mu_0^2 (\mu_p - \mu_n)^2}{\hbar}. \end{aligned} \quad (14)$$

Using the numerical values (6.21-14), (1.21-2, 3), (1.22-6), one has

$$\begin{aligned} A &= 1.16 \cdot 10^{-26} \text{ cm}^2 \\ B &= 1.52 \cdot 10^{-28} \text{ cm}^2. \end{aligned} \quad (15)$$

6.52. Radiative capture of neutrons by protons. When slow neutrons pass through a substance rich in hydrogen, such as paraffin or water, they are not only elastically scattered, but also strongly absorbed. This absorption is due to capture of neutrons by the protons, leading to formation of deuterons with emission of radiation. Since this process is just the inverse of the photodisintegration, its cross-section G is related to that of the latter by an equation resulting from the well-known argument of detailed

balancing. This equation, valid separately for the capture process involving an electric dipole transition as well as for that involving a magnetic dipole transition is

$$G = \frac{3}{2} \left(\frac{\hbar \nu}{p} \right)^2 \Phi; \quad (16)$$

the momentum p/c of the incident neutron is related to that of the emitted photon $\hbar \nu/c$ by the law of energy conservation, so that we may write, with (5.111-13),

$$G = \frac{3}{2} \frac{(\varepsilon + |\varepsilon_0|)^2}{M\varepsilon} \Phi. \quad (17)$$

The factor $\frac{3}{2}$ is the quotient of the statistical weights 2.3 of photon (two states of polarization) and deuteron (multiplicity of ground state) by those, 2.2, of free neutron and proton, arising from the spins. Combining (17) with (9) and (12), we see that for slow neutrons, the "electric" capture cross-section is negligible in comparison with the "magnetic" capture cross-section, which is inversely proportional to $\sqrt{\varepsilon}$, i.e. to the velocity v of the neutrons ("1/ v -law"). (The reason for the smallness of the electric dipole effect is that such transitions entail an initial state of orbital angular momentum $l = 1$ and that a slow neutron with this angular momentum has too little chance to come sufficiently near the proton to be captured.)

The first measurements of the capture cross-section were performed by AMALDI and FERMI [36] according to a method developed theoretically by FERMI [36]. The mean life of a neutron of velocity v in paraffin — from which the capture cross-section immediately follows — is given by lN/v , l being the mean free path and N the mean number of free paths travelled by the neutron before capture. The first of these quantities may be measured directly by letting neutrons pass through increasing thicknesses of paraffin. The second can be deduced from a study of the diffusion of the neutrons in paraffin: the probability for a neutron to reach the surface of a paraffin block from a depth x before being captured is found to be of the form $e^{-x/l'}$, with $l' = l\sqrt{N/3}$; the determination of l' then yields N . For neutrons of ordinary thermal energies, this method gives a cross-section *

$$G = 0.31 \cdot 10^{-24} \text{ cm}^2. \quad (18)$$

Very nearly the same result, viz. $0.33 \cdot 10^{-24} \text{ cm}^2$, has been derived by MANLEY, HAWORTH and LUEBKE [42] from a direct measurement of the mean life-time of the neutrons in water; the accuracy of this value is estimated at about 5 %, the chief uncertainty being that in the neutron velocity.

Other determinations are obtained by comparing the capture cross-section of hydrogen with that of boron, silver or cadmium; the latter may

* Recently, however, GAMERTSFELDER and GOLDBABER [46] have given a value as low as $0.25 \cdot 10^{-24} \text{ cm}^2$, for which they claim 10 % accuracy.

in fact, on account of the smallness of scattering, be identified with the total absorption cross-section, which is readily measured. The relative measurements of the capture cross-sections are made with liquids or solutions containing the substances investigated in different concentrations; various methods of comparison can be used, a survey of which is given by VOLZ [43]. Owing to the uncertainty prevailing until recently in the absolute values of the cross-sections of the comparison substances, the precision of the absolute value of the hydrogen cross-section deduced from such relative measurements was considerably impaired. With boron as comparison element, one obtained for G values of about $0,27$ to $0,31 \cdot 10^{-24}$ cm² (FRISCH, HALBAN and KOCH [37], SCHULZ and GOLDBERGER [45]); with cadmium (HAXEL and VOLZ [43]), one gets $0,235$ to $0,32 \cdot 10^{-24}$ cm², the latter determination corresponding to the most recent cadmium cross-section value (DIEBNER *et al.* [42]). A new determination, however, using monochromatized neutrons from a uranium pile and boron as comparison substance, has definitely confirmed the value (18) of the capture cross-section, which moreover can now be referred to a precise neutron velocity of $2,2 \cdot 10^5$ cm/sec.

The cross-section (18) is much too large to be accounted for by the electric dipole effect. To the same approximation as the above formula (13), we get from (17)

$$G_{el} = C \frac{\sqrt{|\varepsilon_0|} \varepsilon}{\varepsilon + |\varepsilon_0|}, \quad C \equiv 4\pi \frac{e^2}{\hbar} \left(\frac{\hbar}{M} \right)^2 = 4,02 \cdot 10^{-29} \text{ cm}^2, \quad (19)$$

which takes its maximum value $\frac{1}{2}C$ for $\varepsilon = |\varepsilon_0|$. It was just the need to provide for some other capture mechanism capable of explaining the large cross-section observed for slow neutrons which led FERMI [35, 36] to consider for this purpose the magnetic dipole transitions. In fact, the latter effect yields a capture cross-section following the $1/v$ -law, which can thus become very large at small velocities. At this point it must be stressed that this effect exists only in so far as there is a spin dependent nuclear interaction: otherwise, there being no difference between ^1S and ^3S states, the first term in the expression (8) for K_{magn} would vanish because the eigenfunctions occurring in it would be orthogonal, while the last term would disappear because there would then be no exchange magnetic moment (4.43). The large capture cross-section of protons for slow neutrons therefore provides us with an additional qualitative argument of great weight for the existence of a spin dependent nuclear potential.

More quantitatively, we get*, neglecting the exchange effect as in formula (13),

$$G_{\text{magn}} \approx D \sqrt{\frac{|\varepsilon_0|}{\varepsilon}} \frac{\varepsilon + |\varepsilon_0|}{\varepsilon + |^1\varepsilon^{(0)}|} \cdot \left(1 \pm \sqrt{\frac{|^1\varepsilon^{(0)}|}{\varepsilon_0}} \right)^2, \quad (20)$$

$$D \equiv 4\pi \frac{\mu_0^2}{\hbar} (\mu_p - \mu_n)^2 \frac{|\varepsilon_0|}{M} = 0,53 \cdot 10^{-30} \text{ cm}^2.$$

* Formula (20) clearly illustrates the vanishing of the magnetic effect for $^1\varepsilon^{(0)} = -\varepsilon_0$.

Taking (table 6.431-1) $|1\epsilon^{(0)}| = 0,065$ MeV, i.e. $|1\epsilon^{(0)}/\epsilon_0| = 0,03$, this formula yields, for thermal neutrons ($E = 0,026$ MeV, i.e. $\epsilon = 0,013$ MeV),

$$G_{\text{magn}} \approx 0,23 \cdot (1 \pm 0,17)^2 \cdot 10^{-24} \text{ cm}^2$$

$$= \begin{cases} 0,32 \cdot 10^{-24} \text{ cm}^2 & \text{for a virtual } ^1S \text{ level} \\ 0,16 \cdot 10^{-24} \text{ cm}^2 & \text{for an actual } ^1S \text{ level.} \end{cases} \quad (20a)$$

These results are, in the first place, of the right order of magnitude; they further distinctly favour the case of a virtual ^1S level. But (chiefly because of the neglect of the exchange effect) the calculation is not sufficiently accurate to allow us to regard this evidence as entirely conclusive.

6.53. Disintegration of the deuteron by electron impact. In principle, the disintegration of the deuteron can be brought about by the electromagnetic field of an impinging electron just as well as by that of a photon. Experiments to this effect have hitherto not been attempted, owing to the difficulty of producing beams of electrons of the required energies. But the analogous reaction with the ^9Be nucleus, which has a smaller threshold (1,63 MeV), has already been produced by GUTH and his collaborators [39a, b], using electrons of up to 1,75 MeV energy, accelerated by a van de Graaf generator. The theoretical treatment of this process follows the same general lines as that of the photo-effect; modifications arise on the one hand from the different form of electromagnetic field, on the other (and more important still) from the fact that the electron, unlike the photon, is not absorbed in the course of the reaction, but emerges with a diminished energy.

If the transition performed by the electron is from a state i (of momentum $\vec{p}_i^{(e)}$, energy $E_i^{(e)}$) * to a state f , so that the momentum and energy transferred to the nucleus are

$$\vec{p} = \vec{p}_i^{(e)} - \vec{p}_f^{(e)} \quad , \quad E = E_i^{(e)} - E_f^{(e)} \quad , \quad (21)$$

the effective electromagnetic field by which the electron acts on the nucleus may, according to the principles of quantum electrodynamics **, be represented (in a notation similar to that used above in the photon case) by

$$\vec{E}_0 e^{iEct/\hbar} + \text{complex conj.} \quad , \quad \vec{H}_0 e^{iEct/\hbar} + \text{compl. conj.} \quad ,$$

with

$$\vec{E}_0 = \frac{1}{i} \frac{4\pi\hbar e}{v_i^{(e)}} \frac{1}{p^2 - E^2} \cdot [E(i|\varrho_i^{(e)}\vec{\sigma}^{(e)}|f) - \vec{p}(i|1|f)]$$

$$\vec{H}_0 = \frac{1}{i} \frac{4\pi\hbar e}{v_i^{(e)}} \frac{1}{p^2 - E^2} \cdot \vec{p} \wedge (i|\varrho_i^{(e)}\vec{\sigma}^{(e)}|f) \quad . \quad (22)$$

* All quantities referring to the electron are distinguished by an upper index (e) .

** Cf. C. MÖLLER, *Ann. Physik* 14, 531. 1932.

These formulae are only valid for sufficiently small values of the momentum transfer p ; they represent in fact, to the first approximation, the field produced by the effective charge and current distribution of respective densities

$$\frac{e}{v_i^{(e)}} e^{-\frac{i}{\hbar} (\vec{p} \cdot \vec{x}^{(e)} - Ect)} (i | 1 | f) \quad , \quad \frac{e}{v_i^{(e)}} e^{-\frac{i}{\hbar} (\vec{p} \cdot \vec{x}^{(e)} - Ect)} (i | \varrho_1^{(e)} \vec{\sigma}^{(e)} | f).$$

The normalization factor $1/v_i^{(e)}$ ($v_i^{(e)} = c p_i^{(e)} / E_i^{(e)}$ being the initial velocity of the electron) corresponds to an intensity of one electron per cm^2 and per sec.

For a definite transition $i \rightarrow f$ of the electron, the total cross-section of the disintegration process is, exactly like that of the photo-effect (and with the same notations), the sum of an electric and a magnetic contribution

$$\begin{aligned} \Phi_{el}(i \rightarrow f) &= \frac{\pi e^2 c}{6 \hbar} |K_{el}|^2 |\vec{E}_0|^2 \\ \Phi_{mag}(i \rightarrow f) &= \frac{2\pi \mu_0^2 c}{3 \hbar} |K_{mag}|^2 |\vec{H}_0|^2. \end{aligned} \quad (23)$$

(Also the angular dependence of the electric and magnetic effects is the same. The differential cross-sections are obtained by multiplying the total ones by $\frac{3}{8\pi} \sin^2 \vartheta d\Omega$ and $\frac{1}{4\pi} d\Omega$, respectively.) If the impinging electron beam is unpolarized, we have to average these quantities over the initial spin orientations; moreover, we must sum over the final spin orientations. One has, in the usual way,

$$\begin{aligned} \frac{1}{2} \sum_{(spin)} \sum |\vec{E}_0|^2 &= \frac{1}{2} \text{tr} \left[\vec{E}_0 \cdot \frac{E_i^{(e)} + \varrho_1^{(e)} \vec{\sigma}^{(e)} p_i^{(e)} + m \varrho_3^{(e)}}{2 E_i^{(e)}} \vec{E}_0 \frac{E_f^{(e)} + \varrho_1^{(e)} \vec{\sigma}^{(e)} p_f^{(e)} + m \varrho_3^{(e)}}{2 E_f^{(e)}} \right] \\ &= \frac{16\pi^2 \hbar^2 e^2}{v_i^{(e)2} E_i^{(e)} E_f^{(e)}} \left[-\frac{1}{4} + \frac{1}{2} \frac{E_i^{(e)2} + E_f^{(e)2}}{p^2 - E^2} - \frac{E^2 m^2}{(p^2 - E^2)^2} \right] \\ \frac{1}{2} \sum_{(spin)} \sum |\vec{H}_0|^2 &= \frac{16\pi^2 \hbar^2 e^2}{v_i^{(e)2} E_i^{(e)} E_f^{(e)}} \left[\frac{1}{4} + \frac{1}{2} \frac{p_i^{(e)2} + p_f^{(e)2}}{p^2 - E^2} - \frac{E^2 m^2}{(p^2 - E^2)^2} \right]. \end{aligned} \quad (24)$$

Finally, the cross-sections must be summed over all possible values of the momentum $\vec{p}_f^{(e)}$ of the electron after the collision. For this purpose, they have to be multiplied by the number of final states in an element of momentum space, viz.

$$\frac{p_f^{(e)} E_f^{(e)} v_i^{(e)}}{8\pi^3 \hbar^3} dE_f^{(e)} d\Omega_f^{(e)}.$$

and integrated over all angles and over $E_f^{(e)}$ from m to $E_i^{(e)} - |\varepsilon_0|$. The

angle variables occur in the integrand only through

$$p^2 - E^2 = 2 (E_i^{(e)} E_f^{(e)} - \vec{p}_i^{(e)} \vec{p}_f^{(e)} - m^2);$$

performing the angular integration, one gets the general formulae

$$\begin{aligned} \Phi_{el}^{(e)} &= \frac{\pi}{3} \left(\frac{e^2}{\hbar} \right)^2 \int_m^{E_i^{(e)} - |\varepsilon_0|} dE_f^{(e)} |K_{el}|^2 \cdot \\ &\quad \cdot \left[\frac{E_i^{(e)2} + E_f^{(e)2}}{p_i^{(e)2}} \log \frac{E_i^{(e)} E_f^{(e)} + p_i^{(e)} p_f^{(e)} - m^2}{(E_i^{(e)} - E_f^{(e)}) m} - 2 \frac{p_f^{(e)}}{p_i^{(e)}} \right] \\ \Phi_{magn}^{(e)} &= \frac{4\pi}{3} \cdot \frac{\mu_0^2}{\hbar} \cdot \frac{e^2}{\hbar} \int_m^{E_i^{(e)} - |\varepsilon_0|} dE_f^{(e)} |K_{magn}|^2 \cdot \\ &\quad \cdot \frac{p_i^{(e)2} + p_f^{(e)2}}{p_i^{(e)2}} \log \frac{E_i^{(e)} E_f^{(e)} + p_i^{(e)} p_f^{(e)} - m^2}{(E_i^{(e)} - E_f^{(e)}) m}. \end{aligned} \quad (25)$$

The energy ε of the nucleons after the disintegration, which occurs in the K 's, is given by $\varepsilon = E - |\varepsilon_0|$.

For electrons of relatively low energy, we may expand the expressions (25) in powers of $W^{\frac{1}{2}}$, if

$$W = E_i^{(e)} - m - |\varepsilon_0| \quad (26)$$

denotes the energy of the impinging electron above the threshold. To this end, we put

$$E_f^{(e)} = m + Wx$$

and integrate over x from 0 to 1. Since

$$\varepsilon = W(1-x)$$

is also small, we may use the expressions (12) for the K 's. Expanding the integrands (but retaining in $|K_{magn}|^2$ the factor $[|1_{\varepsilon(0)}| + W(1-x)]^{-1}$, since $|1_{\varepsilon(0)}|$ may be of the same order as W), we readily find

$$\begin{aligned} \Phi_{el}^{(e)} &\approx \frac{\pi\sqrt{2}}{6} \cdot \left(\frac{e^2}{\hbar} \right)^2 \cdot \frac{\pi}{8} F_{el}(0) \cdot \frac{|\varepsilon_0|^2 + 2m^2}{|\varepsilon_0|^{\frac{1}{2}} \sqrt{(|\varepsilon_0| + 2m)m}} \cdot W^3 \\ \Phi_{magn}^{(e)} &\approx \frac{2\pi\sqrt{2}}{3} \cdot \frac{e^2}{\hbar} \cdot \frac{\mu_0^2}{\hbar} \cdot 2\pi F_{magn}(0) \cdot \frac{\sqrt{|\varepsilon_0| + 2m}}{\sqrt{|\varepsilon_0|m}} \cdot |1_{\varepsilon(0)}| \cdot \\ &\quad \cdot \left[1 + \frac{W}{2|1_{\varepsilon(0)}|} - \sqrt{1 + \frac{W}{|1_{\varepsilon(0)}|}} \right]. \end{aligned} \quad (27)$$

Using the approximate formula (13), we get *

$$\begin{aligned}
 \Phi_{\text{el}}^{(e)} &\approx \frac{\pi \sqrt{2}}{6} \cdot \left(\frac{e^2}{\hbar}\right)^2 \cdot \frac{\hbar^2}{M |\varepsilon_0|} \cdot \frac{|\varepsilon_0|^2 + 2m^2}{|\varepsilon_0|^4 \sqrt{(|\varepsilon_0| + 2m)m}} \cdot W^3 \\
 &\approx 1,36 \cdot 10^{-30} W^3 \text{ cm}^2 \\
 \Phi_{\text{magn}}^{(e)} &\approx \frac{2\pi \sqrt{2}}{3} \cdot \left(\frac{e^2}{\hbar}\right)^2 \cdot \frac{m}{M} \cdot \frac{\hbar^2}{M |\varepsilon_0|} \cdot (\mu_p - \mu_n)^2 \cdot \left(1 \pm \sqrt{\frac{1_{\varepsilon^{(0)}}}{\varepsilon_0}}\right)^2 \cdot \\
 &\quad \cdot \frac{\sqrt{|\varepsilon_0| + 2m} \cdot |1_{\varepsilon^{(0)}}|}{m^2} \cdot \left[1 + \frac{W}{2|1_{\varepsilon^{(0)}}|} - \sqrt{1 + \frac{W}{|1_{\varepsilon^{(0)}}|}}\right] \\
 &\approx 1,6 \cdot 10^{-31} [1 + 7,5 W - \sqrt{1 + 15 W}] \text{ cm}^2,
 \end{aligned} \tag{28}$$

W being expressed in MeV.

The formulae (28) have been derived by PETERS and RICHMAN [41]. The influence of the magnetic exchange effect has been discussed by LUBANSKI and ROSENFELD [45].

* In the last formula, it has been assumed that the $1S$ level is virtual, with $1_{\varepsilon^{(0)}} = 0,065 \text{ MeV}$ (table 6.43I-1).

CHAPTER VII

CENTRAL INTERACTION BETWEEN PROTONS

7.0. This Chapter will chiefly be concerned with the analysis of the scattering of protons by protons, which yields quite precise information on the effective potential of the nuclear force between two protons in a ^1S state; this effective potential depends only on the central part of the nuclear interaction (4.431). The influence of non-central couplings will only make itself felt for higher proton energies than those hitherto studied in sufficient detail. It has been thought advisable also to include a short account of the interesting reaction in which a collision of two protons leads to the formation of a deuteron with positron emission.

7.1. Proton-proton scattering.

7.11. Theory. The main force acting at large distances between two protons is the electrostatic repulsion, and it will determine the scattering as long as the colliding particles do not come into the range of the proper nuclear force. If E is the kinetic energy of the protons, the distance r_0 of closest approach in a head-on collision (angular momentum $l = 0$) is defined by $e^2/r_0 = E$, whence $r_0 = (e^2/m) (m/E)$. In order that the nuclear force, of range $\approx e^2/m$, can come into play, the kinetic energy of the impinging protons must therefore be of the order of magnitude of several 100 keV. For such energies, the protons of the scatterer may be treated as free; the transition between barycentric and laboratory system of reference will be effected by the formulae (6.212–20). The treatment of the scattering of free protons differs from that of neutrons by protons in two respects. In the first place, the influence of the long range Coulomb interaction makes itself felt even on the asymptotic behaviour of incident and scattered wave; in the second, account must be taken of the peculiar way in which the exclusion principle modifies the scattered intensity.

Let us first consider the pure Coulomb scattering, governed by the radial equation

$$\frac{d^2 R}{dr^2} + \left[k^2 - \frac{2\alpha k}{r} - \frac{l(l+1)}{r^2} \right] R = 0, \quad (1)$$

where k is again (5.111–13) 2π times the wave-number of the incident wave, while

$$\alpha = \frac{Me^2}{2\hbar^2 k} = \frac{Me^2}{2\hbar p} = \frac{e^2 c}{\hbar v}, \quad (2)$$

v denoting the relative velocity; indeed, with our units, v/c is the quotient of the relative momentum p by the reduced mass $\frac{1}{2}M$. For given l and k , this equation has two independent solutions of which the one is regular ($\sim r^{l+1}$), the other irregular ($\sim r^{-l}$) at the origin; the asymptotic forms of these solutions are found* to be (with corresponding normalizations)

$$\begin{aligned} L_l^{(\text{reg})} &\simeq \sin(kr - l\pi/2 + \delta_c^{(l)}) \\ L_l^{(\text{irreg})} &\simeq \cos(kr - l\pi/2 + \delta_c^{(l)}) \end{aligned} \quad (3)$$

with

$$\delta_c^{(l)} = \gamma^{(l)} - a \log 2kr, \quad \gamma^{(l)} = \arg \Gamma(i\alpha + l + 1). \quad (4)$$

They may be compared with the solutions $\sqrt{r} J_{\pm(l+\frac{1}{2})}(kr)$ of the differential equation without field, from which they differ asymptotically by the phase shift $\delta_c^{(l)}$. As an extension of formula (6.21-4), it may be seen that the regular solution

$$\Psi_c^{(+)} = \frac{1}{kr} \sum_l (2l+1) i^l P_l(\cos \vartheta) L_l^{(\text{reg})}(r) e^{i\gamma^{(l)}} \quad (5)$$

is suited to describe the scattering process; in fact, its asymptotic form is**

$$\Psi_c^{(+)} \simeq e^{ikz} e^{i\alpha \log k(r-z)} - \frac{\alpha e^{ikr}}{k(r-z)} e^{-i\alpha \log k(r-z) + 2i\gamma^{(0)}}, \quad (6)$$

representing an incident (plane) wave and an outgoing spherical wave, both distorted by the influence of the Coulomb field; the second term we shall denote by $(\Psi_c^{(+)})_{\text{scatt}}$. We have only to take care of the exclusion principle by suitably symmetrizing the solution $\Psi_c^{(+)}$. For this purpose, introduce

$$\Psi_c^{(-)}(\vartheta) \equiv \Psi_c^{(+)}(\pi - \vartheta), \quad \text{i. e.} \quad \Psi_c^{(-)}(z) \equiv \Psi_c^{(+)}(-z), \quad (7)$$

which represents, as a matter of fact, the other collision partner; if $\Psi_c^{(+)}$ describes the deflection of the impinging proton, $\Psi_c^{(-)}$ will describe the recoil of a proton from the scatterer. Recoil protons also form a part of the scattered wave, and interfere with the impinging wave in the way prescribed by the exclusion principle, according to the relative spin orientation of the two waves; for the total wave, we thus get

$$\Psi_c = {}^1J_0 \cdot {}^1(\sigma)_0 (\Psi_c^{(+)} + \Psi_c^{(-)}) + \sum_{m_s} {}^3J_{m_s} \cdot {}^3(\sigma)_{m_s} (\Psi_c^{(+)} - \Psi_c^{(-)}), \quad (8)$$

with arbitrary amplitudes ${}^gJ_{m_s}$. Owing to the symmetry properties of the

* Cf., e.g., SOMMERFELD [39], Chapt. II, § 7. In the notation of W & W 16.1, the regular solution is $\sim M_{l\alpha, l+\frac{1}{2}}(2ikr)$.

** Cf. SOMMERFELD [39], Chapt. II, § 9. For a deduction of (6) from (5), see W. GORDON, *Z. Physik* 48, 180, 1928.

Legendre polynomials, $\Psi_c^{(+)} \pm \Psi_c^{(-)}$ is simply twice an expression of the form (5), in which the summation over l must be restricted to the even or odd values of l , respectively; this is, of course, precisely the restriction embodied in (4.331-23) when the isotopic eigenfunction is ${}^3(\tau)_{-1}$. Asymptotically, we have, by (6),

$$(\Psi_c^{(+)} \pm \Psi_c^{(-)})_{\text{scatt}} \simeq -\frac{ae^{ikr}}{2kr} e^{-i\alpha \log 2kr + 2i\gamma(0)} \left(\frac{e^{-i\alpha \log \sin^2 \vartheta/2}}{\sin^2 \vartheta/2} \pm \frac{e^{-i\alpha \log \cos^2 \vartheta/2}}{\cos^2 \vartheta/2} \right). \quad (9)$$

From (8) we obtain for the differential scattering cross-section

$$dS_c = r^2 d\Omega [|J_0|^2 \cdot |\Psi_c^{(+)} + \Psi_c^{(-)}|^2_{\text{scatt}} + \sum_{m_s} |J_{m_s}|^2 \cdot |\Psi_c^{(+)} - \Psi_c^{(-)}|^2_{\text{scatt}}],$$

the incident wave being normalized not to unity, but * to 2; for an unpolarized wave, we must therefore make all $|J_{m_s}|^2 = \frac{1}{4}$ and get from (9)

$$dS_c = d\Omega \frac{a^2}{k^2} \left[\frac{1}{(2 \sin^2 \vartheta/2)^2} + \frac{1}{(2 \cos^2 \vartheta/2)^2} + \frac{(\frac{1}{4} - \frac{3}{4}) 2 \cos(\alpha \log \tan^2 \vartheta/2)}{4 \sin^2 \vartheta/2 \cos^2 \vartheta/2} \right],$$

or, going over to the laboratory system,

$$dS_c = 4 \cos \Theta d\Omega_{\text{lab}} \frac{a^2}{4k^2} \left[\frac{1}{\sin^4 \Theta} + \frac{1}{\cos^4 \Theta} - \frac{\cos(\alpha \log \tan^2 \Theta)}{\sin^2 \Theta \cos^2 \Theta} \right]. \quad (10)$$

This is the well-known formula first derived by Mott. The first term is just Rutherford's classical expression; the second is the same expression for the recoil protons; the third results from the exclusion principle.

The modification of the Coulomb scattering, due to a short range potential, can now be derived in a way entirely analogous to the method of 6.21. For given l and k , the asymptotic form of the solution may be written in the form

$$L_l \simeq \sin(kr - l\pi/2 + \delta_c^{(l)} + {}^\sigma\delta^{(l)}), \quad (11)$$

as a linear combination of the two asymptotic solutions (3); the constant phase-shift ${}^\sigma\delta^{(l)}$ depends mainly on the short range potential ${}^\sigma V^{(l)}$ (and to a lesser degree also on the presence of the Coulomb potential). The general solution

$$\Psi = \sum' {}^\sigma R_j^{(l)m} \cdot {}^\sigma Z_j^{(l)m} \cdot \frac{1}{r} L_l(r),$$

in which $\sum'$ denotes the restricted summation over even or odd l according as $\sigma = 1$ or 3, must be determined in such a way that $\Psi - \Psi_c$ reduce asymptotically to an outgoing wave only. This condition, using (8) and

* Cf. N. MOTT and H. MASSEY, *The theory of atomic collisions* (1933), p. 69 (footnote).

(5), gives for the scattered wave

$$\Psi_{\text{scatt}} = \sum_{\sigma, m_s} {}^{\sigma}J_{m_s} \cdot {}^{\sigma}(\sigma)_{m_s} \cdot {}^{\sigma}\Psi_{\text{scatt}}, \quad (12)$$

with

$${}^{\sigma}\Psi_{\text{scatt}} \simeq (\Psi_c^{(+)} \pm \Psi_c^{(-)})_{\text{scatt}} + \sum' \sqrt{4\pi(2l+1)} Y_l^0 \frac{e^{ikr}}{ikr} e^{i(2\gamma^{(l)} - \alpha \log 2kr)} [e^{2i\sigma\delta^{(l)}} - 1], \quad (13)$$

the double sign being + or — according as $\sigma = 1$ or 3; the symbol $\sum'$ has just been explained. By (9) we may write

$${}^{\sigma}\Psi_{\text{scatt}} \simeq \frac{e^{ikr}}{ikr} e^{-i\alpha \log 2kr} \left[\frac{\alpha}{2i} \left(\frac{e^{-i\alpha \log \sin^2 \vartheta/2}}{\sin^2 \vartheta/2} \pm \frac{e^{-i\alpha \log \cos^2 \vartheta/2}}{\cos^2 \vartheta/2} \right) e^{2i\gamma^{(0)}} + \sum' \sqrt{4\pi(2l+1)} Y_l^0 e^{2i\gamma^{(l)}} (e^{2i\sigma\delta^{(l)}} - 1) \right]. \quad (14)$$

For unpolarized waves, the cross-section is built up from the singlet and triplet contributions in the ratio (1 : 3):

$$dS = \frac{1}{4} d^1S + \frac{3}{4} d^3S \quad (15)$$

$$d^{\sigma}S = |{}^{\sigma}\Psi_{\text{scatt}}|^2 r^2 d\Omega.$$

Formula (14) shows that the total scattering consists of the pure Coulomb scattering, as given by Mott's formula (10), and of an "anomaly" due to the scattering of the waves of different angular momenta by the nuclear potential and to interference between these waves and the Coulomb wave. Keeping in mind that the phases $\sigma\delta^{(l)}$ decrease with increasing l , we may from (14), (15) evaluate the cross-section with increasing accuracy by successively taking into account the terms corresponding to $l = 0, 1, 2, \dots$. The main contribution will be the S-wave anomaly, coming from the singlet configuration (antiparallel spins of the colliding protons); it arises from the nuclear scattering of the S-wave and from its interference with the Coulomb wave:

$$dS_{\text{anom}}^{(0)} = 4 \cos \Theta d\Omega_{\text{lab}} \frac{\alpha^2}{4k^2} \left[\frac{4}{\alpha^2} \sin^2 \delta - \frac{2}{\alpha} \sin \delta \left(\frac{\cos(\delta + \alpha \log \sin^2 \Theta)}{\sin^2 \Theta} + \frac{\cos(\delta + \alpha \log \cos^2 \Theta)}{\cos^2 \Theta} \right) \right], \quad (16)$$

where we have used the laboratory variables and put, for simplicity, $\delta \equiv {}^1\delta^{(0)}$. Next in importance comes a *P*-wave anomaly, from the triplet configuration (parallel spins); this anomaly also comprises a pure scattering and an interference term. There follows a *D*-wave anomaly, again from the singlet configuration, which consists of three terms, since there is also interference between the *D*- and the *S*-wave. These anomalies have been calculated by BREIT and his collaborators [36a, 39a], who also give an

elaborate discussion of the procedure best suited to derive from the observed angular distribution of the scattering intensity the values of the phases at the corresponding energy. It appears that for most of the hitherto utilized proton energies (in fact for energies up to a few MeV) practically the whole observed deviation from Mott's formula is due to the *S*-wave anomaly.

About the behaviour of the ratio

$$\mathfrak{R} \equiv \frac{d\mathcal{S}_{\text{obs}}}{d\mathcal{S}_c} \approx 1 + \frac{d\mathcal{S}_{\text{anom}}^{(0)}}{d\mathcal{S}_c}, \quad (17)$$

some theoretical predictions can be made, which, as we shall see, allow us to draw important conclusions on the nature of the nuclear forces. To begin with, we expect, on account of (10) and (16), the ratio $\mathfrak{R}$ at a given energy to exhibit a maximum or minimum for $\Theta = 45^\circ$, about which value it will vary symmetrically with angle, tending to unity for $\Theta \rightarrow 0$ and $\Theta \rightarrow 90^\circ$. But the nature of the extreme, as well as the more detailed behaviour of the angular variation will depend in a characteristic way on the *sign* of the nuclear proton-proton force. If this force is repulsive, it will always enhance the effect of the Coulomb field, so that $\mathfrak{R}$ will always be larger than unity and have a maximum at $\Theta = 45^\circ$. The case of an attractive force, however, is more complicated. If the energy of the impinging protons is sufficiently high, the nuclear field (which now counteracts the Coulomb repulsion) will predominate, giving $\mathfrak{R} > 1$; but this preponderance will depend on the scattering angle, since the Coulomb scattering becomes very large at small angles. As the energy decreases, and the influence of the nuclear potential consequently becomes weaker, $\mathfrak{R}$ will therefore become < 1 at small angles, while remaining > 1 at larger angles, and the region where it is < 1 will progressively expand. At still lower energies, $\mathfrak{R}$ will have become < 1 for all angles; by further decrease of the energy, the maximum of $\mathfrak{R}$ at 45° will change over to a minimum. During all this variation, the value of $\mathfrak{R}$ at any angle will decrease until it reaches a minimum; after which, as the Coulomb field begins to predominate over the nuclear field (at that angle), $\mathfrak{R}$ increases again towards the limiting value unity. We thus see that, in contrast to the case of proton-neutron scattering (the *S*-part of which only depends on the absolute values of the phases), the scattering of protons by protons is quite sensitive to the sign of the nuclear potential; this is due, of course, to the fact that we have here to do with an interference effect between Coulomb and nuclear scattering.

7.12. Analysis of experiments. The technique of absolute measurements of proton-proton scattering has been developed to a high degree of accuracy, chiefly by TUVE, HEYDENBURG and HAFSTAD [36, 38, 39] and by HERB and his collaborators (HERB *et al.* [39], RAGAN *et al.* [41]). The apparatus is clearly described by fig. 7.12-1, which represents TUVE

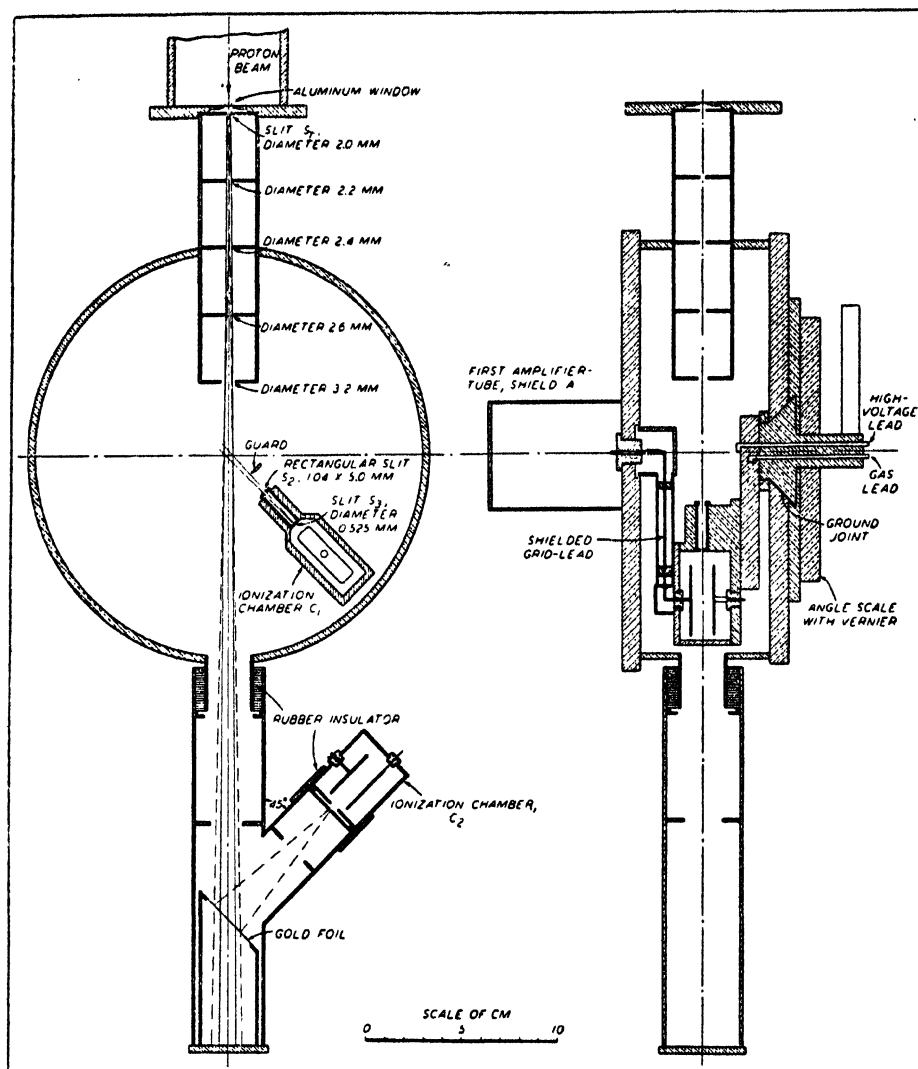


Fig. 7.12-1. Proton-proton scattering apparatus, according to TUVE, HEYDENBERG and HAFSTAD [39].

et al.'s improved design. The scattering chamber, in the form of a flat cylinder, is filled with hydrogen at 12 mm pressure; it contains a movable detector which registers the protons scattered at various angles in a volume defined by the divergence of the primary (incident) proton beam and the convergence of the detector slit-system. As a detector, the ionization chamber connected with a linear amplifier ceases to be satisfactory for protons with a residual range less than about 3 mm; to record slower protons (TUVE *et al.* [38], RAGAN *et al.* [41]), one has to replace it by some variety of point counter.

For each value of the energy of the incident protons, the results of the experiments are conveniently expressed by giving the ratio $\mathcal{R}$, defined by

formula (17), of the observed differential cross-section to that corresponding to pure Coulomb-scattering, in terms of the scattering angle Θ . Since $\mathfrak{R}(\Theta)$ is expected to be symmetrical about $\Theta = 45^\circ$, most of the measurements are limited to the angular range $0^\circ \dots 45^\circ$, but some of them (TUVE *et al.* [38, 39], RAGAN *et al.* [41]) have been extended up to 60° to check the symmetry. Results obtained by this method are now available for the whole energy range between 0.175 and 2.4 MeV. Besides, an investigation of the scattering of 4.2 MeV protons by protons has been carried out by MAY and POWELL (POWELL [47a]), using the latter's photographic method (6.42). All these measurements will be referred to as the "first group".

A "second group" of experiments, performed by WILSON and his collaborators [47a, b, c], covers the range of 8 ... 14.5 MeV proton energies. The protons from a cyclotron are scattered by the hydrogen atoms in a cellophane or Nylon foil. The scattered and the recoiling particles are recorded in coincidence by two counters fixed on rotating arms at 90° , with the scattering foil at the centre of rotation; in this way, spurious particles are eliminated. In this energy region, there is no point in considering the ratio $\mathfrak{R}$: the results of the (relative) measurements are described by giving, for each energy ($E = 8, 10$ and 14.5 MeV), the ratio $\mathfrak{S}(\Theta)$ of the differential cross-section at angle Θ to that at $\Theta = 45^\circ$ (cf. fig. 7.131). For $E = 8$ MeV, the absolute scattering cross-section per unit solid angle at $\Theta = 45^\circ$ has also been determined and found to be $(1.7 \pm 0.1) \cdot 10^{-25} \text{ cm}^2$.

Already the first experiments by TUVE *et al.* [36] (fig. 7.12-2) allowed

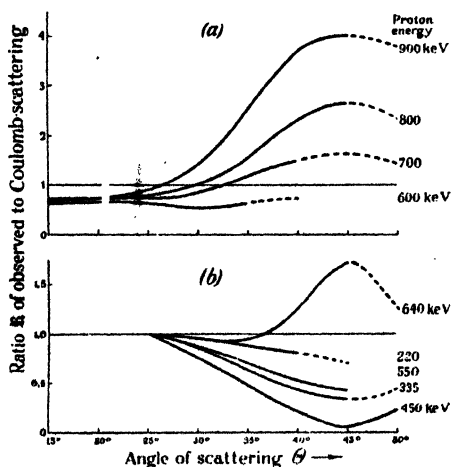


Fig. 7.12-2. Proton-proton scattering. Ratio $\mathfrak{R}$ of observed to pure Coulomb scattering as a function of scattering angle Θ . Experiments by TUVE, HEYDENBURG and HAFSTAD:

(a) according TUVE *et al.* [36],

(b) according TUVE *et al.* [37].

Relative measurements; $\mathfrak{R}$ is adjusted to unity at $\Theta = 25^\circ$.

BREIT, CONDON and PRESENT [36a] to conclude (upon the natural assumption that S-wave scattering is predominant) that *the nuclear proton-proton potential is attractive*: this follows indeed, as explained in 7.11, from the occurrence of $\mathfrak{R}$ -values smaller than unity at small angles, the angular region in question spreading towards larger angles as the proton energy decreases. This qualitative inference was further strikingly confirmed by new experiments (TUVE *et al.* [38]) extending to lower

energies: it was observed (fig. 7.12-2) how the ratio $\Re(45^\circ)$ changed from a maximum to a minimum. For an energy of 0.45 MeV, the scattering intensity at 45° almost vanished, owing to destructive interference between Coulomb and nuclear field. At still lower energies, the ratio $\Re(45^\circ)$ began to increase again, quite in conformity with theoretical expectation.

The subsequent experiments of the first group (HERB *et al.* [39], TUVE *et al.* [39], RAGAN *et al.* [41]) aimed at increasing the precision and extending the range of proton energies explored. The mutual consistency of the results could be checked in overlapping energy regions (see table 7.12 and fig. 7.12-3), and was found to be very satisfactory. The empirical

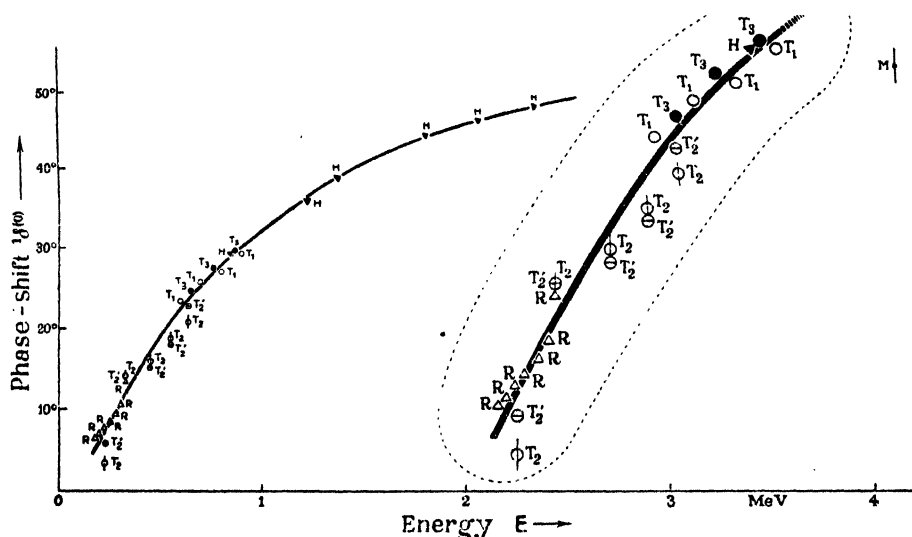


Fig. 7.12-3. Proton-proton scattering. S -phase derived from experiments:

R RAGAN <i>et al.</i> [41]	T_1 TUVE <i>et al.</i> [36]
H HERB <i>et al.</i> [39]	T_2 TUVE <i>et al.</i> [38]
M MAY and POWELL	T_2' the same, using only the scattering at $\Theta = 45^\circ$
(POWELL [47a])	T_3 TUVE <i>et al.</i> [39]

data were carefully analysed by BREIT and his collaborators (BREIT [36a, 39a], CREUTZ [39]) in order to derive the phases $\delta^{(l)}$ as functions of the proton energy. The S -wave scattering — in the first instance assumed to be responsible for the whole effect — is in fact found to account for it with deviations of at most a few percent (in many cases less than 1 %). These deviations cannot consistently be attributed to P - or D -wave scattering, which leads to the conclusion that the latter effects, in the energy range under consideration, are too small to be disclosed even by such accurate measurements.

From (10), (16), (17) it follows that

$$\Re(45^\circ) = 1 - \frac{2}{a} \sin \delta \cos(\delta - a \log 2) + \frac{1}{a^2} \sin^2 \delta, \quad (18)$$

and from this formula, with $\alpha^{-1} \approx 5 \dots 10$, two essentially different values of the S -phase δ may in general be deduced, one of them lying in the first quadrant, the other in the fourth. The latter would correspond to a repulsive potential. Quite apart from the behaviour of $\Re$ at smaller angles, it can be excluded, however, by application of a useful criterion due to Wheeler (cf. BREIT *et al.* [36a]): If the radial wave-function R keeps the same sign in the interval $0 \dots r'$, the ratio $\left(\frac{1}{R} \frac{dR}{dr}\right)_{r=r'}$ decreases with increasing energy. [The proof results immediately from the first formula in the footnote to (5.12-21).] The condition for R is fulfilled in the lowest (nodeless) S state up to the distance r' at which the function is sufficiently well represented by its asymptotic form, so that $\left(\frac{1}{R} \frac{dR}{dr}\right)_{r=r'}$ can be calculated in terms of δ . It is then seen that the fourth quadrant value contradicts Wheeler's criterion. This gives additional support to the above deduction that the proton-proton potential is attractive.

The results of the analysis of the first group of experiments are summarized in the following table and represented on fig. 7.12-3.

7.12. S -phases derived from first group of proton-proton scattering experiments			
E keV	$\delta(0)$ degrees	Estimated error on phase degrees	Reference
176,5 200,2 225,9 249,5 275,3 298,3 321,4	6,3 6,7 7,4 8,5 9,5 10,5 13,4	0,5	RAGAN <i>et al.</i> [41]
220 325 450 550 640	3,5 14,1 16,0 18,7 20,8	0,8 — 0,4 0,6 0,7	TUVE <i>et al.</i> [38]
670 776 867	24,80 27,42 29,41	0,4	TUVE <i>et al.</i> [39]
600 700 800 900	23,5 25,87 27,13 29,55	1,6 0,71 0,44 0,25	TUVE <i>et al.</i> [36]
860 1220 1390 1830 2105 2392	29,28 35,94 38,96 44,02 46,18 48,08	0,4	HERB <i>et al.</i> [39]
4200	54,0	2,5	MAY and POWELL (POWELL [47a])

The experiments of the second group have not been analyzed in the same detailed way, but the observed differential cross-sections have been

directly compared with the theoretical ones; we shall come back to this point in 7.131 below.

7.13. Derivation of nuclear potential parameters from the observed phases. There is unfortunately no analytical method (at least in quantum theory*) to derive the expression for the scattering potential from the known values of the phases at various energies. The only possibility is to start from some assumed type of potential, involving undetermined parameters, and to try to adjust the values of these parameters in such a way that the calculated phases agree with observation. This is a very unsatisfactory procedure, since — besides involving tedious numerical integrations of the wave-equation — it can at best only yield several possible types of nuclear potential, without any means (on internal evidence) of removing the resulting ambiguity. An encouraging feature, however, is that these various solutions all lead, as we shall see, to practically equivalent general conclusions.

For a detailed account of the special procedures required for the calculation of the phases corresponding to a given form of nuclear potential, the reader is referred to the papers already cited of BREIT and his collaborators [36a, 39a]**. In the latter paper [39a], refined methods of series expansions have been elaborated in order to check the results of numerical integrations. We shall just mention here that phases of higher angular momentum can be estimated by a formula

$$\delta^{(l)} \approx - \frac{M}{\hbar^2 k} \int_0^\infty \mathcal{V}^{(l)}(r) [L_l^{(\text{reg})}(r)]^2 dr, \quad (19)$$

entirely analogous to (5.33–37) and derivable by the same argument, starting from an appropriate identity, similar to (5.121–25). A slight modification of the argument permits an estimation of the change $d\delta$ of the phase-shift δ resulting from a small change $d\mathcal{V}$ of the nuclear potential***; if R is the radial wave-function for the original choice of the potential, we have to a first approximation

$$d\delta \approx - \frac{M}{\hbar^2 k} \int_0^\infty d\mathcal{V}(r) \cdot R^2 dr. \quad (20)$$

More accurate formulae have been derived by BREIT *et al.* [36a, 39a, 39d], and used in particular [39d] to compare the effects due to potentials of different shapes, — the difference arising from a change either of the parameter values or of the potential type itself.

Let us now survey the results which have been obtained with various

* On the analogous classical problem, see HOYT and FRYE [40]. In this paper, the semi-classical approximation based on an expansion of the phase of the wave-function in powers of $\hbar$ is also treated; but it is unsuited for nuclear problems.

** LANDAU and SMORODINSKY [44] have obtained by a semi-empirical method a general formula roughly representing the variation of the phases with energy.

*** This method is due to Mott; it has first been worked out by H. TAYLOR, *Proc. R. S. A* 136, 605, 1932.

simple types of potentials; the most extensively studied are the well, Gauss and meson potentials. For the first two types, calculations of S -wave scattering and estimates of P -wave phases have been extended up to 10 MeV by THAXTON and HOISINGTON [39b]. In all cases, the best values of the range constant κ and the strength $^1J_{-1}^{(0)}$ have been determined so as to fit the measurements of the first group at higher proton energies (HERB *et al.* [39], TUVE *et al.* [39]); later on (THAXTON [41]), the test was extended to the low energy results (RAGAN *et al.* [41]). As to May and Powell's results for 4.2 MeV protons, they agree in a general way with the theoretical predictions, but are not accurate enough for the purpose of discriminating between different potentials.

Potential well. In this case, the high energy values of the phases can be fitted to two potentials of different widths (and depths); but the low energy measurements just quoted are useful in removing this ambiguity: they definitely establish that the best fit at all energies is obtained with a well-width very nearly equal to the electron radius d . The corresponding depth is found* to be 11.35 MeV (BREIT *et al.* [36a, 39a, 39d]).

Gauss potential. Also for this type of potential, BREIT *et al.* [39a, d] were able to determine uniquely the constants fitting all experimental results, thanks to the remarkable accuracy of these results. From calculations of the S -wave scattering, they found $\kappa^{-1} = 1.9 \cdot 10^{-13}$ cm, $J = 26.3$ MeV. But the small P -wave anomaly has also been calculated by THAXTON and MONROE [39a] and shown not to be significant.

Meson potential. In contrast to the types just discussed, the meson potential does not allow of unique determination by comparison with the observed phases. BREIT *et al.* [39d] showed that no sufficiently close fit with these phases could be achieved with a meson mass of 200 m , and they concluded that the best possible fit required a meson mass as high as 326 m . However, in a renewed investigation, LUBANSKI and DE JAGER find

* It is convenient first to calculate the scattering upon the assumption that the Coulomb potential does not act inside the well; the depth so determined, viz. 10.5 MeV, has then to be increased, in order to compensate the effect of the Coulomb repulsion. The correction (of the order of magnitude $e^2/D \approx e^2/d = m$) can easily be estimated with any desired accuracy (BREIT *et al.* [39a]). A first approximation, for instance, is obtained from the expression (20) for the change of phase resulting from an assumed change of potential: for this change of potential, inside the well, we take the sum of e^2/r and a constant $-dJ$, and express that the resulting total change of phase is zero; this yields for the increase of depth needed

$$dJ = \frac{e^2}{D} \cdot \frac{a \int_0^{\alpha} \frac{\sin^2 u}{u} du}{\int_0^{\alpha} \sin^2 u du} = \frac{e^2}{D} \cdot \frac{a \left(\log 2a + 0.5772 + \int_{2a}^{\infty} \frac{\cos u}{u} du \right)}{a - \sin \alpha \cos \alpha},$$

with

$$a \equiv \frac{\sqrt{M(J + \frac{1}{2}E)}}{\hbar} \cdot D.$$

just as close a fit with $M_m = 278 m$, whereas even with a mass value $M_m = 248 m$, the deviations of the theoretical figures from the empirical ones do not exceed four times the estimated errors of measurement. Still, this would seem to point to a range of nuclear force rather smaller than that corresponding to the mass of the cosmic ray mesons observed at sea level. But although the available experimental results exhibit a very satisfactory internal consistency, they might be affected by small systematic errors, so that it is difficult to judge whether this discrepancy is real or spurious. To decide this issue by new proton-proton scattering experiments is one of the most important tasks in the field of research bearing on nuclear forces.

For later reference, all data thus far obtained on proton-proton potentials are here collected in tabular form:

7.13. Parameters of proton-proton potentials from scattering data				
Type of potential	Reference	Best parameter values		Observations
		Range κ^{-1} 10^{-13} cm	Strength J MeV	
Well (5.11-2)	BREIT <i>et al.</i> [36a, 39a, d]	2,8	11,35	Poor fit with data No close fit with data Close fit with data Close fit with data Unsuited
Exponential (5.11-3)	BREIT <i>et al.</i> [39d]	1,73	70,3	
MORSE (5.11-4)	THAXTON and MONROE [40a]	1,4	61,0	
Gauss (5.11-5)	BREIT <i>et al.</i> [39a, d]	1,9	26,3	
Meson (5.11-6) $M_m/m = \begin{cases} 200 \\ 248 \\ 278 \\ 326 \end{cases}$	BREIT <i>et al.</i> [39d]	1,9	17,45	
	LUBANSKI and DE JAGER	1,55	25,8	
	id.	1,37	32,6	
	BREIT <i>et al.</i> [39d]	1,17	45,8	
$w(\xi) = \xi^{-2}$	id.	—	—	

7.131. Comparison of second group of scattering data with theory. The second group of scattering data, relating to proton energies in the range 8 ... 14,5 MeV, might be expected to give evidence of the incipient influence of P -scattering and accordingly to yield information on the effective potential in the ${}^3P$ -configuration of the two protons. To this end, the empirical curves giving $\mathcal{S}(\theta)$ (7.12) have been compared with the values calculated on the assumption that the effective potential in the ${}^3P$ configuration is a well of the same width as the ${}^1S$ one. The calculation is carried out both for an attractive and for a repulsive ${}^3P$ potential, while the ${}^1S$ potential is taken to be attractive, in conformity with the foregoing analysis. Unfortunately, the results of such a comparison, as exemplified by fig. 7.131, do not as yet allow us to draw any far-reaching conclusions. If anything, they are in favour of a *repulsive* ${}^3P$ -potential.

Nevertheless, PEIERLS and PRESTON [47], attempting a more detailed analysis of the 10 MeV results, show that these results would indicate a ${}^1S$ phase of $52,5^\circ$ and a ${}^3P$ phase of about $-0,8^\circ$. Assuming well potentials of somewhat smaller width, viz. $D = 2,5 \cdot 10^{-13}$ cm, they find that the ${}^3P$ phase can be accounted for by a repulsive potential of about 10 ± 3 MeV;

this figure may be compared (11.21) with the depth $^3J = 25$ MeV of the ^3S potential for the same width.

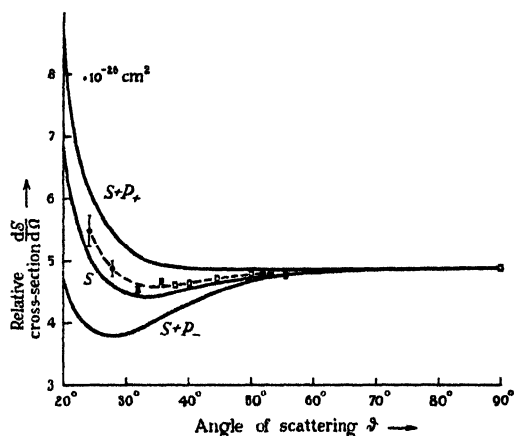


Fig. 7.131. Scattering of 10 MeV protons by protons (WILSON [47b]). Comparison of measured relative cross-section $\mathfrak{S}(\theta)$ with theory. (The data are adjusted to the theoretical value of the absolute cross-section at $\theta = 45^\circ$ i.e. $\vartheta = 90^\circ$.) Curve S: theoretical S-wave scattering. Curve $S + P_+$: theoretical $S + P$ scattering assuming repulsive P -potential. Curve $S + P_-$: same, assuming attractive P -potential. The depth assumed for the P -potential is not clearly specified.

7.14. Range of potential from disintegration "stars". An interesting check on the value derived for the range of proton-proton force from the analysis of the scattering experiments has come from quite a different quarter. The disintegration "stars" produced in a photographic emulsion by the impact of cosmic radiation* afford indirect evidence of nucleon scattering for energies of a totally different order of magnitude from those that can be produced in the laboratory. In fact, the tracks forming such stars are interpreted as due mostly to protons knocked out of a heavy nucleus (Ag or Br) of the emulsion by the impact of a nucleon of very high energy (≈ 200 MeV and more). The energy distribution of these protons will primarily be determined by the cross-section for collisions between the impinging fast nucleon and the constituent particles of the nucleus, and this cross-section will in its turn depend sensitively on the range of the interaction between the colliding nucleons: we thus have here, as pointed out by HEISENBERG [37], the possibility of a rather precise determination of the range.

A statistical analysis, performed by ORTNER [40], of empirical material obtained under homogeneous conditions**, leads to a very simple exponential law for the average number $\bar{n}_0(E_0)$ of star particles of energy $\geq E_0$:

$$\log_{10} \bar{n}_0(E_0) \approx \text{const} - (0.0252 \pm 0.0045) E_0 \quad (21)$$

(E_0 expressed in MeV)

We shall carry out the comparison with theory for a law of force of the Gauss type***. It may be shown (12.35) that $\bar{n}_0(E_0)$ is proportional to a certain function f_G of the range of the force, which in the case considered

* The subject of disintegration stars is reviewed by BAGGE [43].

** New material, published by PERKINS [47], fits Ortner's formula fairly well for $E_0 \approx 20 \dots 80$ MeV.

*** As observed in 12.33, Heisenberg's theory is unfortunately not applicable in its present form to the case of the meson potential.

may be approximated by just an exponential function of the same form as (21), viz. (12.36-43)

$$\log_{10} f_G \approx \text{const} - \left[0.0288 \left(\frac{1}{\kappa d} \right)^2 + 0.0096 \left(\frac{2r_0}{d} \right)^2 \right] E_0 \quad (22)$$

(E_0 expressed in MeV).

Adopting as before (2.22-7) $r_0 = \frac{1}{2}d$, we immediately get from (21) and (22)

$$\frac{1}{\kappa d} = 0.72 \pm 0.1,$$

i.e.

$$\kappa^{-1} = (2.0 \pm 0.3) \cdot 10^{-13} \text{ cm},$$

in excellent agreement with the value resulting from the proton-proton scattering experiments*. Moreover, it will be seen that a change by 10 % of the value chosen for r_0 would only imply a change of about ± 0.04 in $(\kappa d)^{-1}$, which is negligible in view of the experimental uncertainty.

7.2. Deuteron formation by proton collisions

Although it has no immediate bearing on our subject, we wish to call attention to the reaction



in which two protons combine to form a deuteron, the extra charge being emitted as a lepton field (positon and neutrino). This process, which is of considerable astrophysical importance, has been treated by BETHE and CRITCHFIELD [38]. We may restrict ourselves to the initial state in which the two-proton system has no orbital momentum, for this will be the most favorable for the envisaged proton combination; on account of the exclusion principle, the spins of the protons must then be antiparallel, so that we have to do with a β^+ transition from a ^1S state of the two-proton system to the ^3S ground-state of the deuteron. Assuming the interaction between nucleons and lepton field to be of the Gamow-Teller type (A 1.11), such a transition, which belongs to class (b) of A 1.12, is an allowed one. (On Fermi interaction, it would be forbidden; cf. A 1.132.) The cross-section S_d for the formation of a deuteron by two protons of relative velocity v may then be written in the form

$$S_d = \frac{1}{4} \cdot \frac{1}{v} \cdot \frac{1}{\tau_0} I(W) |G|^2, \quad (2)$$

the factor $\frac{1}{4}$ representing the *a priori* probability of the protons having antiparallel spins, while the expression for the probability per unit time of the β^+ transition is taken over from A 1.11 with the notations explained there. Here one has

$$W = \frac{2M_p - M_d}{m} = 1.8 \text{ whence } I(W) = 0.132, \quad (3)$$

by (A 1.11-3,4).

* BAGGE's [43] contention that the present method would yield the most precise range determination can obviously not be substantiated.

The matrix-element $|G|^2$ can be decomposed into a factor arising from the summation over spin and isotopic variables, the value of which, according to table A 1.132, is 2, and another factor containing the integration over the relative space coordinates; in the latter, angular integration simply yields 4π , so that

$$|G|^2 = 2 \cdot (4\pi)^2 \left| \int_0^\infty R_0(r) \cdot {}^1L_i^{(0)}(r) dr \right|^2. \quad (4)$$

In this formula, R_0 is the radial part of the deuteron eigenfunction, ${}^1L_i^{(0)}$ that of the two-proton system of kinetic energy ϵ . The latter has the asymptotic form

$${}^1L_i^{(0)}(r) \simeq \frac{\sqrt{2}}{k} e^{i\delta} \cos \delta [L_0^{(\text{reg})}(r) + L_0^{(\text{irr})}(r) \operatorname{tg} \delta], \quad (5)$$

the L 's on the right being defined by (7.11-3). In fact, (5) is just the asymptotic form of the $l=0$ component of the normalized solution of the two-proton problem set up in 7.11, multiplied by $e^{-i\gamma^{(0)}}/\sqrt{2}$; the phase factor $e^{-i\gamma^{(0)}}$ is without significance, but the other factor $1/\sqrt{2}$ is necessary because the solution used in 7.11 was normalized to 2, while we here want a wave-function normalized to unity (for a proton pair).

Let us assume, as effective potentials, wells of the same width $D = d$ and of depths ${}^3J_0^{(0)}, {}^1J_{-1}^{(0)}$ given in tables 6.432 and 7.13, respectively. The functions $R_0, {}^1L_i^{(0)}$ can then be written down explicitly and the integration in (4) performed. The essential velocity dependence of ${}^1L_i^{(0)}$, for small values of v , is given by a factor

$$\sqrt{2\pi a} e^{-2\pi a} \quad , \quad a \equiv e^2 c / \hbar v,$$

contained in $L_0^{(\text{reg})} + L_0^{(\text{irr})} \operatorname{tg} \delta$, the square of which expresses the penetrability of the Coulomb potential barrier. The final result can be written

$$\left| \int_0^\infty R_0(r) \cdot {}^1L_i^{(0)}(r) dr \right|^2 = 2 a e^{-2\pi a} ({}^3a_0)^3 \Lambda^2, \quad (6)$$

where ${}^3a_0$ is given by (6.21-14) and Λ^2 is a complicated function of $D, {}^3J_0^{(0)}, {}^1J_{-1}^{(0)}$ and ϵ_0 , representing the sum of the probabilities for the reaction to take place when the two protons are at different mutual distances. The main contribution to Λ^2 is found to come from the distances $\approx {}^3a_0$ and yields, with the adopted numerical values of the parameters, $\Lambda^2 \approx 8.1$.

The cross-section (2) finally takes the form

$$S_d = 2 (2\pi a)^2 e^{-2\pi a} \cdot \Lambda^2 \cdot \frac{2}{\tau_0} I(W) \frac{\hbar}{e^2 c} {}^3a_0 \cdot ({}^3a_0)^2: \quad (7)$$

the meaning of the two first factors has already been explained; the next one, $\frac{2}{\tau_0} I(W) \frac{\hbar}{e^2 c} {}^3a_0$, is just the chance of the positron emission taking place during the time taken by a proton of velocity $e^2 c / \hbar$ to go through the distance ${}^3a_0$. Inserting the above numerical values, and taking (A 1.21-4) $\tau_0 \approx 3 \cdot 10^3$ sec, formula (7) yields

$$S_d = a^2 e^{-2\pi(\alpha-1)} \cdot 4 \cdot 10^{-50} \text{ cm}^2. \quad (8)$$

This result, based on the assumption of free proton collisions, is valid under the conditions prevailing in the interior of the stars of the main sequence. For white dwarfs, however, the effective electrostatic interaction between the closely packed protons becomes very different from the Coulomb potential and ${}^1L_i^{(0)}$ takes a quite different form; an estimate of S_d in this case has been obtained by WILDHACK [40]. The astrophysical implications are discussed by WILDHACK [40], CHERTOCK [44] and SCHATZMAN [46] *.

See, in this connexion, A 1.22.

CHAPTER VIII

THE CHARGE INDEPENDENCE OF NUCLEAR INTERACTION

8.1. Comparison of proton-neutron and proton-proton potential

We are now in a position to make a quantitative comparison between the nuclear proton-proton potential, as determined in the preceding Chapter, and the potential between proton and neutron in the same (viz. the ^1S) configuration of the two-nucleon system. We begin by assuming that both effective potentials have the same range. This assumption is admittedly not free from uncertainty; in order to test it, the only available calculations are those concerning the well and meson potentials, of which the latter (7.13, 8.33) are quite inconclusive. In the case of the well potential (6.431), if we took the proton-neutron scattering results at their face value, we should have to adopt a range of proton-neutron potential essentially smaller than that of the proton-proton nuclear force. But, as already observed, the evidence is not so precise as to exclude the possibility of both ranges being equal. Once we accept this equality, however, we meet (as a comparison of the tables 6.432 and 7.13 will show) with the striking feature that in all cases investigated, the strengths of both potentials are also about the same. More precisely, as shown by table 8.1, the proton-

8.1. Comparison of proton-proton and proton-neutron potential		
Type of potential		$\frac{^1J_0^{(0)}}{^1J_{-1}^{(0)}} - 1$
Well		+ 0.039
Morse		+ 0.033
Gauss		+ 0.034
Meson	$\frac{M_m}{m} = \left\{ \begin{array}{l} 200 \\ 248 \\ 278 \\ 326 \end{array} \right.$	(- 0.043)
		+ 0.023
		+ 0.012
		+ 0.017

neutron potential seems to be about 1 ... 4 % larger than the other. The approximate constancy of this ratio for widely different shapes of the potential would perhaps testify to the reality of the difference; but a definite opinion on this point would certainly be premature*. At any rate,

* The conclusion reached by BREIT *et al.* [39b] and by THAXTON and MONROE [40a] as to the exact equality of both potentials was based on Simons' inadequate value of the proton scattering cross-section for zero energy neutrons (6.22).

such a difference would be a secondary effect, which we may neglect to a first approximation.

Assuming further (3.3) the neutron-neutron potential to be the same as the proton-proton one, we are thus led to the conclusion that *in even singlet configurations (type ^1S), the nuclear interaction between two nucleons is independent of the charges of these nucleons* (at any rate to a first approximation). On the other (^3P i.e. odd triplet) type of configurations compatible with the exclusion principle for nucleons of like charge (4.331), we have no clear-cut empirical information (7.13, 7.131). But it can easily be proved that *any operator of nuclear potential, non-vanishing and charge independent in even singlet configurations, has the same property in all configurations that are at all possible for nucleons of like charge*, i.e. all "charge triplet" configurations. In fact, the matrix-elements of the nuclear interaction operator (4.331–26) in such a configuration have, with respect to the isotopic quantum numbers, the diagonal form $(m_t; \tau = 3 | \dots | m_t; \tau = 3)$; and the charge independence property is simply expressed by saying that these matrix-elements are independent of m_t : in other words, the interaction operator itself (with exclusion of the Coulomb repulsion term) is invariant for *all* rotations in isotopic space. This, again, means that all its terms either are independent of the isotopic variables $\tau^{(1)}$, $\tau^{(2)}$, or involve these variables through the only rotation invariant combination $\tau^{(1)}\tau^{(2)}$. Clearly such an operator,

$$\mathcal{V}_{\text{nuc}} = \mathcal{A} + \mathcal{B} \tau^{(1)} \tau^{(2)} \quad (1)$$

($\mathcal{A}$ and $\mathcal{B}$ being functions of position and spin coordinates), is charge independent in any configuration.

It must be observed, however, that this does not at all mean that the general potential operator is necessarily of the form (1), i.e. charge independent in any possible configuration. In the first place, the charge independence in ^1S configurations does not tell us anything about any possible term of non-central interaction: for the expectation value of such a term would vanish in a ^1S state, irrespective of its dependence on the isotopic variables; and on the other hand, the derivation of the numerical value of the potential strength $^1J^{(0)}$ from the experimental data on slow neutron scattering, and, therefore, the establishment of the charge independence property of even singlet states, is not appreciably affected by the very small modification of the triplet scattering cross-section brought about by non-central forces (16.13). But even if we keep to central interactions, the addition to the interaction potential of any term involving the factor $3 + \frac{\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}}{\sigma^{(1)} \sigma^{(2)}}$, the expectation value of which vanishes in singlet states, does not impair the charge independence property of such states. Through the occurrence of additional terms containing factors of these types, and depending on an arbitrary way on the isotopic variables, the nuclear potential might be charge dependent in ^3P states. This logical

possibility cannot be dismissed by any physical argument; but it does not seem to correspond to any simple feature of nuclear field theory (an example* will be treated in 16.311). We shall therefore, in the following, adopt as a general form of dependence of the nuclear potential on the isotopic variables the expression (1) above, which expresses the *charge independence property* of the potential in all possible configurations. One should, however, not lose sight of the fact that this form is only a plausible extrapolation, and that even in the case of even singlet states its empirical foundation leaves room for small deviations.

8.2. General form of central nuclear potential

Exactly in the same way as the charge independence of the nuclear potential restricts its dependence on the isotopic variables to the form (8.1-1), the assumption of central interactions further limits each one of the functions $\mathcal{A}$ and $\mathcal{B}$ to the type $a + b \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$, a and b being functions of the distance only: indeed, any non-central coupling being excluded, there remain just the rotation invariant spin functions 1 and $\vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$. The most general expression of the central potential would thus involve four independent functions for the distance; but a further assumption will from now on be introduced, viz. that the law of distance dependence is the same for the four possible types of dependence on spin and isotopic variables. This assumption is, of course, made primarily for the sake of simplicity. But the consistency of the results obtained on this basis in the discussion of even (^3S and ^1S) configurations (6.431, 6.432, 8.1) may be considered to afford some partial justification for it. The nuclear potential may finally be written in the form

$$\mathcal{V}_{\text{nuc}} = J(r) \mathcal{O} \quad (1)$$

$$\mathcal{O} = a_0 + a_\sigma \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} + a_\tau \tau^{(1)} \tau^{(2)} + a_{\tau\tau} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \tau^{(1)} \tau^{(2)},$$

the a 's being *constant* coefficients.

We may also decompose the potential (1) into a combination of the four types of exchange interactions (4.341-34). Remembering that (4.13-26)

$$\vec{\sigma}^{(1)} \vec{\sigma}^{(2)} = 2P_\sigma - 1, \quad \tau^{(1)} \tau^{(2)} = 2P_\tau - 1 \quad (2)$$

and consequently

$$\vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \tau^{(1)} \tau^{(2)} = 4P_\sigma P_\tau - 2P_\sigma - 2P_\tau + 1, \quad (3)$$

we may in fact write

$$\mathcal{O} = W + BP_\sigma - HP_\tau - MP_\sigma P_\tau, \quad (4)$$

* See also HULTHÉN [43b], § 4.

with

$$\begin{aligned} W &= a_0 - a_\sigma - a_\tau + a_{\sigma\tau} \\ B &= 2(a_\sigma - a_{\sigma\tau}) \\ H &= 2(a_{\sigma\tau} - a_\tau) \\ M &= -4a_{\sigma\tau}. \end{aligned} \quad (5)$$

From these formulae, it appears that the absence of any dependence on spin and isotopic variables is characterized by

$$a_\sigma = a_\tau = a_{\sigma\tau} \quad (6)$$

(there is then only an ordinary and a Majorana interaction), so that the occurrence of such a dependence is revealed by unequal values of these parameters.

Another mode of decomposition consists in expressing $\mathcal{O}$ as a combination of the effective potentials for the four different types of configurations ${}^3S$, ${}^1S$, ${}^3P$, ${}^1P$, which, on account of (4.331-23), may be characterized by the eigenvalues ± 1 of P_σ and P_τ ; thus the effective potential for the ${}^3S$ configurations will depend on spin and isotopic variables through the combination $\frac{1}{2}(1 + P_\sigma) \cdot \frac{1}{2}(1 - P_\tau)$. The operator $\mathcal{O}$ accordingly takes the form

$$\begin{aligned} \mathcal{O} = & -\frac{1}{4} [(1 + P_\sigma)(1 - P_\tau) {}^3s + (1 - P_\sigma)(1 + P_\tau) {}^1s \\ & + (1 + P_\sigma)(1 + P_\tau) {}^3p + (1 - P_\sigma)(1 - P_\tau) {}^1p], \end{aligned} \quad (7)$$

with

$$-{}^3s = a_0 + a_\sigma - 3(a_\tau + a_{\sigma\tau}) \quad (8)$$

$$-{}^1s = a_0 - 3a_\sigma + a_\tau - 3a_{\sigma\tau},$$

$$-{}^3p = a_0 + a_\sigma + a_\tau + a_{\sigma\tau} \quad (9)$$

$$-{}^1p = a_0 - 3a_\sigma - 3(a_\tau - 3a_{\sigma\tau}).$$

It will be convenient to normalize the constants by taking for $J(r)$ the *absolute value* of the effective ${}^3S$ potential. This means ${}^3s = 1$; the constant ${}^1s$ will further be equal to the ratio

$$q = {}^1J_0^{(0)} / {}^3J_0^{(0)}, \quad (10)$$

which, according to table 6.432, is remarkably insensitive to the law of force adopted for $J(r)$, and roughly equal to 0.6. Formula (7) then becomes

$$\begin{aligned} \mathcal{O} = & -\frac{1}{4} [(1 + P_\sigma)(1 - P_\tau) + (1 - P_\sigma)(1 + P_\tau) q \\ & + (1 + P_\sigma)(1 + P_\tau) {}^3p + (1 - P_\sigma)(1 - P_\tau) {}^1p]; \end{aligned} \quad (11)$$

we also re-write the formulae (8):

$$\begin{aligned} -1 &= a_0 + a_\sigma - 3(a_\tau + a_{\sigma\tau}) \\ -q &= a_0 - 3a_\sigma + a_\tau - 3a_{\sigma\tau}. \end{aligned} \quad (12)$$

The parameters ${}^3p$, ${}^1p$ pertaining to the odd configurations cannot be directly given numerical values. We shall presently show how experiments on scattering of fast neutrons by protons, when extended to sufficiently high energies, will eventually provide the information needed to fix the odd-state potentials (8.33). It will appear later (14.12) that the analysis of the scattering of fast neutrons or protons by deuterons may serve the same purpose. But for the moment, we must make use of less direct arguments: the general saturation requirements for heavy nuclei (2.22) impose on the interaction parameters further limitations, which practically amount to their complete determination (11.33).

8.21. Traces of operators related to the nuclear potential. In a later section, we shall need the expressions for the traces of the operators $\mathcal{O}$, $\mathcal{O}P_\sigma P_\tau$, $\mathcal{O}^2$ and $\mathcal{O}^2 P_\sigma P_\tau$. They are easily found with the help of elementary properties of dichotomic variables (4.13). From (1) one immediately gets

$$\begin{aligned}\text{tr } \mathcal{O} &= 16 a_0, & \text{tr } \mathcal{O} P_\sigma P_\tau &= 4 [a_0 + 3a_\tau + 3(a_\tau + 3a_{\sigma\tau})], \\ \text{tr } \mathcal{O}^2 &= 16 [a_0^2 + 3a_\tau^2 + 3(a_\tau^2 + 3a_{\sigma\tau}^2)].\end{aligned}\quad (13)$$

To calculate $\text{tr } \mathcal{O}^2 P_\sigma P_\tau$, it is convenient to start from (11); since

$$(1 + P)(1 - P) = 0, \quad (1 \pm P)^2 = 2(1 \pm P), \quad (14)$$

one has

$$\begin{aligned}\mathcal{O}^2 &= \frac{1}{4} [(1 + P_\sigma)(1 - P_\tau) + q^2(1 - P_\sigma)(1 + P_\tau) + {}^3p^2(1 + P_\sigma)(1 + P_\tau) \\ &\quad + {}^1p^2(1 - P_\sigma)(1 - P_\tau)].\end{aligned}\quad (15)$$

The expressions for $\mathcal{O} P_\sigma P_\tau$, $\mathcal{O}^2 P_\sigma P_\tau$ differ from those for $\mathcal{O}$ and $\mathcal{O}^2$, respectively, only by the sign of the first two terms. We thus obtain the desired traces in a very symmetrical form:

$$\begin{aligned}\text{tr } \mathcal{O} &= -3(1 + q) - (9 {}^3p + {}^1p) \\ \text{tr } \mathcal{O} P_\sigma P_\tau &= 3(1 + q) - (9 {}^3p + {}^1p) \\ \text{tr } \mathcal{O}^2 &= 3(1 + q^2) + 9 {}^3p^2 + {}^1p^2 \\ \text{tr } \mathcal{O}^2 P_\sigma P_\tau &= -3(1 + q^2) + 9 {}^3p^2 + {}^1p^2.\end{aligned}\quad (16)$$

8.3. Charge independent interaction and meson field *

8.31. Neutral and symmetrical meson theories. The charge independence property of nuclear interaction has an important consequence for the meson field theory of this interaction: it implies that — besides the charged meson fields, the production of which by nuclear systems is testified by the phenomena of cosmic radiation — there must also occur (at least virtually, in the sense of 1.321) neutral meson fields *. Indeed, to a first approximation, a charged meson field will only give rise to an interaction between a proton and a neutron; this is a simple consequence of the

* This has first been pointed out by H. FRÖHLICH, W. HEITLER and N. KEMMER, *Proc. R. S. A* 166, 154, 1938.

principle of conservation of charge, as a (virtual) emission or absorption of a charged meson is necessarily accompanied by a transition from proton to neutron state or vice-versa. A force between two nucleons of like charge will first appear in the next approximation, as the result of a double virtual process of meson emission and absorption, and will consequently be of a smaller order of magnitude*.

It will be necessary to consider somewhat more closely the formal aspect of this situation. As is well-known**, the description of a charged field involves two sets of Hermitian components, the relation of which to the charge and current distribution is completely fixed by their properties with respect to gauge transformations. Such a transformation is defined by the addition to the electromagnetic potential of the gradient of an arbitrary function:

$$\frac{e}{\hbar} A_i \rightarrow \frac{e}{\hbar} A_i + \frac{\partial \alpha}{\partial x^i}. \quad (1)$$

For our present purpose, it will suffice to consider the simplest case of a meson field of spin 0, represented by scalar or pseudoscalar components (1.32); insofar as the charge properties of the field are concerned, the following considerations apply just as well to spin 1 mesons, which require for their representation sets of components possessing vector or pseudo-vector properties. Let, therefore, ψ_1, ψ_2 be the Hermitian components of a charged meson field of spin 0. It will be convenient to regard them as the orthogonal projections of a symbolic vector in a fixed plane; when the transformation (1) is performed, these components undergo a linear transformation, which may be described as a rotation of the symbolic vector in the fixed plane through the angle α . This property may be concisely expressed by stating that the non-Hermitian quantity $\psi = \frac{1}{\sqrt{2}} (\psi_1 + i\psi_2)$ is multiplied by the phase factor $e^{i\alpha}$:

$$\psi \rightarrow \psi e^{i\alpha}. \quad (1a)$$

It will not be necessary for our argument to explain further how the requirement of invariance of the Lagrangian for the transformation (1), (1a) yields the expression for the charge and current density in terms of ψ and its adjoint $\psi^\dagger$ (or, what amounts to the same, in terms of ψ_1 and ψ_2). The only property of the field quantity ψ which we need is the following: if ψ is expanded in a Fourier series, i.e. in a series of eigenfunctions of the meson states of definite momentum,

$$\psi = \sum_{\vec{p}} \left[b_+(\vec{p}) e^{\frac{i}{\hbar} \vec{p} \cdot \vec{x}} + b_-^\dagger(\vec{p}) e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{x}} \right], \quad (2)$$

* One might think that this argument would be restricted to the case of weak coupling (1.35), but it has been verified (A 3.24) that even a strong coupling theory involving only charged mesons is unable to account for the charge independence of the nuclear interaction.

** See, e.g., MøLLER and ROSENFELD [43].

the Fourier amplitudes $b_+(\vec{p})$, $b_-(\vec{p})$ are such that when operating on the wave-function of any state of the field characterized by definite numbers of mesons of given momenta, they decrease by one the respective numbers of positive and negative mesons of momentum $\vec{p}$; the adjoint operators $b_+^\dagger, b_-^\dagger$ accordingly increase these numbers by one. Every term of the expansion (2) of ψ is thus connected with a decrease of the charge of the meson field by one (positive) unit, either by absorption of a positive or emission of a negative meson.

The interaction energy between the meson field and a system of nucleons may be written

$$\mathcal{V}_\psi = - \int (\psi^\dagger s + s^\dagger \psi) dv, \quad (3)$$

the functions $s, s^\dagger$ of the nucleon variables being just the source densities which enter into the fundamental field equations of the form (1.32–10) for ψ and $\psi^\dagger$. Now, it is clear that in order to satisfy the requirement of charge conservation of the total system of nucleons and mesons for every emission or absorption process, the coefficients of $\psi^\dagger$ and ψ in (3) must be linear combinations of the operators $\Pi^{(i)\dagger}$ and $\Pi^{(i)}$, respectively, expressing that the change of charge of the meson field is compensated by an accompanying transition of the i -th nucleon from the proton to the neutron state, or vice-versa (4.12–24). In other words, we may write the density s in the form

$$s = \sqrt{2} \sum_i \Pi^{(i)\dagger} s^{(i)}, \quad (4)$$

the symbol $s^{(i)}$ denoting a certain function of the space and spin variables of the i -th nucleon, and also of the point of space-time at which the density s is taken, while (4.12–24) $\Pi^{(i)\dagger} = \frac{1}{2}(\tau_1^{(i)} + i\tau_2^{(i)})$. In all cases, the $s^{(i)}$ are Hermitian, so that the source densities of the Hermitian components ψ_1, ψ_2 are

$$s_1 = \sum_i \tau_1^{(i)} s^{(i)}, \quad s_2 = \sum_i \tau_2^{(i)} s^{(i)}, \quad (5)$$

respectively; the interaction energy (3) takes the form

$$\mathcal{V}_\psi = - \int (\psi_1 s_1 + \psi_2 s_2) dv. \quad (6)$$

The formulae (5) exhibit an extremely simple and direct connexion between the symbolic isotopic space (4.22) and that which has just been introduced to express the gauge transformation properties of the charged field. In particular, the gauge invariance of the expression (6) is secured by the fact that the dichotomic variable τ_1 , whose components are τ_1, τ_2 , is free to rotate in its plane (4.12).

When we consider the interactions between nucleons under circumstances prevailing in atomic nuclei or in scattering experiments involving

energies not much higher than those hitherto used, we may to a first approximation disregard all dependence of the source density on the time variations of the various coordinates of the nucleons and accordingly neglect the retardation of the force transmission through the meson field; this approximation will yield the *static* interaction. Higher terms form the *non-static* part of the interaction, which includes not only the effect of the *translation* motions of the nucleons, but also those of the *precession of their spins* and of the *exchanges of electric charge* between them (transitions between proton and neutron states, mathematically described by precessions of the isotopic vectors $\tau^{(i)}$ in isotopic space). From the expression (3) or (6) one obtains by the usual methods of field theory (15.31), with the help of the Green's function (1.31, 1.32-9) $\varphi = \frac{1}{r}e^{-\kappa r}$, the total static interaction energy of a system of nucleons in the form of a sum of interactions between pairs

$$\mathcal{V}_{\psi}^{(12)} = 2 \mathcal{T}^{(12)} \int \overset{\circ}{s}^{(1)}(P') \varphi(P'; P) \overset{\circ}{s}^{(2)}(P) dv_P dv_{P'}; \quad (7)$$

in this expression, $\overset{\circ}{s}^{(i)}(P)$ denotes the static approximation of the density function $s^{(i)}$ occurring in (4), (5), taken at point P ; the *isotopic factor* $\mathcal{T}^{(12)}$, defined by (4.41-1), has a non-vanishing expectation value only for a proton-neutron pair. Clearly, then, if static forces are predominant, charged meson fields alone will be unable to account for the charge independence property.

An obvious way of arriving at a law of interaction independent of the charges would be to introduce a purely neutral meson field, the source density of which would simply be independent of the isotopic variables of the nucleons. Since a neutral field is characterized by its invariance with respect to gauge transformations, i.e. rotations in the plane (1, 2) of isotopic space, it can be represented by a Hermitian component ψ_3 directed along the 3-axis of this space. If s_3 is the corresponding source density, the interaction energy of the neutral field with the nucleons will be of the type

$$\mathcal{V}_{\psi_3} = - \int \psi_3 s_3 dv. \quad (8)$$

The hypothesis just formulated consists in putting, similarly to (5),

$$s_3 \sim \sum_i s^{(i)}, \quad (9)$$

which leads to an interaction between nucleons completely independent of their charges in any approximation. However, it must be stressed that the simplicity of this picture is only apparent: for there are, anyhow, charged meson fields, produced by the nucleons, which will inevitably give rise to a certain additional amount of proton-neutron interaction. One should therefore assume that the source densities of the charged fields are

sufficiently small compared with those of the neutral field to account for the almost complete charge independence of the total interaction deduced from the experiments; the small and uncertain deviation from exact charge independence, exhibited by these experiments, should from this point of view be regarded as real and would indicate an upper limit to the ratio of the magnitudes of the source intensities of charged and neutral fields. It is true that the smallness of the charged meson fields would not come into conflict with any known fact concerning cosmic radiation. Nevertheless, such a predominantly neutral meson theory would be very unsatisfactory indeed, inasmuch as in it the connexion between nuclear forces and cosmic mesons, the most beautiful and significant feature of Yukawa's idea, would be completely lost. In fact, the assumptions made about the mass and other properties of the neutral mesons would be entirely arbitrary, and the whole scheme would be little more than a rather clumsy and roundabout way of putting forward a special form of interaction potential, distinguished from any other only by its unwieldiness. In spite of this objection, we shall in all following discussions retain the *neutral* type of nuclear field theory as a logical possibility to be tested by experiment.

There is, however, an alternative possibility of securing the charge independence of the nuclear potential. It is the so-called *symmetrical* type of meson theory, pointed out by KEMMER [38a]. As we have seen above (8.1), the total static interaction will be charge independent if the contribution from the neutral meson field is such as to add to the isotopic factor $2T^{(12)} = \tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)}$ of the charged field interaction (7) the term $\tau_3^{(1)}\tau_3^{(2)}$, completing the scalar product $\tau^{(1)}\tau^{(2)}$. Comparing (5), (6) with (8), we see that this result is achieved by taking

$$s_3 = \sum_i \tau_3^{(i)} s^{(i)}. \quad (10)$$

In this combination of charged and neutral fields, the three Hermitian components ψ_1, ψ_2, ψ_3 enter in an entirely symmetrical way; we may regard them as the three projections of a vector ψ in isotopic space, and the source densities are likewise components of a symbolic vector

$$s = \sum_i \tau^{(i)} s^{(i)}. \quad (11)$$

The neutral and the symmetrical theories correspond so to speak to opposite extreme cases of the general law of interaction (8.1-1), viz. $B=0$ and $A=0$. From the point of view of meson theory, these two extreme cases are the only simple ones, for the general expression for the interaction, with A as well as $B \neq 0$, could only be obtained by assuming two independent kinds of neutral mesons, which could only be distinguished from each other by the different ways in which they interact with nucleons, viz. according as their source density is of the form (9) or (10). Indeed, the most general function of the isotopic variables which might occur in the source density of a neutral field would be a linear combination of operators T_+ and T_- , which express that the nucleon does not change its proton or

neutron state when emitting or absorbing a neutral meson. This amounts to a source density of the form

$$s = \sum_i (a + b \tau_3^{(i)}) s^{(i)}, \quad (12)$$

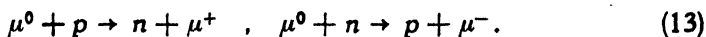
in which the coefficients a and b are not necessarily real. This leads to an isotopic factor in the law of interaction

$$|a|^2 + |b|^2 \tau_3^{(1)} \tau_3^{(2)} + (ab^* + a^*b) (\tau_3^{(1)} + \tau_3^{(2)}),$$

and the condition that the interaction be symmetrical with respect to the charges implies that the coefficient of $\tau_3^{(1)} + \tau_3^{(2)}$ should vanish. Unless, therefore, either a or b vanishes, the source density (12) would be essentially non-Hermitian, which means that we should have two independent Hermitian field functions, representing two kinds of neutral fields differently coupled to the nucleons. This would, of course, be a very artificial state of affairs.

To sum up, we see that while the introduction of neutral meson fields besides the charged ones provides the possibility of deriving a law of static interaction between nucleons of the general charge-independent form (8.1-1), the special cases of *purely neutral* and *symmetrical* theories, characterized by the respective isotopic factors 1 and $\tau^{(1)} \tau^{(2)}$, appear more natural from the point of view of meson theory.

8.311. Instability of the neutral meson. If there is any truth in the field conception of nuclear forces and their approximate charge independence, a large proportion of the mesons produced by the primary cosmic radiation in the atmosphere must be expected to be neutral. If the life-time of the neutral mesons were comparable with that of the charged ones, they could be detected thanks to a *charge transfer effect*, by which a neutral meson can emerge as a charged one from a collision with a nucleus, according to the scheme *



Such non-ionizing agents, capable of producing ionizing secondaries, can in principle be disclosed by a *coincidence method*, the main features of which go back to Rossi, and which has been applied to the present problem both by ROSSI and his collaborators [40] and by NISHINA and BIRUS [41b]. We shall outline the somewhat simpler arrangement used by the latter (fig. 8.311-1): the rates of coincidences of two counters A and B are compared when a lead absorber (sufficiently thick to rule out effects due to photons) is placed either between the counters (position (1)) or above them (position (2)); a non-ionizing agent giving rise to penetrating charged secondaries in the absorber would indeed manifest itself by additional coincidences when the absorber is in position (2). However, this inter-

* The symbols μ^0 , $\mu^\pm$ denote neutral and charged mesons, respectively.

pretation of a possible positive effect cannot be accepted without a control provided by the set of counters *C* placed above the apparatus and covering the whole area of incidence of the rays capable of passing through both counters *A* and *B*: clearly, if the difference in twofold coincidence is due to a neutral agent, the rate of threefold coincidences of the counters *A*, *B*

Fig. 8.311-1.
Coincidence method
of detection of non-
ionizing agents pro-
ducing ionizing
secondaries.

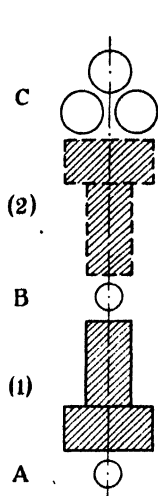
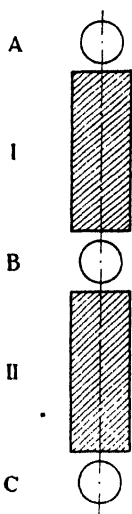


Fig. 8.311-2.
Principle of arrange-
ment for detecting
charge transfer effect
of meson.



and *C* recorded during the same period as the twofold coincidences, should be the same for both positions of the absorber. In the actual experiments, both the rates of threefold and twofold coincidences were larger with the absorber in position (2), the increase being nearly the same for threefold and twofold coincidences; there was consequently hardly any change in the rate of anticoincidences (i.e. coincidences of *A* and *B* not accompanied by a discharge of *C*) when the absorber was moved from position (1) to position (2). This result strongly points to some other origin of the effect than the production of charged secondaries by non-ionizing rays; and previous experiments, in which the precaution of the control counters *C* was not taken, lose any significance. The same conclusion is reached by ROSSI *et al.* [40], who further analyze the nature of the spurious effects due to meson scattering and shower production in the neighbourhood of the apparatus, and confirm the predominance of such effects by cloud chamber photographs*.

The charge transfer effect (13), together with its inverse, gives rise to an alternation of the charged and neutral states of a meson during its passage through matter. This phenomenon can be studied by an arrangement of counters and absorbers, very simple in principle (fig. 8.311-2). A charged meson, detected by counter *A*, may in absorber *I* go over to the neutral state and traverse counter *B* without discharging it; in a second

* See further TRUMPY and BJØRDAL [42], and CLAY and LEVERT [46].

absorber *II*, identical to *I*, it will have equal chance to recover its charge and cause a coincidence of counter *C* with counter *A*. The mean free path of a meson with respect to charge transfer will therefore (on the assumption of sufficient stability of the neutral state) be very simply related to the ratio of the rates of anticoincidences in which *A* and *C* but not *B* are discharged to the twofold coincidences of *A* and *C*. The experiment has been performed by NISHINA *et al.* [41a, b] with negative result: within the experimental error, no genuine anticoincidences could be discerned.

The conclusion to be drawn from these unsuccessful attempts at detecting neutral mesons* is that the life-time of these particles must be considerably shorter than that of charged mesons. This would seem to imply, from a theoretical point of view, that the interaction between mesons and lepton pairs should be highly unsymmetrical with respect to charged and neutral mesons: the coupling of the neutral meson field to the lepton field should be much stronger than that of the charged meson field, — in a way somewhat analogous to the situation in the neutral theory of nuclear interaction (in which the coupling with the nucleons is assumed to be much stronger for neutral than for charged mesons). There is in fact no objection to such an assumption; in particular, as has been stressed by PAIS [45], we have no reason to expect that the very small interactions between two leptons on account of their meson fields should exhibit any particular symmetry with respect to the charges of the leptons, since the relativistic effects, which will in this case predominate because of the smallness of the lepton masses, will (even if the source densities are symmetrical with regard to charge) be quite different for electrons and neutrinos owing to the large difference between their masses.

According to this view, the short-lived neutral mesons would decay into a pair of neutrinos (and thus be utterly lost to observation) or into a pair of electrons of both signs. The latter would have sufficient energy to initiate showers; an indirect detection of neutral mesons could therefore eventually be obtained if the analysis of the soft component of cosmic radiation, especially at high altitudes, could be sufficiently improved to allow such showers to be recognized. However, even if the decay of neutral mesons into lepton pairs does not have a larger probability than the corresponding decay of charged mesons, there is still another mode of decay, the estimated probability of which is so large that it may possibly be preponderant in bringing about the rapid disappearance of the neutral mesons. It is a process, pointed out by SAKATA and TANIKAWA [40], by which a neutral meson ultimately decays into a certain number of photons. This process may be described as follows: the meson transforms, by a *virtual* transition, into a pair consisting of a proton and an antiproton; these two virtually annihilate each other with photon emission. The number of

* Prof. Heitler has kindly pointed out to me that the negative results of the experiments just described might simply be due to the fact that the cross-section for the charge transfer effect becomes extremely small for mesons of high energy.

photons emitted is at least two, except in the case of a vector meson, when owing to the special symmetry properties of the matrix-element corresponding to the virtual transformation of the meson into a proton pair, the minimum number of emitted photons is three. These photons will again initiate showers, which should form a not inconsiderable part of the soft component of cosmic radiation.

A very rough estimate of the order of magnitude of the life-time t'_0 of the neutral meson, according to the last decay process, is given by

$$\frac{1}{t'_0} \approx \frac{g^2}{\hbar} \left(\frac{e^2}{\hbar} \right)^n c \kappa, \quad (14)$$

where n ($= 2$ or 3) is the least number of photons emitted, while g represents, as in 1.35, the order of magnitude of the constants analogous to the electric charge e , occurring in the nucleon source densities of the meson field. Numerically (assuming the same mass for the neutral as for the charged mesons), formula (14) gives *

$$t'_0 \approx 10^{-18} \dots 10^{-16} \text{ sec} \quad (15)$$

(according as $n = 2$ or 3). The life-time corresponding to decay into leptons would depend on the choice of the constants $\tilde{g}$, analogous to g for lepton source densities. If the same values are adopted for neutral and charged mesons, the life-times will also be the same, i.e. (1.332) of the order of 10^{-6} sec. The photon decay of the neutral meson will therefore be the predominant one, unless rather extreme assumptions are made about the order of magnitude of the coupling constants $\tilde{g}$ for neutral mesons.

8.32. Central static interactions on meson theory. We have seen how the charge independence requirement of the static interaction between two nucleons could be fulfilled either by a purely neutral meson field or by a symmetrical combination of charged and neutral fields. The next question is, whether this static interaction can be reduced to a central potential. This point will, however, be more suitably treated in a later Chapter (16.42); at this stage, it will suffice to know that a central interaction will result from a suitable mixture of meson fields of spin 0 and 1, both in the case of a purely neutral and in that of a symmetrical theory. In the notation of (8.2-1), this central interaction thus takes either of the following forms:

$$\mathcal{V}_{\text{neut}}^{(12)} = (a_0 + a_\sigma \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}) J(r) \quad (16)$$

$$\mathcal{V}_{\text{symm}}^{(12)} = \tau^{(1)} \tau^{(2)} (a_\tau + a_{\sigma\tau} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}) J(r); \quad (17)$$

and we shall now be concerned with the task of confronting the consequences of these two laws of nuclear interaction with relevant experiments.

The main point here is that in both (16) and (17) the parameters a are completely fixed by the properties of the S states of the two-nucleon systems, and that the effective potentials for odd states can therefore

A detailed calculation by FINKELSTEIN [47] leads to $t'_0 \approx 10^{-16} \dots 2 \cdot 10^{-11}$ sec.

uniquely be deduced from them. A comparison will thus be possible between theoretical predictions and experimental results concerning phenomena in which states of higher orbital momenta play a part. Since there is no such state of binding for the two-nucleon system, we are limited to the investigation of the scattering of sufficiently fast nucleons (neutrons or protons) by protons. The experimental study of proton-proton scattering has perhaps the advantage of greater accuracy, but the interpretation of the results is complicated by the fact that we have to do with interference of the proper nuclear scattering with the effect of Coulomb interaction; on the other hand, just this circumstance makes it possible to use *relative* measurements of the differential cross-section for a comparison between nuclear and Coulomb forces. The scattering of fast neutrons by protons, though more difficult of approach, yields very direct evidence on the nuclear potential. Especially the angular distribution of the scattered neutrons (or of the recoil protons) discloses by its deviation from isotropy in the barycentric system of reference the contribution of states of higher orbital momenta to the scattering process. It is this effect which will mainly occupy our attention.

It is clear at the outset that the neutral and symmetrical theories must lead to very different results, for the properties of the odd-state potentials are quite different in the two cases *. In the neutral theory, the odd-state potentials are evidently equal to the corresponding ones for even configurations:

$$\sigma \mathcal{V}_{\text{neut}}^{(\text{odd})} = \sigma \mathcal{V}_{\text{neut}}^{(\text{even})} \quad (\sigma = 1, 3). \quad (18)$$

In the symmetrical theory, on the other hand, the occurrence of the factor $\tau^{(1)} \tau^{(2)}$ in (17), in conjunction with the correspondence expressed in (4.331–23), immediately leads to the relations

$$\begin{aligned} 3\mathcal{V}_{\text{symm}}^{(\text{odd})} &= -\frac{1}{3} 3\mathcal{V}_{\text{symm}}^{(\text{even})} \\ 1\mathcal{V}_{\text{symm}}^{(\text{odd})} &= -3 1\mathcal{V}_{\text{symm}}^{(\text{even})}. \end{aligned} \quad (19)$$

Qualitatively, we see that *while in a neutral theory all effective potentials are attractive, in a symmetrical theory the potentials are attractive for even configurations and repulsive for odd configurations*. We shall now proceed to a closer discussion of the implications of these properties for the phenomenon of fast neutron scattering by protons.

8.33. Scattering of fast neutrons by protons. The differential scattering cross-section is given by the fundamental formula (6.21–10), which corresponds to an angular distribution of the general form

$$d\mathcal{S} = \frac{1}{16k^2} d\Omega \sum_n ({}^1A_n + 3 \cdot {}^3A_n) \cos^n \vartheta, \quad (20)$$

* The validity of the relations (18), (19) is not restricted to meson theory, but covers all central interactions of the forms (16) and (17), with any distance dependence $j(r)$.

the $^{\sigma}A$'s being certain functions of the phases $^{\sigma}\delta^{(l)}$. For each value of n , all phases give a contribution to the corresponding $^{\sigma}A_n$; we may write

$$^{\sigma}A_n = \sum_{l+l' \geq n} a_{ll'} ^{\sigma}a^{(ll')}, \quad (21)$$

$$^{\sigma}a^{(ll')} \equiv 4 \sin ^{\sigma}\delta^{(l)} \sin ^{\sigma}\delta^{(l')} \cos (^{\sigma}\delta^{(l)} - ^{\sigma}\delta^{(l')}),$$

with certain numerical coefficients $a_{ll'}$. Explicit formulae, including the phases up to $l = 5$, will be found in a paper by PAIS [46]. The calculation of the phases can be carried out by the methods explained in 5.3; for each value of the energy of the incident neutrons, the ^3S and ^1S phases have to be computed by the variational method (5.31), the higher phases by the approximate procedure of 5.331 or simply by Born's approximation. The reduced strength ^1b of the effective ^1S meson potential can be taken equal to the critical value (5.231) $^1b \approx 1.68$, independent of the value of the meson mass; this only amounts to neglecting the small energy of the virtual ^1S state (5.323, 6.23). The strength ^3b of the effective ^3S potential, on the other hand, must be calculated, for each assumed value of the meson mass, by means of formula (5.231-25), which, using the value (6.11-1) of the energy of the deuteron ground state, takes the form

$$^3b \approx 1.68 + 201.2 \frac{m}{M_m}, \quad (22)$$

with sufficient accuracy.

The main features of the resulting angular distribution, in the case of the

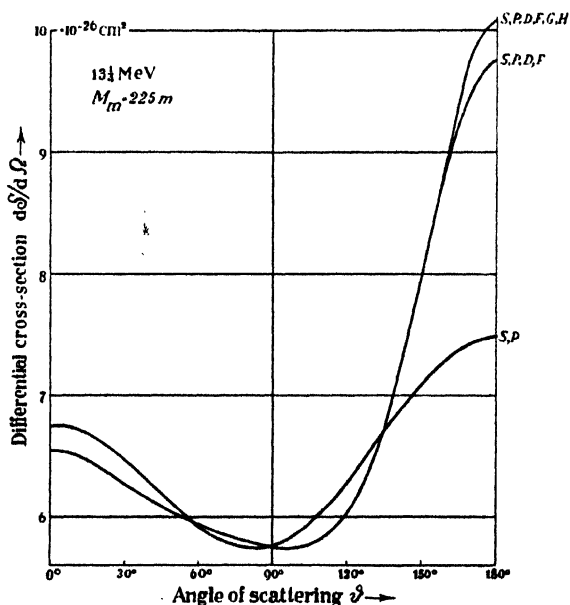


Fig. 8.33-1. Contribution of different orders in the case of a meson mass of $200m$ and an energy of $13\frac{1}{2}$ MeV.

* If n is even, only terms for which l and l' are of the same parity occur in the series (21); if n is odd, there occur only terms with l and l' of different parities.

symmetrical meson theory, are clearly brought out in the figures 8.33-1, 2, 3, due to FRÖHLICH, RAMSEY and SNEDDON [46] *. Fig. 1 shows how the successive inclusion of higher and higher phases in (21) modifies the angular distribution; in particular, it strikingly exhibits the insufficiency of an approximation limited to the P phases. Fig. 2 illustrates the influence

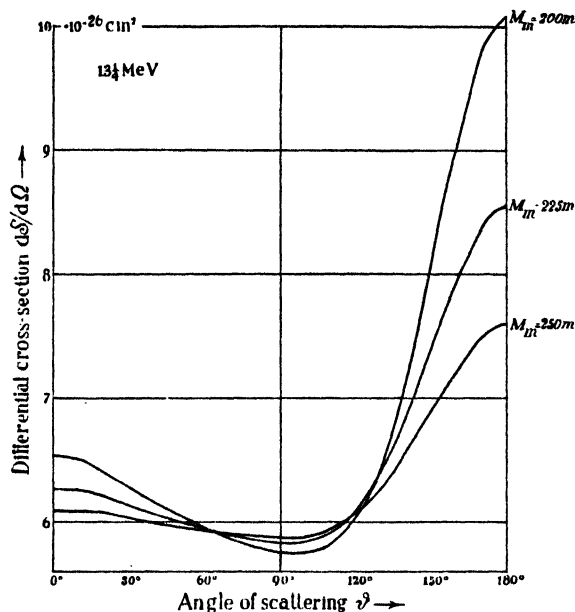


Fig. 8.33-2. Angular distribution of scattering to be expected for 13 1/4 MeV neutrons.

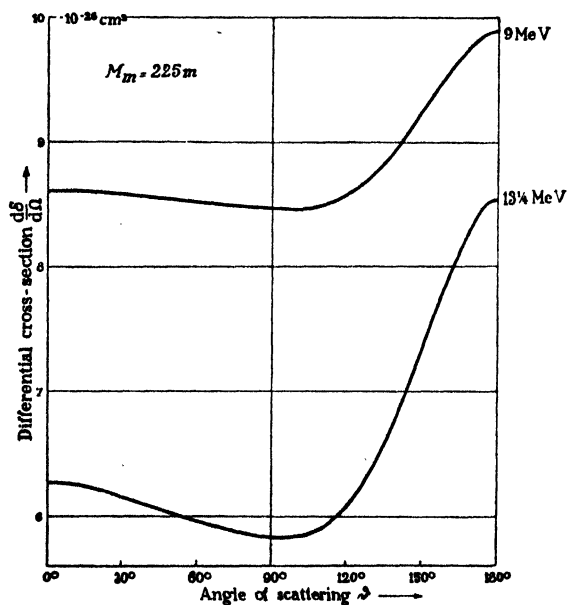


Fig. 8.33-3. Angular distribution of scattering to be expected for 9 MeV and 13 1/4 MeV neutrons on assumption that meson mass is 225 m.

* See further RAMSEY [47].

of the range of the force (i.e. the value of the meson mass) on the differential cross-section. Finally fig. 3 shows how the deviation from isotropy becomes more and more accentuated with increasing energy of the incident particles: it gives the angular distribution calculated upon the assumption of a meson mass $M_m = 225m$ for the energies 9 and 13.25 MeV. The comparison of these curves with the results of the Bristol experiments (after correction for the "loss" of tracks from the emulsion, as explained in 6.42) is exhibited by fig. 6.42-3, 4. It appears from these diagrams that the experiments indeed seem to agree somewhat better with the predictions of symmetrical meson theory than with a purely isotropic distribution; but the deviation from isotropy is still too small at these energies to permit a clear-cut conclusion*.

The theory shows, however, that for only slightly larger energies one can hope that the comparison with observation will enable us (provided we assume the nuclear interaction to be mainly central) to decide between a neutral or a symmetrical form of isotopic factor. This results from extensive calculations, carried out by HULTHÉN and PAIS [46b], of the anisotropy ratio

$$\Delta = S(\pi)/S(\frac{1}{2}\pi), \quad (23)$$

which conveniently expresses the main character ("forward" or "backward") of the anisotropy and permits an easier visualization of its variation with the different parameters of the problem. In terms of the $\sigma_a^{(l)}$ introduced in formula (21), the ratio Δ takes the form (PAIS [46])

$$= \frac{\sum_{m,n} (-1)^{m+n} (2m+1)(2n+1) (1a^{(mn)} + 3 \cdot 3a^{(mn)})}{\sum_{m,n} (-1)^{m+n} (4m+1)(4n+1) \frac{1 \cdot 3 \cdot 5 \dots (2m-1)}{2 \cdot 4 \cdot 6 \dots 2m} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} (1a^{(mn)} + 3 \cdot 3a^{(mn)})}. \quad (24)$$

From this formula, a result of extended validity can be obtained upon the assumption that the S phases (and consequently the absolute values of all other phases) are smaller than π , and that all differences $|\sigma_\delta^{(l)} - \sigma_\delta^{(l')}| < \pi/2$. In a symmetrical theory, owing to the alternation of attractive and repulsive potentials in even and odd states, one has then (5.131, 5.132)

$$(-1)^{m+n} \cdot \sigma_a^{(m,n)} > 0,$$

whence $\Delta > 1$: under the conditions stated, a symmetrical theory leads to a predominantly backward scattering. A neutral theory, in which all phases are positive, does not imply any such general property of the anisotropy ratio.

* Similar calculations have been carried out by HULTHÉN [43a, 44d] for an energy of 14.5 MeV and a meson mass of 200 m . However, he applied to the b -values for the S states certain corrections which later investigation has invalidated, so that these results cannot be retained.

The calculation of A has been performed for a range of energies from 5 to 25 MeV and two different values of the meson mass, viz. 200 and 300 m , in both the symmetrical and the neutral theory. The results are collected, together with some others, concerning the 9 and 13 MeV

8.33. Scattering of fast neutrons by protons; theoretical results						
Energy MeV	Meson mass M_m/m	Symmetr. theory		Neutral theory		Reference
		A	S 10^{-24} cm^2	A	S 10^{-24} cm^2	
5	200	1.07	1.63	1.17	1.65	HULTHÉN and PAIS [46b]
10		1.34	0.97	1.06	1.02	
15		2.01	0.68	0.96	0.74	
20		2.76	0.52	0.80	0.59	
25		3.73	0.43	0.73	0.49	
5	300	1.03	1.60	1.06	1.60	HULTHÉN and PAIS [46b]
10		1.09	0.93	1.05	0.94	
15		1.22	0.65	1.02	0.66	
20		1.42	0.50	0.96	0.51	
25		1.68	0.41	0.90	0.42	
9	200	1.3	1.1	1.09	1.15	HULTHÉN and PAIS [46b]*
	225	1.17	1.03	—	—	RAMSEY [47]
	300	1.08	1.06	1.05	1.07	HULTHÉN and PAIS [46b]*
13.5	177	2.58	0.82	0.86	0.89	PAIS [46]*** } RAMSEY [47] HULTHÉN and PAIS [46b]* RAMSEY [47]
13.25	200	1.70	0.75	1.0**	0.85**	
	225	1.45	0.74	—	—	
	250	1.29	0.74	—	—	
	300	1.17	0.76	1.03	0.77	
18	225	1.90	0.56	—	—	RAMSEY [47]
* Interpolated. ** Interpolated from HULTHÉN and PAIS [46b]. *** The S phases have been computed by FERRETTI [43a].						

neutrons, in table 8.33; their characteristic features are clearly exhibited by fig. 8.33-4. In the table, values of the total cross-section S have also been listed; the comparison with the empirical values collected in table 6.413 is shown by fig. 8.33-5. While the total cross-section is quite insensitive to the range of the interaction and to its dependence on the isotopic variables, the angular distribution, as expressed by the anisotropy factor, becomes, at higher energies, very markedly different for neutral and symmetrical theories, and in the latter case it also varies appreciably with the meson mass. The study of the angular distribution of very fast neutrons scattered by protons is therefore, as already stated, one of our most powerful means of inquiring into the nature of nuclear forces.

8.34. Photodisintegration of the deuteron and meson theory. As explained in 6.51, the photodisintegration differential cross-section consists

of two terms of different origins, the photoelectric and the photomagnetic effect, which can essentially be distinguished from each other by their different dependence on the angle of ejection of the nucleons with respect to the direction of the impinging photon. In fact, since the photomagnetic

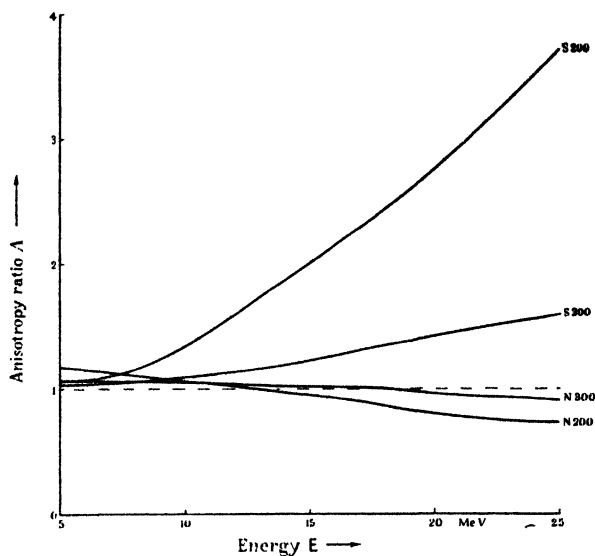


Fig. 8.33-4. Anisotropy ratio, according to different theories, neutral (*N*) or symmetrical (*S*), assuming a meson mass $M_m = 200$ or $300 m$ (as indicated by the number following the symbol *N* or *S*).

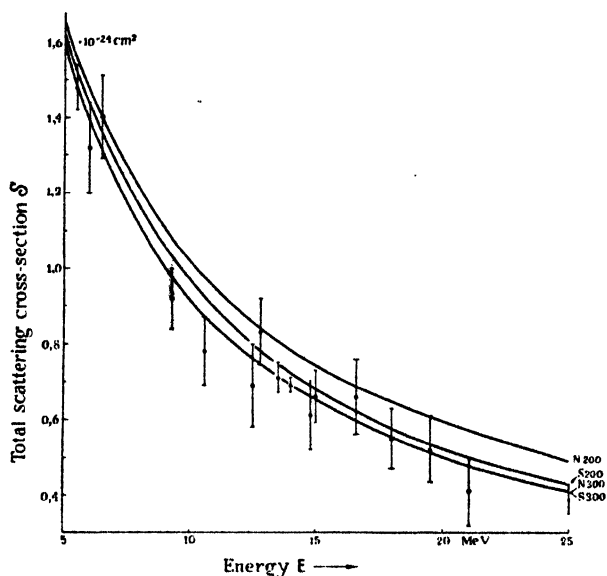


Fig. 8.33-5. Total proton-neutron scattering cross-section according to meson theory. (The curve N_{300} nearly coincides with S_{300} .)

effect is spherically symmetrical (in the barycentric system*), while the photoelectric effect has a $\sin^2\theta$ -distribution, the ratio of the intensities

* For photons of not too high frequency (and therefore sufficiently small momentum), the barycentric system of reference can be identified with the laboratory system.

of ejected nucleons in directions parallel and perpendicular to that of incidence of the radiation will be given by

$$\begin{aligned}
 H &= \frac{d\Phi_{\text{magn}}}{d\Phi_{\text{magn}} + d\Phi_{\text{el}}(\pi/2)} \\
 &= \frac{\Phi_{\text{magn}}}{\Phi_{\text{magn}} + \frac{1}{2}\Phi_{\text{el}}}.
 \end{aligned}
 \tag{25}$$

Thus, the measurement of H , together with that of the total cross-section, would give us the contributions Φ_{el} and Φ_{magn} separately. Unfortunately, owing to the smallness of the effects concerned, such measurements are very difficult, and it has not yet been possible to attain the accuracy required for the isolation of Φ_{magn} from the empirical data. All one can do at present is to compare the value of H directly measured, for a given frequency, with that calculated from theory.

This parameter H is essentially suited to give definite indications about the isotopic factor of the interaction potential. In the first place, the eigenfunctions of the ${}^3P$ states entering into the expression of the photoelectric cross-section Φ_{el} will be different in a symmetrical and in a neutral theory, because of the different behaviours of the corresponding effective potentials; and this may be expected to affect appreciably the value of Φ_{el} . In the second place the exchange magnetic moment occurring in Φ_{magn} , being obviously determined by the charged meson fields only, will vanish in a neutral theory. In the computations carried out by PAIS [43], the former effect was neglected, and the latter exaggerated owing to a slip in the basic formula *, so that his results are not reliable.

The diagrams published by Pais are here reproduced (fig. 8.34-1, 2) only to give a general qualitative idea of the relative magnitudes of the two cross-sections Φ_{el} and Φ_{magn} and of their variation with the excitation energy. More accurate calculations have been started by HULTHÉN; for photons of 2.62 MeV energy, he finds, assuming a meson mass $M_m = 200m$:

8.34-1. Theoretical photodisintegration cross-sections for $h\nu = 2.62 \text{ MeV}$			
	Φ_{el} 10^{-28} cm^2	Φ_{magn} 10^{-28} cm^2	H
Neutral theory	13,76	3,08	0,130
Symmetrical theory	12,07	3,075	0,145

Thus, the exchange effect proves to be quite negligible, and the change in Φ_{el} produces only a relatively small variation of H : a great accuracy would indeed be required to bring it out.

* Dr. Pais has kindly informed me that in the formula on p. 18, l. 10 from bottom, of his paper, the factor $\beta + n$ in the argument of the arctg should be $\beta + n + 1$.

The first experiments on the angular distribution of the ejected particles consisted in investigating cloud chamber tracks of the photo-protons. Neither CHADWICK, FEATHER and BRETSCHER [37] using 2.62 MeV (Th C'') γ -rays, nor RICHARDSON and EMO [38] with 3 MeV γ -rays from the Na (d, p) reaction, could in this way establish with certainty the existence of an isotropic component, but the accuracy was very poor owing

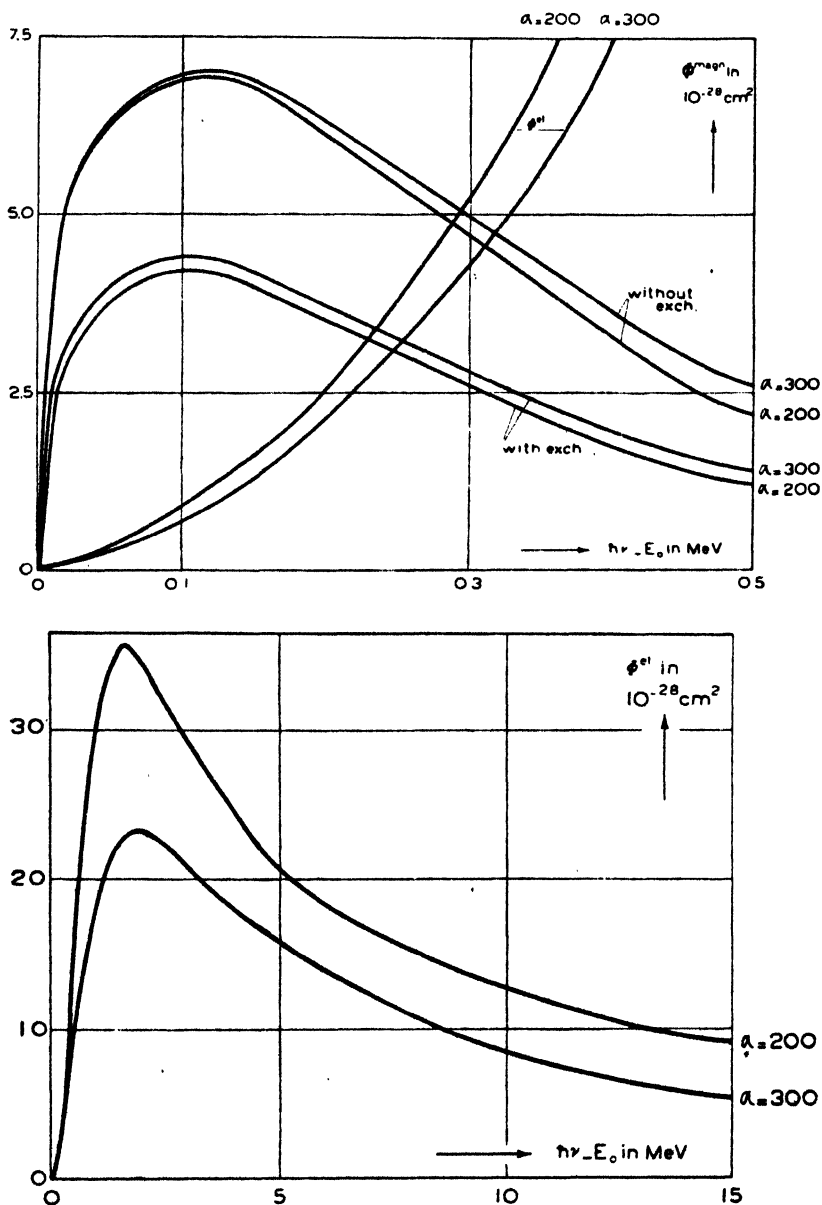


Fig. 8.34-1, 2. Photoelectric and photomagnetic disintegration cross-sections of the deuteron, according to PAIS [43]. (α means M_m/m .)

to the small number of tracks (about 60 in either case). The only indications about the parameter H which can be derived from this work are as follows:

$$\begin{aligned} \text{for } h\nu = 2.62 \text{ MeV, } H &< \frac{1}{4} \\ h\nu = 3.0 \text{ MeV, } H &\lesssim 0.16. \end{aligned} \quad (26)$$

The direct measurement of the angular distribution of the photo-neutrons produced by 2.62 MeV (Th C'') γ -rays was first attempted by HALBAN [38] and repeated with an improved technique by GRAHAM and HALBAN [45]; the same method has also been applied by MYERS and VAN ATTA [42], who, however, used an ill-defined continuum of X-rays reaching up to 2.43 MeV. A glass sphere filled with heavy water is irradiated by a suitable source and the ejected neutrons are collected by a system of boron chamber detectors, the orientation of which relative to the axis source-sphere can be varied; rather large and not quite certain corrections must be applied to the measured intensities owing to the geometrical conditions and the scattering of the photo-neutrons within the sphere. The figure given for H by Myers and Van Atta, viz. 0.87 ± 0.07 , has little significance; it may perhaps be compared with the theoretical values for the weighted average energy of 2.28 MeV. As to Halban's results for 2.62 MeV, they are unfortunately not concordant: his first experiments, with a sphere of 1 cm diameter, gave $H \approx 0.07 \pm 0.06$; the second, in collaboration with Graham, were performed with two spheres of diameters 1.6 and 0.77 cm, and the result, extrapolated to zero diameter was $H \approx 0.20 \pm 0.05$ (with some indications in favour of a low value). Further improvement of these difficult measurements is thus needed before any theoretical conclusion can safely be drawn.

At higher γ -ray energies, the angular distribution of the ejected nucleons is expected to follow very nearly the $\sin^2\theta$ -law corresponding to the photo-electric process. For 6 MeV γ -rays, the distribution of the ejected protons has been investigated by the photographic plate method (6.42) (GIBSON, GREEN and LIVESEY [47]). The analysis of 60 tracks in the angular range $70^\circ \dots 180^\circ$ gives a result compatible with the $\sin^2\theta$ -law.

The total cross-section of the photodisintegration of the deuteron is hardly known at all with any precision. The only existing estimates are the following:

8.34-2. Measurements of total photodisintegration cross-section			
Reference	$h\nu$ MeV	Total cross-section 10^{-28} cm^2	Observations
HALBAN	2.62	13	Provisional value, not yet published May be wrong by a factor 2
RICHARDSON and EMO [38]	3.0	(11)	
ALLEN and SMITH [41]	6.2	11.6	

For the capture cross-section of neutrons of $2.2 \cdot 10^5$ cm/sec velocity, Hulthén finds the following results:

$$\begin{aligned} \text{for } M_m = 197m, \quad G &= 0.294 \cdot 10^{-24} \text{ cm}^2 \\ \text{for } M_m = 295m, \quad G &= 0.303 \cdot 10^{-24} \text{ cm}^2; \end{aligned}$$

the exchange effect is again entirely negligible. The comparison with the empirical value (6.52–18) would perhaps favour a high value of the meson mass. However, a slight modification of the ^1S wave-function, within the uncertainty of the empirical value (6.22–32) of the slow neutron-proton scattering cross-section (assuming 22 instead of 21 for S in units 10^{-24} cm^2), is sufficient to bring the value of G for $M_m = 197m$ to $0.308 \cdot 10^{-24} \text{ cm}^2$.

8.4. Central nuclear potential: summary of argument

At this stage, it will not be superfluous to review the line of argument followed in the preceding analysis of the properties of two-nucleon systems. This analysis starts from the assumption that we have to do with a *central nuclear potential*; it is further necessary to assume a definite form of distance dependence, which in the simplest cases involves two parameters, viz. the range constant and the strength. The available evidence then gives fairly complete information on the effective potentials in *triplet and singlet even configurations*; and in particular it supports the view that the analytical form of distance dependence and range value are the same in both cases, the strength alone being different (8.2).

The *proton-neutron potential* is markedly spin dependent. On the very natural assumption that the lowest stationary states to which this potential gives rise are S states (6.11), a large spin dependence is disclosed by the fact that the ground state of the deuteron belongs to the triplet system, as well as by the large cross-section of the proton for scattering or radiative capture of very slow neutrons. Indeed, the magnetic interaction between proton and neutron would depress the ^1S state below the ^3S state (6.13); whereas the large contribution of ^1S waves to the mentioned scattering and capture processes requires the existence of some (actual or virtual) ^1S level of very low energy (6.23, 6.52). The virtual character of this level, made probable by the study of the radiative capture process (6.52), is definitely established by the scattering properties of ortho- and para-hydrogen for slow neutrons (6.33). Thus, both the ^3S and ^1S effective potentials are attractive, but only the former leads to a stable stationary state of binding.

For a given law of distance dependence and a given value of the range of force, we can determine the strength of the ^3S potential from the observed binding energy of the corresponding stationary state; the strength of the ^1S potential then follows from the value of the scattering cross-section of protons for very slow neutrons (6.431, 6.432). In principle, the value of the range might also be derived from the scattering cross-sections for faster neutrons, especially the $D(d, n)$ -neutrons of 2 .. 3 MeV energy;

but the data available at present still leave a large uncertainty about the quantity in question. They are at any rate compatible with the assumption that the range of the proton-neutron force is the same as that of the proton-proton force (6.431).

The *proton-proton potential* (exclusive of Coulomb force) can be determined with somewhat better accuracy, in range as well as in strength, from the experiments on proton-proton scattering; the meson type of distance dependence, however, constitutes a conspicuous exception: the theory of the scattering process proves to be very insensitive to large variations of the value of the meson mass (which fixes the potential parameters) (7.13). The ^1S proton-proton potential is found to be attractive, and in fact (on the *assumption* of equal ranges) very nearly the same as the ^1S proton-neutron potential (8.1): this property is well-established for the well and Gauss types of potentials, whereas the meson potential, owing to the above-mentioned uncertainty, hardly permits any significant conclusion for the time being.

Finally, there is good reason to believe that the *neutron-neutron potential* is identical with the proton-proton one (apart from the Coulomb energy). This conclusion is reached chiefly by a comparison of the masses of isobars with neutron excesses $+1$ and -1 : it is verified that the mass difference of such isobaric pairs consists solely of the electrostatic energy of the excess proton in the latter nucleus (3.3). Altogether, from the evidence on even states one can surmise with great probability that in all (even or odd) configurations compatible with the exclusion principle the effective potential is (with a possible deviation of a few percent) *independent of the charges* of the interacting nucleons (8.1). The general form of the central potential operator is then given by (8.2-1); in this formula, two of the four parameters a remain as yet undetermined, due to insufficient evidence on the odd configurations.

In fact, although the study of the scattering of fast nucleons is in principle suited to give us direct information about the odd state potentials, it has not yet reached the extent and accuracy needed for this purpose (7.131, 8.33). In order to complete our knowledge of nuclear interactions, we must therefore turn to other phenomena, pertaining to more complex nuclear systems.

PART III

NUCLEAR MODELS AND SATURATION PROPERTIES ON CENTRAL FORCE HYPOTHESIS

CHAPTER IX

SURVEY OF NUCLEAR MODELS

9.0. Going over to the study of complex nuclei, we shall in the first place be concerned with the question as to which models can be used to describe the nuclear systems considered with more or less accuracy. We may state at the outset that no entirely adequate model has as yet been found; different approaches can be tried according to the point of view chosen and in each case the way is barred by mathematical difficulties. Nevertheless, many interesting results have been obtained, which we shall try to summarize in this Part.

9.1. Description of heavy nuclei

9.11. *Bohr's droplet model.* It has already been stressed (2.22) that heavier nuclei — in contrast with atoms — must be regarded as systems of closely coupled particles, in some way comparable to liquid droplets; the analogy, however, is very rough, since the mobility of the closely packed nucleons is a consequence of their unclassical zero-point motion (2.3). Additional confirmation of this picture is afforded by the discussion of nuclear reactions, initiated by BOHR [36]. The large probability of the capture of fast neutrons by heavy nuclei, together with the sharpness of the γ -rays emitted in such processes, forces us to conclude that the duration of the collision leading to capture is very much longer than the time which the neutron would take to cross the nucleus. In other words, the collision first leads to the formation of a *compound nucleus* of great stability, in a highly excited state; the ensuing emission of radiation, by which the compound nucleus goes over to its normal state, is a secondary process, occurring quite independently of the way in which the compound nucleus has been formed. This behaviour is easily understood just by realizing that the particles within the nucleus interact so strongly with each other and with the impinging neutron, that the energy of the latter is rapidly distributed among all nucleons; the possibility for any of these to escape from the compound nucleus is then conditioned by the occurrence of a casual fluctuation leading to a sufficient concentration of energy on that particular nucleon. These considerations clearly apply to any type of nuclear reaction, initiated by the impact of some elementary or compound particle (deuteron, α -particle) or a photon on a nucleus with emission of some other particle or of radiation. Such a reaction will take place in two independent stages, of which the first will be the formation of the compound nucleus; the second will be one of a number of competing modes of

disintegration of the compound nucleus or of radiative transitions to its ground state. From the study of nuclear reactions, much information has in particular been gathered about the distribution of the excited energy levels of compound nuclei; this distribution exhibits just the characteristic features to be expected for a system with close coupling: a rapid decrease of the mean distance between neighbouring levels with increasing excitation energy; at the same time, an increase of the breadth of the levels (owing to the various possibilities of transitions from the excited states), so that the level distribution becomes continuous (fig. 9.11).

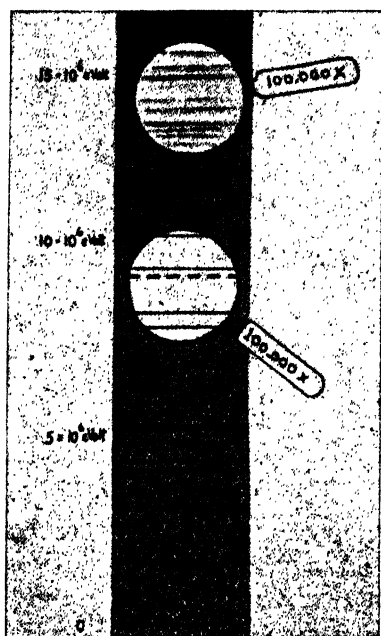


Fig. 9.11. Schematic level system of a heavy nucleus. The dotted line in the lower "100,000 times magnifying glass" indicates the binding energy of a neutron in the nucleus, so that the levels above this line exhibit resonance with neutrons whose kinetic energy is given by the distance between the level considered and the dotted line.

Of course, such a system of closely coupled particles, as an actual nucleus proves to be, is rather unmanageable by present methods of quantum theory, since the usual approximation procedures of this theory have chiefly been developed for the treatment of loosely coupled atomic systems of electrons. On the other hand, methods derived from the analogy with a classical liquid droplet cannot claim more than qualitative validity (BOHR and KALCKAR [37]). In this connexion, *statistical considerations*, first advocated by FRENKEL [36], have proved most fruitful. According to this point of view, the motion of the constituent nucleons is compared with the thermal agitation of ordinary matter: to the nucleus in any excited state of energy E^* is attributed a temperature T , related to E^* according to the rules of quantum statistics. Assuming e.g. the excited states of the

* We denote by T the temperature in the dynamic scale, i.e. the usual temperature in degrees times Boltzmann's constant k .

nucleus to consist of a superposition of harmonic vibrations, the frequencies of which are uniformly distributed, and using Planck's formula for the mean energy of a harmonic oscillator, one finds

$$E^* = \frac{1}{2} \gamma_A T^2, \quad (1)$$

the constant γ_A^{-1} representing $3/\pi^2$ times the average energy difference between two successive proper vibrations.

The entropy $S(E^*)$ of the nucleus in its state of energy E^* is simply related to the density $\varrho(E^*)$ of the energy levels at that energy by a formula of the type

$$\varrho(E^*) \sim e^{S(E^*)/k}, \quad (2)$$

while the well-known relation

$$\frac{1}{k} \frac{dS}{dE^*} = \frac{1}{T} \quad (3)$$

obtains between the derivative of the entropy and the temperature. Combining this last formula with the given relation, such as (1), between temperature and energy, we get an explicit expression for the function $S(E^*)$ or $\varrho(E^*)$. In our example, we get

$$\varrho(E^*) \sim e^{\sqrt{2} \gamma_A E^*}, \quad (4)$$

a formula* in good agreement with the empirically estimated level densities, if we take (WEISSKOPF [37])

$$\gamma_A^{-1} \approx 0.1 \text{ MeV}. \quad (5)$$

The emission of particles by the excited compound nucleus can be compared (FRENKEL [36], WEISSKOPF [37]) with an evaporation process, and for sufficiently high excitations, its probability can be estimated by well-known statistical arguments. To take only the simplest case of neutron emission, it is found, in particular, that the mean energy of the emitted neutrons will be $2T$, if T denotes the temperature of the nucleus left after the process. Owing to the relatively small number of constituent nucleons, this residual nucleus is considerably "cooled down" by the evaporation process: if, e.g., the excitation of the compound nucleus was 20 MeV, that of the residual nucleus will be only about 12 MeV. According to formulae (1) and (5), its temperature will be ≈ 1.5 MeV, and the average

*. This formula receives an interesting interpretation if we put the excitation energy E^* equal to n^* times the mean energy difference between successive proper vibrations; the resulting expression

$$\varrho(n^*) \sim e^{\pi \sqrt{2n^*/3}}$$

is just the asymptotic estimate of Hardy and Ramanujan for the number of partitions of the integer n^* . See also BOHR and KALCKAR [37], p. 34, and VAN LIER and UHLENBECK [37].

energy of the emitted neutrons about 3 MeV, thus very much smaller than the initial excitation energy. This striking consequence of the statistical conception is likewise in qualitative and rough quantitative agreement with experiment.

The above considerations will suffice to illustrate the kind of arguments which may be based on the droplet model and which in many cases have led to a very helpful global understanding of properties of heavy nuclei. The detailed form of nuclear interaction, however, clearly lies outside the scope of a description which treats nuclear matter as a continuous medium. We now turn our attention to another nuclear model which, though admittedly less rational than the droplet model, nevertheless in many respects gives a sufficiently adequate picture of heavy nuclei and allows of a much more elaborate treatment.

9.12. The Fermi gas model. It turns out that the behaviour of heavy nuclei is to a large extent so insensitive to the precise form of the interactions at close distances between the nucleons that it is correctly described — at any rate qualitatively — even if we begin by completely disregarding this interaction. To describe an excited nuclear state of temperature T , we thus start from an ideal gas of nuclear matter at this temperature, consisting of a given number A of nucleons enclosed in a volume of nuclear dimensions. In this initial approximation, the limitation of the system to a given volume is the only way in which the nuclear forces come into play. It may seem paradoxical that such an apparently inadequate picture of a nucleus should still possess fundamental features embodied in the liquid droplet model. The reason is that we are dealing with an assembly of particles obeying Fermi statistics in a strongly degenerate state: in fact, for all states of excitation commonly considered, the excitation energy (of 20 MeV, say) of an assembly of 50 or 100 particles of mean kinetic energy ≈ 20 MeV each (2.3) only means a slight departure from complete degeneracy. As a matter of fact, such a degenerate state resembles a liquid more than a gas, since the exclusion principle prevents two nucleons with the same charge and spin from occupying the same place within the nucleus; this has just the effect, even in the absence of any forces, of keeping the nucleons of the kind considered a certain mean distance apart, a feature which we have recognized as characteristic of nuclear structure (2.22).

In particular, it can be verified, as we shall see (9.43), that the degenerate gas of nucleons possesses the properties required to account for the course of nuclear reactions sketched above (9.11). For temperatures of a few MeV (corresponding to usual excitation energies), its viscosity is found to be so large that any vibration excited in it will be aperiodically damped, so that the excitation energy can only be kept in the form of heat. The heat conductivity is also very large: a local concentration of heat (such as produced by the impact of a particle) will rapidly spread over the whole nucleus.

Moreover, it is well-known (9.42) that the excitation energy of a degenerate Fermi gas depends on the temperature just by a relation of the form (1); the numerical value of the co-factor γ_A , calculated on this model for $A \approx 100$, is about twice as large as that corresponding to the empirical equation (5), but the discrepancy can be attributed to the influence of the nuclear forces (9.42).

If we now turn to the consideration of the ground state of a heavy nucleus, the Fermi gas model will again offer us a useful starting point, at least for a qualitative discussion; for the actual calculation of mass-defects, it is, of course, less satisfactory. The nuclear interactions (for which some analytical expression, e.g. of the type (8.2-1), is assumed) are introduced as perturbations, and the calculation of the total energy of the system may be improved by applying a variational procedure, in which the assumed nuclear radius is treated as the quantity to be varied. Calculations of this type have been carried out as far as the second approximation (12.2); from a quantitative point of view, their outcome is rather disappointing, but they lead to qualitative conclusions of considerable interest.

On the one hand, even the first approximation treatment suffices (together with a few additional considerations) for a general discussion of the saturation properties of the nuclear bonds, which enables us to complete the determination of the spin and charge dependence of the nuclear potential (Chapt. XI). On the other hand, the second approximation discloses a peculiar correlation between any four nucleons with different charges and spins, leading to the temporary formation of clusters resembling α -particles (12.23). This tendency to " α -clustering" illustrates from another side the inadequacy of the Fermi gas model: indeed, the considerable binding energy of such clusters is not properly taken into account in the computations based on the perturbation method. At the same time, it points to an approach to the problem of nuclear structure starting so to speak from the opposite end, viz. the so-called " α -particle model", in which the α -clusters (together with a few odd nucleons if necessary) are taken as fundamental units, held together by appropriate forces. For heavy nuclei, however, the latter model has no obvious advantage over the Fermi gas picture: its interest will first appear in the domain of lighter nuclei, which we shall now proceed to consider.

9.2. Description of light nuclei

9.21. The quasi-atomic model. As is well-known, we possess for the treatment of many-electron systems in the electric field of an atomic nucleus a powerful method initiated by Hartree and modified by Fock to take account of exchange interactions (2.22). Each particle is assumed to be acted upon by a potential field of suitable form, representing in a certain sense some average of the total field exerted on the particle by the other particles of the system: this potential involves no other dynamical variables

than those of the particle envisaged, while it depends on the other particles in a schematical way, through certain quantities treated as parameters. The states of the particles in this field are thus described by *individual* wave-functions, and the eigenfunctions of the total system, in the initial approximation, are linear combinations of Slater-determinants (4.14) built up of appropriate individual wave-functions. The eigenvalue problem can then be solved by the usual variation principle, the parameters of the potential field being varied and their "best" values determined.

The Hartree-Fock method has also been extensively applied to the treatment of light nuclei, in spite of the fact that it obviously loses any justification for such closely coupled systems; it is not surprising that the quantitative results obtained in this way are inconsistent and unreliable. The general scheme indeed differs essentially from the atomic case by the absence of any analogue of the central field of the atomic nucleus which constitutes a very considerable part of the Hartree field of the atom. The choice of the individual eigenfunctions of the nucleons is therefore to a large extent arbitrary; they are made to depend on certain adjustable parameters, which may be regarded as related to a fictitious average nuclear field. For instance, this field may be assumed to be quasi-elastic, so that the individual wave-functions of the nucleons have the form of (spatial) harmonic oscillator eigenfunctions, with arbitrary frequency parameters. The general picture of the nucleus corresponding to a Hartree-Fock procedure will be called the *quasi-atomic model*.

Formally, the quasi-atomic model and the Fermi gas model are very similar: they have in common the essential feature of starting from the consideration of individual states for the single nucleons; we may say that they belong to the class of *individual nuclear models*. In the Fermi gas picture, the individual wave-functions are simply plane waves confined to a region of nuclear dimensions; the parameters defining this region (nuclear radius, e.g.) play a similar role to that of the parameters of the Hartree-Fock wave-functions, viz. in schematically representing the influence of the nuclear forces in the initial approximation. It is clear that, so far as they go, the quasi-atomic model is better adapted to lighter nuclei, the Fermi gas model to heavier ones. But no more than qualitative indications can be expected from either.

9.22. Nuclear models of the collective type. Any adequate nuclear model must necessarily be of the *collective* type: i.e. the description of any state of the model, as embodied in the corresponding wave-function, must depend essentially on the variables of all the constituent nucleons. In some way the droplet model might be regarded as belonging to this class; but it is perhaps more correct to say that it falls outside the alternative: individual-collective, as it treats nuclear matter as a continuum. The most general and comprehensive form of collective nuclear model is the *resonating group model* (13.3). It attempts to analyze the states of nuclear systems

into a continually changing medley of all possible groupings of nucleons. In more technical language, the wave-function of any state of the system is set up as a superposition of all possible products of wave-functions representing smaller groups of nucleons. A workable approximation is then eventually obtained by judiciously weighing the probabilities of the various groupings and dropping the unlikely ones.

The α -particle model (13.1), for instance, falls under this general scheme: only α -clusters are retained as constituent groupings. It can be estimated that the stability of the temporary α -clusters is sufficient to justify such a treatment in many cases. Especially the study of the lightest nuclei which can be decomposed into a whole number of α -clusters (" α -nuclei") may be carried out in great detail by methods borrowed from the theory of molecules. As the number of α -clusters increases, their cohesion becomes loosened by the interplay of the nuclear forces and the periodicity due to α -clustering, which is so marked in the domain of light nuclei, gets more and more blurred out.

9.23. The Wigner approximation. The preceding approximate treatments of the nuclear problem do not involve any restrictive hypothesis regarding the law of nuclear force itself: they are rather concerned with some schematization of the structure of the system of nucleons assumed as a starting point for the computation of its binding energy or other properties. Another principle of approximation consists in leaving the structure *a priori* undetermined, but introducing simplifying assumptions about the nuclear interactions. One can then consider the most general collective model from the point of view of its transformation properties for the groups allowed by the assumed form of the nuclear potential; and by the methods of group theory, one can derive qualitative or semi-quantitative features of nuclear states which have a wide range of validity. Wigner, who has especially developed this method (10.14), has shown that a convenient starting point is obtained by first retaining only the ordinary and Majorana terms in the nuclear potential; the explicit spin and charge dependence, although by no means small, nevertheless appears to be of secondary importance for this kind of consideration. The Coulomb energy of the protons, however, will tend to disturb the symmetry properties essential for the application of the method and will therefore limit its validity for the heavier nuclei. The usefulness of *Wigner's approximation* for the study of light and intermediate nuclei is quite comparable with that of the analogous methods of group theory in atomic and molecular physics.

9.3. A table of nuclear models

It will be convenient to summarize in tabular form the conclusions of the preceding sections, the nuclear models discussed being classified from two distinct points of view: (a) according to their individual or collective character: (b) according as they are better adapted to the treatment of

lighter or heavier nuclei. Further, the position of Wigner's approximation has been indicated, the arrows pointing in the direction of increasing accuracy of the models.

9.3. Nuclear models		
	Heavy nuclei	Light nuclei
<i>Continuous nuclear matter</i>	Liquid droplet model	
<i>Collective models</i>	Resonating group model	
		α -particle model
<i>Individual models</i>	Fermi gas model	quasi-atomic model

Wigner approx.

←

9.4. Elementary properties of Fermi gas model

9.40. In this section, we shall discuss some properties of a strongly degenerate ideal Fermi gas, in a form adapted to the case of a gas of nucleons enclosed in a region of nuclear dimensions. This forms the initial approximation of the Fermi gas model.

9.41. *Kinetic energy of ground state (absolute zero).* The ground state of the nuclear model under discussion corresponds, in the initial approximation, to the absolute zero of temperature. All the lowest individual states of neutrons and protons are occupied up to those with a maximum momentum given by the condition that the total number of occupied states be equal to the number of neutrons or protons, as the case may be. For each value of the momentum vector $\vec{p}$ there are two individual states, corresponding to the two possible orientations of the spin of the nucleon, and in the approximation here considered, these two states will have the same energy. On account of the pairing tendency of the nucleon spins (3.12), we shall expect that in the ground state as many spins as possible are saturated, i.e. that as many individual proton and neutron states as possible are doubly occupied. Since we are only interested in asymptotic formulae for large numbers of protons and neutrons, we shall suppose these numbers even, and accordingly all spins saturated in the ground state. For the same reason, the form assumed for the enclosure does not matter and we can take a cube having the same volume $\frac{4\pi}{3} r_0^3 A$ as a spherical nucleus of radius $R = r_0 A^{1/3}$ (2.1-1). The constant r_0 may be treated as a variation parameter, no assumption being made about its eventual dependence on the mass number A .

Consider now, e.g., the N neutrons of such a nucleus. The maximum value $p_m^{(N)}$ of the momentum of the occupied states is given by

$$p_m^{(N)} = \frac{(9\pi)^{1/3}}{2r_0} \left(\frac{2}{g_s} \right)^{1/3} \left(\frac{N}{\frac{1}{2}A} \right)^{1/3}; \quad (1)$$

in this formula, g_s denotes the number of occupied states of given

momentum $\vec{p}$: in the ground state, therefore, $g_s = 2$; but the formula is also applicable to the case ($g_s = 1$) of all spins parallel, — a case which will occur in the course of a later discussion. The maximum kinetic energy $p_m^{(N)2}/2M$ we denote by $E_m^{(N)}$, and we find that the average kinetic energy $K^{(N)}$ may be written

$$K^{(N)} = \frac{3}{5} N E_m^{(N)}. \quad (2)$$

Similar formulae obtain for the proton gas.

It will be convenient to refer all quantities pertaining to any nucleus of N neutrons and Z protons to the corresponding ones for a fictitious nucleus, of dimensions such that $r_0 = \frac{1}{2}d$ (2.22–7), and consisting of equal numbers of protons and neutrons ($N = Z = \frac{1}{2}A$) with entirely saturated spins. The constitution of such a *standard heavy nucleus* corresponds to a complete neglect of Coulomb forces. In this case, the common value of $p_m^{(N)}$ and $p_m^{(Z)}$ is

$$p_m = (9\pi)^{\frac{1}{2}} \frac{\hbar}{e^2} m, \quad (3)$$

so that

$$K = \frac{3}{5} E_m = \frac{3}{10} (9\pi)^{\frac{1}{2}} \left(\frac{\hbar}{e^2} \right)^2 \frac{m}{M} \cdot m \\ = 14.5 \text{ MeV}, \quad (4)$$

$$E_m = 24.2 \text{ MeV}. \quad (5)$$

In terms of these standard values, the general expression for the kinetic energy takes the form

$$K = K \left(\frac{2}{g_s} \right)^{\frac{3}{2}} \left(\frac{\frac{1}{2}d}{r_0} \right)^2 \frac{1}{(\frac{1}{2}A)^{\frac{3}{2}}} [N^{\frac{3}{2}} + Z^{\frac{3}{2}}]; \quad (6)$$

this shows, in particular, that in the standard nucleus, the kinetic energy is proportional to the number of nucleons and has the value K per nucleon. More generally, if we express N and Z as $\frac{1}{2}(A \pm n)$ in terms of the mass number A and the neutron excess n , and if we expand $N^{\frac{3}{2}}$, $Z^{\frac{3}{2}}$ in powers of n/A , we get

$$K = K \left(\frac{2}{g_s} \right)^{\frac{3}{2}} \left(\frac{\frac{1}{2}d}{r_0} \right)^2 A \left[1 + \frac{5}{9} \left(\frac{n}{A} \right)^2 \right]. \quad (7)$$

The deviation of K from linearity in A thus always remains quite small for stable nuclei, since even for the heaviest ones, the ratio n/A is only ≈ 0.2 .

* From (3.22–2), we estimate, for large A ,

$$\frac{n}{A} \approx \frac{\frac{3}{10} \frac{e^2}{r_0} A^{\frac{1}{2}}}{2\epsilon_1 \gamma + \frac{3}{10} \frac{e^2}{r_0} A^{\frac{1}{2}}} = \frac{A^{\frac{1}{2}}}{136 + A^{\frac{1}{2}}},$$

on account of (2.21–5,6).

The order of magnitude of the mean kinetic energy per particle, as given by (4), corresponds to expectation on general grounds (2.3). In this connexion, it should be noted that, for a given nuclear volume, the initial approximation here discussed clearly yields the smallest possible value for the average kinetic energy. The effect of the neglected nuclear interactions will indeed necessarily be to cause transitions of certain nucleons to unoccupied states of higher momentum, resulting in an increase in kinetic energy; a more general formal proof has been given by WATANABE [39].

As regards the analogy of nuclear matter with He II, alluded to in 2.3, it should be noted that on the individual model an important difference arises with respect to the statistics obeyed by the constituent particles: Fermi statistics for the nucleons, Bose statistics for the He atoms. From this point of view, the analogy is much closer on the α -particle model.

9.42. Kinetic energy of excited states. As is well-known, the kinetic energy of an ideal Fermi gas at temperature T differs from the zero-point energy just calculated by a term proportional to T^2 :

$$E^* = \frac{1}{2} \gamma_A^0 T^2; \quad (8)$$

The co-factor γ_A^0 has the general form

$$\gamma_A^0 = \frac{\pi^2}{2} \left[\frac{N}{E_m^{(N)}} + \frac{Z}{E_m^{(Z)}} \right] = \frac{\pi^2}{2E_m} \left(\frac{g_s}{2} \right)^{\frac{3}{2}} \left(\frac{r_0}{\frac{1}{2}d} \right)^2 \left(\frac{1}{2}A \right)^{\frac{3}{2}} (N^{\frac{1}{2}} + Z^{\frac{1}{2}}), \quad (9)$$

which for the standard nucleus reduces to

$$\gamma_A^0 = \frac{\pi^2 A}{2E_m} \quad (\text{standard heavy nucleus}). \quad (10)$$

If we take the composition of the standard nucleus as representative of "nuclear matter", the specific heat per unit mass of the latter is accordingly

$$C_V = \frac{k}{MA} \gamma_A^0 T = \frac{\pi^2}{2} \frac{k}{M E_m} T. \quad (11)$$

For $A = 100$, the expression (10) gives, with (5), $(\gamma_A^0)^{-1} \approx 0.05$ MeV, i.e. about half the required value (9.11-5). As already stated (9.12), it can be seen (BARDEEN [37]) that the effect of the nuclear interactions will be just to decrease γ_A^0 . In fact, the average force exerted on a nucleon in a specified individual state by all the others can be derived, as we shall see (13.3), from an effective potential which depends explicitly on the individual state in question. Thus the total energy of a nucleon in this state will have a further dependence on the momentum; and the (negative) potential energy will naturally decrease in absolute value with increasing momentum. This means that the derivative dE/dp of energy with respect to momentum will be larger than when the nuclear forces are entirely neglected. Therefore, the constant γ_A^0 , which is proportional to the average number of individual states per unit energy, i.e. to dp/dE , will have a smaller value than in the case of no force.

9.43. Kinetic properties of nuclear matter. Let us, as above, consider a standard heavy nucleus as a "lump" of nuclear matter of density *

$$\varrho = M / \frac{4\pi}{3} r_0^3 \quad (12)$$

fixed by the nuclear dimensions. We may then investigate the "mechanical" and "thermal" properties of this nuclear matter by the ordinary methods of kinetic theory, modified to apply to a strongly degenerate Fermi gas. The coefficients of viscosity η and of heat conductivity κ may be written in the familiar form

$$\begin{aligned} \eta &= \alpha_\eta \cdot \frac{1}{3} \varrho \bar{v} A_\eta \\ \kappa &= \alpha_\kappa \cdot \frac{1}{3} \varrho \bar{v} C_V A_\kappa; \end{aligned} \quad (13)$$

in these formulae, $\bar{v}$ is the mean velocity of the nucleons, C_V the specific heat per unit mass, given by (11), the α 's numerical factors and the A 's mean free paths associated with the transport phenomena considered. In the case of a degenerate Fermi gas, the latter quantities show a marked increase with decreasing temperature, because the number of possible collisions is very much reduced on account of the exclusion principle. Indeed, at a temperature T the possible transitions are limited to the states of the "Maxwell tail" beyond the maximum zero-point energy E_m : in velocity-space, such states roughly fill a spherical shell of radius $v_m = \sqrt{2E_m/M}$ and thickness $\Delta v \approx \Delta E/p_m \sim T/\sqrt{2ME_m}$ (v_m , p_m are the maximum zero-point velocity and momentum, respectively). If S is the average collision cross-section corresponding to the transport process envisaged, we therefore have

$$A \sim \frac{1}{\frac{\varrho}{M} \cdot S \cdot \left(\frac{\Delta v}{v_m}\right)^2} \sim \frac{M}{\varrho} \cdot \frac{1}{S} \cdot \left(\frac{E_m}{T}\right)^2. \quad (14)$$

Further using (11) and $\bar{v} \sim v_m$, we may put (13) in the form

$$\begin{aligned} \eta &= \alpha_\eta \cdot \frac{1}{3} M^{\frac{1}{2}} E_m^{\frac{1}{2}} \frac{1}{S_\eta} \left(\frac{E_m}{T}\right)^2 \\ \kappa &= \alpha_\kappa \cdot \frac{1}{3} M^{-\frac{1}{2}} E_m^{\frac{1}{2}} \frac{\pi^2 k}{2} \frac{1}{S_\kappa} \frac{E_m}{T}. \end{aligned} \quad (15)$$

We see how the peculiar temperature dependence of the mean free paths determines a large increase of viscosity and heat conductivity at low temperatures.

The rigorous calculation of η and κ , starting from the Maxwell-Boltz-

* In this subsection, it will be convenient to take the velocity of light as unit of velocity.

mann equation, has been performed by TOMONAGA [38]. Averaging over the scattering angles and relative velocities yields

$$\begin{aligned} S_\eta &= \int_0^1 \frac{x^5 dx}{\sqrt{1-x^2}} [\bar{S}^{(0)}(2v_m x) - \bar{S}^{(2)}(2v_m x)] \\ S_x &= \int_0^1 \frac{x^3 dx}{\sqrt{1-x^2}} \bar{S}^{(0)}(2v_m x); \end{aligned} \quad (16)$$

in these formulae, $\bar{S}^{(m)}(v)$ has the following meaning: if the differential collision cross-section for any nucleon pair is expanded in a series of zonal harmonics:

$$dS = d\Omega \sum_l c_l(v) Y_l^0(\vartheta), \quad (17)$$

one has

$$\bar{S}^{(m)}(v) = \sum_l^{(\text{even})} c_l(v) \int_0^\pi \frac{4\pi}{2l+1} |Y_l^m(\vartheta=0)|^2, \quad (18)$$

the summation extending over the even values of l only.

To estimate the coefficients $c_l(v)$, we have to express the differential cross-sections dS in terms of the phases according to the formulae (6.21–10) and (7.11–15) for collisions between neutron and proton and between like nucleons, respectively. According to (18), only even l -values in the expansion (17) of dS , i.e. even powers of $\cos \vartheta$, are of importance. Putting therefore, as in (6.21–10), with the notation (8.33–21),

$$\sigma_S^{(\text{even})} = \frac{\pi}{4k^2} \sum_l^{(\text{even})} \sum_{l'}^{(\text{even})} \sqrt{(2l+1)(2l'+1)} Y_l^0 Y_{l'}^0 \sigma_a^{(ll')} \quad (19)$$

and introducing an analogous definition of $\sigma_S^{(\text{odd})}$, we write the proton-neutron cross-section in the form

$$\begin{aligned} dS_{p-n} &= 2 d\Omega [3({}^3S^{(\text{odd})} + {}^3S^{(\text{even})}) + {}^1S^{(\text{odd})} + {}^1S^{(\text{even})} \\ &\quad + S(\text{odd powers of } \cos \vartheta)]; \end{aligned} \quad (20)$$

the factor 2 accounts for the two possible relative orientations of the spins of the colliding nucleons. Assuming charge independence of the nuclear forces (8.1) and neglecting the Coulomb interaction altogether, we can express the like-particle collision cross-section in a quite similar manner; formulae (7.11–14, 15) reduce in this case to

$$dS_{p-p} = dS_{n-n} = d\Omega \cdot 4 (3 \cdot {}^3S^{(\text{odd})} + {}^1S^{(\text{even})}); \quad (21)$$

there is here no factor 2 because the formulae quoted already contained such a factor, expressing that in the proton-proton scattering experiments the recoil protons are also recorded. In all, we thus get

$$\begin{aligned} dS &= 2 d\Omega [9 \cdot {}^3S^{(\text{odd})} + 3 \cdot {}^3S^{(\text{even})} + 3 \cdot {}^1S^{(\text{even})} + {}^1S^{(\text{odd})} \\ &\quad + S(\text{odd powers of } \cos \vartheta)]. \end{aligned} \quad (22)$$

For the computation of the phases, Tomonaga assumes a nuclear potential of the form (8.2-1), $J(r)$ being a well of width $D = 2.2 \cdot 10^{-13}$ cm and depth $J = 30.35$ MeV. For the interaction parameters a , he makes the inadequate choice $a_0 = a_\tau = 0$ (11.32), whence, by (8.2-9, 12), ${}^3p = -\frac{1}{3}q$, ${}^1p = -3$, and he takes $q = \frac{2}{3}$. He further chooses for r_0 a value $0.53 d$ somewhat larger than the one here adopted. But a choice of more likely numerical values would not appreciably alter the results.

Evaluation of the expressions (16), (18) leads to

$$S_\eta \approx 10^{-26} \text{ cm}^2, \quad S_\nu \approx 2 \cdot 10^{-26} \text{ cm}^2. \quad (23)$$

As to the numerical factors a_η, a_ν one gets

$$a_\eta = \frac{2\sqrt{2}}{5\pi^3}, \quad a_\nu = \frac{7\sqrt{2}}{16\pi^3}. \quad (24)$$

It should be noted that the collision radii derived from (23), as well as the particle wave-lengths, are of the same order of magnitude as the average distance between neighbouring particles, so that the kinetic methods employed strictly speaking lose their validity. Still, we may use the above results to get a general idea of the effects of the large viscosity and heat conductivity of nuclear matter. For this purpose, Tomonaga discusses two simple examples:

(1) An elastic vibration of circular frequency ν and mean energy $\bar{E}$ undergoes, owing to the viscosity, a mean energy dissipation $d\bar{E}/dt$, such that the corresponding time of relaxation τ_η is given by

$$\tau_\eta = \frac{\bar{E}}{d\bar{E}/dt} = \frac{3}{2} \frac{\rho}{\eta} \frac{C^2}{\nu^2}, \quad (25)$$

C being the velocity of sound (referred to the velocity of light). The latter quantity can easily be estimated (12.21-39) and one finds

$$C^2 \approx 0.81 \cdot 10^{-2}. \quad (26)$$

Inserting (15), (23), (24), (26) into (25), and remembering the definition of E_m , one gets

$$\tau_\eta \nu \approx 1.35 \cdot 2\pi \cdot \frac{T}{E_m} \cdot \frac{T}{\hbar \nu}. \quad (27)$$

In an actual nucleus, the smallest vibration frequency is $\approx C/R$, which, for $R \approx 10^{-12}$ cm, means a quantum $\hbar \nu \approx 11$ MeV. Since, by (5), $E_m \approx 24$ MeV and the nuclear temperatures for usual excitations do not exceed $T \approx 2$ MeV (say), we see that the vibration will be damped within a time considerably shorter than its period.

(2) Consider in unlimited nuclear matter a plane of discontinuity $(0|T)$ of temperature; assume for simplicity that the heat conductivity and specific heat have constant values corresponding to the temperature $\frac{1}{2}T$.

The time τ_x necessary for the temperature variation at a distance l from the plane of discontinuity to reach the value $\frac{1}{2}T$ is readily found to be

$$\tau_x = \frac{1}{0.92} \cdot \frac{\rho C_V l^2}{\kappa}. \quad (28)$$

Comparing this with the time l/v_m in which a nucleon would on the average travel the same distance if there were no collisions, one gets, using (12), (15), (24),

$$\frac{\tau_x v_m}{l} \approx 28.2 \cdot \frac{S_x l}{r_0^3} \cdot \left(\frac{T}{E_m} \right)^2; \quad (29)$$

with (23), $r_0 = 1.42 \cdot 10^{-13}$ cm, $l \approx R \approx 10^{-12}$ cm, this becomes

$$\frac{\tau_x v_m}{l} \approx 200 \left(\frac{T}{E_m} \right)^2, \quad (30)$$

i.e. for usual excitations ($T/E_m \approx 1/12$),

$$\tau_x v_m / l \approx 1.4,$$

illustrating the great rapidity of heat propagation. For very high excitations, however, such as are produced by impact of fast nucleons (12.3), the local increase of temperature at the point of impact may reach ≈ 7 MeV (excitation energy ≈ 250 MeV) and $\tau_x v_m / l \approx 17.5$: in this case, a local "evaporation" may set in before the excitation has spread over the whole nucleus.

CHAPTER X

NUCLEAR MULTIPLETS

10.1. Classification of nuclear states

10.10. The stationary states of atomic nuclei can be classified according to the invariance properties of the Hamiltonian with respect to different groups of transformation in a way closely analogous to that followed in atomic theory; this point of view has been developed independently by HUND [37] and by WIGNER [37a, b]. The classification will naturally be the more detailed as more specific assumptions are made on the form of nuclear interaction. We shall proceed in order of increasing specialization of this form.

10.11. General invariance properties. In the first place, it follows quite generally from the invariance of nuclear structure with respect to spatial rotations, that the *total angular momentum* in any nuclear state is characterized by a quantum number J , taking either integral or half-integral values, in such a way that the square of the angular momentum is equal to $J(J+1) \hbar^2$. If the mass number A is even, the allowed values of J are 0, 1, 2, ...; if A is odd, one may have $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$.

Likewise, the invariance of the Hamiltonian for a spatial symmetry transformation with respect to a point entails a definite *parity* of the nuclear terms. In such a transformation, the eigenfunction is multiplied by $+1$ or -1 , and the term is then called even or odd, respectively.

10.12. Charge multiplets. There is a further quantum number of universal validity, expressing the invariance of the electric charge of a nuclear configuration. For this purpose, we can take an eigenvalue of the projection on the 3-axis of the total isotopic vector

$$T = \frac{1}{2} \sum_i \tau^{(i)}, \quad (1)$$

i.e.

$$T_3 = \frac{1}{2} \sum_i \tau_3^{(i)}, \quad (2)$$

which can simply be interpreted as half the neutron excess of the configuration. The eigenvalues of T_3 are either integers or half integers, according as the mass number is even or odd.

The charge independence property of the nuclear forces allows us to proceed a step further: as long as we disregard Coulomb energy (and any possible deviation from charge independence of the proper nuclear interaction), we can avail ourselves of the invariance of the (approximate)

Hamiltonian for all rotations in isotopic space (8.1). The operator T^2 then assumes an eigenvalue $T(T+1)$, with integral or half-integral T (according as A is even or odd). To a given T correspond $2T+1$ terms of the same energy, forming a (degenerate) *charge multiplet*; these terms belong to the eigenvalues of T_3 , comprised between $-T$ and $+T$, i.e. to isobaric nuclei of different neutron excesses. Such a charge multiplet is split up into its $2T+1$ components by the Coulomb interaction.

10.13. Spin multiplets. If we assume that the coupling between the spins and the translatory motions of the constituent nucleons is sufficiently small, we get the usual grouping of terms into *spin multiplets*, characterized by the quantum numbers L, S of total orbital momentum and total spin; L can have integral values, S integral or half-integral values, according as A is even or odd. The terms of the multiplet have total angular momenta comprised between $|L-S|$ and $L+S$; the multiplicity is $2S+1$.

Following Hund's proposal, nuclear terms belonging to a spin and charge multiplet can be denoted by symbols similar to those used in atomic spectroscopy, with a slight extension to take account of charge multiplicity: e.g. $^{53}P_0$ is a term of charge multiplicity 5 ($T=2$), spin multiplicity 3 ($S=1$), orbital momentum $L=1$ and total angular momentum $J=0$.

In the quasi-atomic model (9.21), we may introduce individual nucleon states with definite principal and azimuthal quantum numbers n, l . All states with given n and l constitute an "orbital shell", which can be filled by $4(2l+1)$ nucleons. In any configuration built up of such states, the resultant orbital momentum L has a value derived from vector addition of the individual l 's. A characteristic difference from the atomic case lies in the general tendency in nuclei for the stationary states with lower L -values to be the more tightly bound: the reason is that the main interaction between the nucleons, in contrast to that between the atomic electrons, is an attractive one.

10.14. Wigner supermultiplets. It has been especially emphasized by WIGNER [37a]* that it may be useful to consider a more comprehensive grouping of nuclear multiplets, obtained by neglecting all explicit dependence of the nuclear potential on spin and isotopic variables, i.e. by retaining only ordinary and Majorana forces ("Wigner approximation", 9.23). The eigenfunctions of any (degenerate) spin and charge multiplet have the general form

$$\Psi(\vec{x}^{(i)}, \sigma_z^{(i)}, \tau_3^{(i)}) = \frac{1}{\sqrt{f}} \sum_{x=1}^f \Phi_x(\vec{x}^{(i)}) X_x(\sigma_z^{(i)}, \tau_3^{(i)}) \quad (3)$$

of a linear combination of products of (orthogonal and normalized) spatial wave-functions Φ_x and functions X_x of spin and isotopic variables; the linear combination being such as to make Ψ antisymmetric in all

* Minor corrections to this paper are to be found in WIGNER [37b], p. 951, footnote 9..

coordinates. On Wigner's approximation, the energies of a number of spin and charge multiplets will coincide; and the resulting *supermultiplet* will be entirely characterized by the symmetry properties of the spatial wave-functions Φ_x with respect to permutations of the nucleons, i.e. by their *symmetry character* *.

The symmetry properties in question are described by a *partition*

$$\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 = A \quad (\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \Lambda_4) \quad (4)$$

of the total number A of particles into four integers expressing that the Φ_x may be made antisymmetrical with respect to groups of at most $\Lambda_1, \dots, \Lambda_4$ particles separately, or (what amounts to the same, since the total wave-function Ψ must be antisymmetric) that the corresponding factors X_x are symmetrical with respect to groups of at most $\Lambda_1, \dots, \Lambda_4$ nucleons. Symbolically, we denote the "antisymmetrized normal form" of Φ_x by $A(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)$ and the "symmetrized normal form" of X_x by $S(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)$. The reason why the partition (4) contains at most 4 terms is that the X_x obviously cannot be antisymmetrical in more than four particles: the *antisymmetrized* normal form of X_x is therefore of the type

$$A(\underbrace{4 + \dots + 4}_{k_4} + \underbrace{3 + \dots + 3}_{k_3} + \underbrace{2 + \dots + 2}_{k_2} + \underbrace{1 + \dots + 1}_{k_1}),$$

which just implies that the symmetrized normal form has the character $S(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)$, with

$$k_1 = \Lambda_1 - \Lambda_2, \quad k_2 = \Lambda_2 - \Lambda_3, \quad k_3 = \Lambda_3 - \Lambda_4, \quad k_4 = \Lambda_4. \quad (5)$$

The Λ 's can simply be interpreted as the highest numbers of proton or

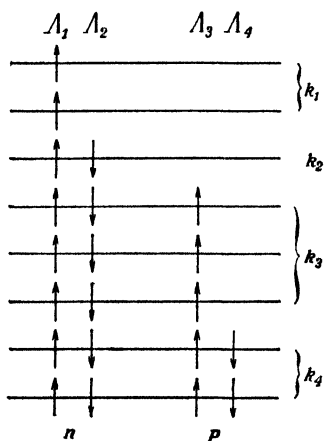


Fig. 10.14. Configuration of symmetry character $\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$. The schematic representation used in this figure is especially adapted to individual nuclear models. The horizontal lines symbolize individual states with the same spatial wave-function; the arrows indicate the spin orientations with respect to a fixed spatial direction (z-components of spins).

n = neutron states, p = proton states.

neutron states with definite spin orientation occurring in the supermultiplet; a configuration exemplifying this interpretation is represented by fig. 10.14, which at the same time illustrates the connexion just utilized between

* A very clear and elementary account of the theory of symmetry characters has been given by HUND [27].

A - and S -symmetry characters, as well as the relations (5) between the k 's and A 's.

Let us introduce, besides T_3 defined by (2), the operators

$$S_z = \frac{1}{2} \sum_i \sigma_z^{(i)} \quad (6)$$

and

$$Y_z = \frac{1}{2} \sum_i \tau_3^{(i)} \sigma_z^{(i)}, \quad (7)$$

representing, respectively, the z -component of the total spin and the difference of the z -components of the total neutron spin and the total proton spin. With the choice of the A 's indicated by fig. 10.14, the quantities

$$\begin{aligned} P &= \frac{1}{2} (A_1 + A_2 - A_3 - A_4) \\ P' &= \frac{1}{2} (A_1 - A_2 + A_3 - A_4) \\ P'' &= \frac{1}{2} (A_1 - A_2 - A_3 + A_4) \end{aligned} \quad (8)$$

have the following meaning: P is the highest value of T_3 (and, therefore, of T) occurring in the supermultiplet, P' is the highest value of S_z (and of S) compatible with the value P of T_3 , while P'' represents the highest value that Y_z can take when the other two operators have the respective values P and P' . Since, however, the correspondence between the A 's and the spin orientations of the proton and neutron states can be established in different ways, by permuting the eigenvalues $+1$ and -1 of τ_3 and σ_z , a more general interpretation of the quantum numbers P, P', P'' is possible: P represents the highest value of any one of the quantities T_3, S_z, Y_z occurring in the supermultiplet; P' is then the highest value of a second one of the three quantities compatible with the value P of the first, and P'' the highest value of the third compatible with the values P, P' of the other two.

On account of the definition (4) of the A 's, one has (WIGNER [37b])

$$\frac{1}{2} A \geq P \geq P' \geq |P''|. \quad (9)$$

It further follows from (4) and (8) that the numbers P, P', P'' are integers or half integers according as A is even or odd, and that

$$P + P' + P'' \text{ is } \begin{cases} \text{even if } A = 4a \\ \text{odd if } A = 4a + 2 \end{cases} \quad (10)$$

(since $P + P' + P'' = 2A_1 - \frac{1}{2}A$). Clearly the set P, P', P'' gives, together with A , a characterization of the supermultiplet equivalent to that expressed by the A 's. It will be noted that the values of P, P', P'' given by (8) remain unchanged when the A 's all change by the same amount. Physically, this change means adding to the original nucleus a certain number of sets of four nucleons with different charges and spins*; formally, as shown by (5), the antisymmetrized normal form of the X_x contains an additional number of sets of four coordinates with respect to which it is antisymmetrical.

* Such sets of nucleons might loosely be described as " α -particles".

The substates of a supermultiplet belong to eigenvalues of S_z , T_3 and Y_z , but — in contrast to the well-known case of the ordinary spin multiplets — such a set of eigenvalues may belong to more than one substate. These substates are in fact split up by the forces depending on the spin and isotopic variables into different spin and charge multiplets, generally giving rise to several states with the same set of eigenvalues of S_z and T_3 . (In this splitting, states with different values of Y_z will in general combine, so that Y_z ceases to furnish a quantum number.)

10.2. Stationary states of light nuclei

10.20. The quantum numbers and energies of the stationary states to be expected for light nuclei can be determined by means of the *quasi-atomic model*. Quantitative energy estimates can hardly be trusted, but the resulting order of the levels, i.e. the assignment of L , S , T values to the successive terms, might still be expected to come out right. The ground states and excited terms have been found in this way for nuclear configurations corresponding to the successive completion of the first s and p orbital shells (as well as for some configurations in which also the first d shell is partly filled up). These results are summarized in the following tables, compiled from HUND's [37] paper; similar results have been obtained independently by WIGNER [37a] and his collaborators (FEENBERG and WIGNER [37a], FEENBERG and PHILLIPS [37b]). We have only retained the enumeration of the possible stationary states in their probable order of excitation; for the (unreliable) energy computations, the reader is referred to the papers quoted.

From the present point of view, the nuclei are conveniently divided into three classes, according as A is of the form $4a \pm 1$, $4a + 2$ or $4a$. In the first column of each table the uncompleted shell is indicated by its symbol s , p , d with the number of nucleons in this shell as an exponent; e.g. d^{19} represents the configuration $s^4p^{12}d^{19}$; configurations with k and f — k nucleons in a shell comprising f individual states have the same types and arrangement of levels (the latter configuration has k "holes" in the last shell). The second column gives the Wigner supermultiplets (P, P', P''), the ground state lowest. In the third column are found the corresponding partitions of the X_x ; the A 's composing the partition $S(A_1 + A_2 + A_3 + A_4)$ are only defined *modulo* 1, and there is a corresponding indeterminacy in the number of 4's occurring in the reciprocal symmetry character $A(4 + \dots)$. The subsequent columns contain, for the various possible values of the (absolute) neutron excess $|n|$, the possible types of levels, also arranged in order of (probable) excitation; levels connected by a $=$ -sign coincide on Wigner's approximation.

Auxiliary tables give probable attributions of definite configurations to the known light nuclei of each class; the justification for such attributions will be found in 10.32. Finally, a small table is added to bring out the regularities concerning the ground states of the nuclei considered; for odd nuclei, it has been assumed that the spin dependent forces depress the triplet term (11.41).

10.21. Nuclei of odd mass number

Shell filling	Wigner super-multiplet	Partitions	$ n = 1$	$ n = 3$	$ n = 5$	Partitions	Wigner super-multiplet	Shell filling
s	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(1), S(1)$	${}^2S$			$A(3), S(1+1+1)$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	s^3
p	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(1), S(1)$	${}^2p$			$A(3), S(1+1+1)$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	p^{11}
p^3	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}$ $\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$ $\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$A(1+1+1), S(3)$ $A(2+1), S(2+1)$ $A(3), S(1+1+1)$	$\left\{ \begin{array}{l} {}^4S = {}^2S \\ {}^4D = {}^2D = {}^2p \\ {}^4p = {}^2p = {}^2p \end{array} \right\}$ $\left\{ \begin{array}{l} {}^2F \\ {}^2p \end{array} \right\}$	${}^2S$ ${}^4D$ ${}^4p$		$A(3+3+3), S(3+3+3)$ $A(3+2), S(2+2+1)$ $A(1), S(1)$	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}$ $\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$ $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	p^9
p^5	$\frac{5}{2}, \frac{1}{2}, \frac{1}{2}$ $\frac{3}{2}, \frac{3}{2}, \frac{1}{2}$ $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$ $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(2+2+1), S(3+2)$ $A(3+1+1), S(3+1+1)$ $A(3+2), S(2+2+1)$ $A(4+1), S(2+1+1+1)$	$\left\{ \begin{array}{l} {}^6p = {}^2p = {}^4p = {}^4p \\ \quad \quad \quad = {}^2p = {}^2p \end{array} \right\}$ $\left\{ \begin{array}{l} {}^4D = {}^4D = {}^2D = {}^2D \\ {}^4S = {}^4S = {}^2S = {}^2S \end{array} \right\}$ $\left\{ \begin{array}{l} {}^4F = {}^2F = {}^2F \\ {}^4D = {}^2D = {}^2D \\ {}^4p = {}^2p = {}^2p \end{array} \right\}$ $\left\{ \begin{array}{l} {}^2G \\ {}^2F \\ {}^2D \\ {}^2p \end{array} \right\}$	${}^6p = {}^4p = {}^4p$ ${}^4D = {}^4D$ ${}^4S = {}^4S$ ${}^4F$ ${}^4D$ ${}^4p$	${}^6p$	$A(3+2+2), S(3+3+1)$ $A(3+3+1), S(3+2+2)$ $A(2+1), S(2+1)$ $A(3), S(1+1+1)$	$\frac{5}{2}, \frac{1}{2}, \frac{1}{2}$ $\frac{3}{2}, \frac{3}{2}, \frac{1}{2}$ $\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$ $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	p^7
d	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(1), S(1)$	${}^2d$			$A(3), S(1+1+1)$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	d^{19}

Parity: when s or d shell is filling, all terms even; when p shell is filling, all terms odd.

10.211. Light nuclei of odd mass number					
	n =				
	- 1	+ 1	- 1	+ 1	
s	$^{17}_9\text{F}$	$^{17}_8\text{O}$	^3_2He $^{19}_{10}\text{Ne}$	^3_1H $^{19}_9\text{F}$	s^3
p		^5_2He	$^{15}_8\text{O}$	$^{15}_7\text{N}$	p^{11}
p ³	^7_4Be	^7_3Li	$^{13}_7\text{N}$	$^{13}_6\text{C}$	p^9
p ⁵	^9_5B	^9_4Be	$^{11}_6\text{C}$	$^{11}_5\text{B}$	p^7
d	$^{21}_{11}\text{Na}$	$^{21}_{10}\text{Ne}$	$(^{39}_{20}\text{Ca})$	$^{39}_{19}\text{K}$	d^{19}

10.221. Light nuclei of mass number A = 4a+2					
	n =				
	0	2	0	2	
s ²	^2_1H $^{18}_9\text{F}$	$^{18}_8\text{O}$			
p ²	^6_3Li	^6_2He	$^{14}_7\text{N}$	$^{14}_6\text{C}$	p^{10}
p ⁶	$^{10}_5\text{B}$	$^{10}_4\text{Be}$			

10.22. Nuclei of mass number $A = 4a + 2$						
Shell filling	Wigner super-multiplet	Partitions	$n = 0$ (odd nuclei)	$ n = 2$ (even nuclei)	$ n = 4$ (odd nuclei)	$ n = 6$ (even nuclei)
s^2	1, 0, 0	$A(2), \quad S(1+1)$	${}^3_1S = {}^{13}_1S$	${}^3_1S$		
p^2, p^{10}	$\left. \begin{array}{l} 1, 1, 1 \\ 1, 0, 0 \end{array} \right\}$	$\left. \begin{array}{l} A(1+1), \quad S(2) \\ A(2), \quad S(1+1) \end{array} \right\}$	$\left. \begin{array}{l} {}^{11}_1P = {}^{33}_1P \\ {}^{31}_1D = {}^{13}_1D \\ {}^{31}_1S = {}^{13}_1S \end{array} \right\}$	$\left. \begin{array}{l} {}^{33}_1P \\ {}^{31}_1D \\ {}^{31}_1S \end{array} \right\}$		
p^6	$\left. \begin{array}{l} 3, 0, 0 \\ 2, 1, 0 \\ 1, 1, -1 \\ 1, 1, 1 \\ 1, 0, 0 \end{array} \right\}$	$\left. \begin{array}{l} A(2+2+2), \quad S(3+3) \\ A(3+2+1), \quad S(3+2+1) \\ A(3+3), \quad S(2+2+2) \\ A(4+1+1), \quad S(3+1+1+1) \\ A(4+2), \quad S(2+2+1+1) \end{array} \right\}$	$\left. \begin{array}{l} {}^{71}_1S = {}^{17}_1S = {}^{53}_1S = {}^{35}_1S \\ \quad \quad \quad = {}^{31}_1S = {}^{13}_1S \\ {}^{53}_1D = {}^{35}_1D = {}^{51}_1D = {}^{15}_1D \\ \quad \quad \quad = {}^{33}_1DD = {}^{31}_1D = {}^{13}_1D \\ {}^{53}_1P = {}^{35}_1P = {}^{51}_1P = {}^{15}_1P \\ \quad \quad \quad = {}^{33}_1PP = {}^{31}_1P = {}^{13}_1P \\ {}^{33}_1FF = {}^{11}_1FF \\ {}^{33}_1PP = {}^{11}_1PP \\ {}^{33}_1FF = {}^{11}_1FF \\ {}^{33}_1PP = {}^{11}_1PP \\ {}^{31}_1G = {}^{13}_1G \\ {}^{31}_1F = {}^{13}_1F \\ {}^{31}_1D = {}^{13}_1D \\ {}^{31}_1D = {}^{13}_1D \\ {}^{31}_1S = {}^{13}_1S \end{array} \right\}$	$\left. \begin{array}{l} {}^{71}_1S = {}^{53}_1S = {}^{35}_1S = {}^{31}_1S \\ {}^{53}_1D = {}^{35}_1D = {}^{51}_1D \\ \quad \quad \quad = {}^{33}_1DD = {}^{31}_1D \\ {}^{53}_1P = {}^{35}_1P = {}^{51}_1P \\ \quad \quad \quad = {}^{33}_1PP = {}^{31}_1P \\ {}^{33}_1FF \\ {}^{33}_1PP \\ {}^{33}_1FF \\ {}^{33}_1PP \\ {}^{31}_1G \\ {}^{31}_1F \\ {}^{31}_1D \\ {}^{31}_1D \\ {}^{31}_1S \end{array} \right\}$	$\left. \begin{array}{l} {}^{71}_1S = {}^{53}_1S \\ {}^{53}_1D = {}^{51}_1D \\ {}^{53}_1P = {}^{51}_1P \\ {}^{33}_1FF \\ {}^{33}_1PP \\ {}^{33}_1FF \\ {}^{33}_1PP \\ {}^{31}_1G \\ {}^{31}_1F \\ {}^{31}_1D \\ {}^{31}_1D \\ {}^{31}_1S \end{array} \right\}$	${}^{71}_1S$

Parity: all terms even.

10.23. Nuclei of mass number $A = 4a$					
Shell filling	Wigner super-multiplet	Partitions	$n = 0$ (even nuclei)	$ n = 2$ (odd nuclei)	$ n = 4$ (even nuclei)
s^4	0, 0, 0	$A(4), \quad S(1+1+1+1)$	^{11}S		
p^4, p^8	2, 1, 1	$A(2+1+1), S(3+1)$	$^{53}P = ^{35}P = ^{33}P = ^{31}P = ^{13}P$	$^{53}P = ^{35}P = ^{33}P = ^{31}P = ^{13}P$	^{53}P
	2, 0, 0	$A(2+2), \quad S(2+2)$	$^{51}D = ^{15}D = ^{33}D = ^{11}D$ $^{51}S = ^{15}S = ^{33}S = ^{11}S$	$^{51}D = ^{33}D$ $^{51}S = ^{33}D$	^{51}D ^{51}S
	1, 1, 0	$A(3+1), \quad S(2+1+1)$	$^{33}F = ^{31}F = ^{13}F$ $^{33}D = ^{31}D = ^{13}D$ $^{33}P = ^{31}P = ^{13}P$	$^{33}F = ^{31}F$ $^{33}D = ^{31}D$ $^{33}P = ^{31}P$	
	0, 0, 0	$A(4), \quad S(1+1+1+1)$	^{11}G ^{11}D ^{11}S		
p^{12}	0, 0, 0	$A(4), \quad S(1+1+1+1)$	^{11}S		
d^{20}	0, 0, 0	$A(4), \quad S(1+1+1+1)$	^{11}S		

Parity: all terms even.

10.231. Light nuclei of mass number $A = 4a$				
	$n =$			
	0	2	0	2
s^4	^4_2He $^{20}_{10}\text{Ne}$			
p^4 p^{12}	^8_4Be $^{16}_8\text{O}$	^8_3Li	$^{12}_6\text{C}$ $^{12}_5\text{B}$	p^8
d^{20}	$^{40}_{20}\text{Ca}$			

10.24. Ground states of light nuclei		
Type of nucleus		Ground state
odd-mass nuclei	$\left\{ \begin{array}{l} s \text{ shell filling} \\ p \text{ shell filling} \end{array} \right.$	^2S ^2P
	even nuclei	^1S
odd nuclei	$\left\{ \begin{array}{l} n = 0 \\ n \neq 0 \end{array} \right.$	^3S ^3P

10.3. Energies of nuclear states

10.31. *Nuclear potential energy in a supermultiplet state.* The nuclear interaction operator which underlies Wigner's approximation is of the form (4.341)

$$\mathcal{V}_W = \frac{1}{2} \sum_{i,k} J(\vec{x}^{(i)} - \vec{x}^{(k)}) (W + M P_x^{(ik)}). \quad (1)$$

Since it does not affect the factors X_r of the eigenfunction (10.14-3), its expectation value in the corresponding state may be written

$$\begin{aligned} \text{av } \mathcal{V}_W = \frac{1}{f} \sum_{x=1}^f \frac{1}{2} \sum_{i,k} \int \Phi_x^* (\vec{x}^{(1)}, \dots, \vec{x}^{(A)}) J(\vec{x}^{(i)} - \vec{x}^{(k)}) \\ \cdot (W + M P_x^{(ik)}) \Phi_x (\vec{x}^{(1)}, \dots, \vec{x}^{(A)}) d\nu^{(1)} \dots d\nu^{(A)}. \end{aligned} \quad (2)$$

The evaluation of this expression may be effected by a method due to WIGNER [37b]. It is based on the fundamental property of the functions Φ_x of given symmetry character to transform linearly among themselves when a permutation R is performed on the space coordinates:

$$R \Phi_x (\vec{x}^{(l)}) \equiv \Phi_x (R \vec{x}^{(l)}) = \sum_{\lambda=1}^f D_{x\lambda}(R) \Phi_\lambda (\vec{x}^{(l)}); \quad (3)$$

the unitary matrices $D(R)$ are determined (apart from a common transformation $D \rightarrow S^{-1}DS$) by the symmetry character of the Φ_x : — in the language of group theory they form an irreducible representation of degree f of the symmetric group. The expectation value of any operator $\mathcal{W} = \frac{1}{2} \sum \mathcal{W}^{(ik)}$, with $\mathcal{W}^{(ik)} \equiv \mathcal{W}(\vec{x}^{(i)}, \vec{x}^{(k)})$, can then be expressed in terms of $\mathcal{W}^{(12)}$ only, as follows:

$$\begin{aligned} \text{av } \mathcal{W} &= \frac{1}{f} \sum_{x=1}^f \frac{1}{2} \sum_{i,k} \int \Phi_x^* (P_x^{(i1)} P_x^{(k2)} \vec{x}^{(1)}) \mathcal{W}^{(12)} \Phi_x (P_x^{(i1)} P_x^{(k2)} \vec{x}^{(1)}) d\nu^{(1)} \dots d\nu^{(A)} \\ &= \frac{1}{f} \sum_{x,\lambda} \sum_{\mu=1}^f \frac{1}{2} \sum_{i,k} D_{x\lambda}^* (P_x^{(i1)} P_x^{(k2)}) D_{x\mu} (P_x^{(i1)} P_x^{(k2)}) \int \Phi_\lambda^* (\vec{x}^{(1)}) \mathcal{W}^{(12)} \Phi_\mu (\vec{x}^{(1)}) d\nu^{(1)} \dots d\nu^{(A)}; \end{aligned}$$

or, since $\sum_x D_{x\lambda}^* D_{x\mu} = \delta_{\lambda\mu}$,

$$\text{av } \mathcal{W} = \frac{A(A-1)}{2} \frac{1}{f} \sum_{x=1}^f \int \Phi_x^* (\vec{x}^{(1)}) \mathcal{W}^{(12)} \Phi_x (\vec{x}^{(1)}) d\nu^{(1)} \dots d\nu^{(A)}. \quad (4)$$

We may now make use of the above-mentioned arbitrariness in the representation D to make all Φ_x either symmetric or antisymmetric in $\vec{x}^{(1)}, \vec{x}^{(2)}$, while retaining their orthogonality; if the first s functions Φ_x are symmetric in $\vec{x}^{(1)}, \vec{x}^{(2)}$ and the others antisymmetric, the matrix $D(P_x^{(12)})$ will be diagonal, the first s elements being $= +1$ and the others $= -1$, so that s is defined in an invariant way by

$$\text{tr } D(P_x^{(12)}) = 2s - f. \quad (5)$$

(The trace of the matrix $D(P_x^{(12)})$ is called in group theory the "character" of the operation $P_x^{(12)}$ in the representation in question.) Let us denote by

$$\begin{aligned} g_s(\vec{x}^{(1)}, \vec{x}^{(2)}) &= \frac{1}{s} \sum_{z=1}^s \int |\Phi_z|^2 dv^{(3)} dv^{(4)} \dots dv^{(A)} \\ g_a(\vec{x}^{(1)}, \vec{x}^{(2)}) &= \frac{1}{f-s} \sum_{z=s+1}^f \int |\Phi_z|^2 dv^{(3)} dv^{(4)} \dots dv^{(A)} \end{aligned} \quad (6)$$

the probabilities for two symmetrically or antisymmetrically coupled particles to be at a distance $|\vec{x}^{(1)} - \vec{x}^{(2)}|$ from each other.

Turning back to (2), we may write it successively, using (4), (3) and (6),

$$\begin{aligned} \text{av } \mathcal{V}_W &= \frac{A(A-1)}{2} \frac{1}{f} \sum_{z=1}^f \int \Phi_z^* J(\vec{x}^{(1)} - \vec{x}^{(2)}) (W + M P_x^{(12)}) \Phi_z dv^{(1)} \dots dv^{(A)} \\ &= \frac{A(A-1)}{2} \frac{1}{f} \sum_{z=1}^f \int \Phi_z^* J(\vec{x}^{(1)} - \vec{x}^{(2)}) [W + M D_{xz}(P_x^{(12)})] \Phi_z dv^{(1)} \dots dv^{(A)} \quad (7) \\ &= \frac{A(A-1)}{2} \frac{1}{f} [(W+M)s \int g_s J dv^{(1)} dv^{(2)} + (W-M)(f-s) \int g_a J dv^{(1)} dv^{(2)}]. \end{aligned}$$

Putting

$$\mathcal{L}^\pm \equiv \frac{1}{2} \int (g_s \pm g_a) J dv^{(1)} dv^{(2)}, \quad (8)$$

and taking account of (5), we finally get

$$\text{av } \mathcal{V}_W = \frac{A(A-1)}{2} (W \mathcal{L}^+ + M \mathcal{L}^-) - \mathcal{E} (W \mathcal{L}^- + M \mathcal{L}^+) \quad (9)$$

with

$$\mathcal{E} = - \frac{A(A-1)}{2} \frac{1}{f} \text{tr } D(P_x^{(12)}).$$

The interest of this formula lies in the fact that the coefficient $\mathcal{E}$ is entirely determined by the symmetry character of the supermultiplet and can thus be calculated without knowing the eigenfunctions.

The integrals $\mathcal{L}^\pm$ cannot be evaluated on general lines, but they may reasonably be expected to vary smoothly with the mass number and the supermultiplet considered: the rapid irregular variations of nuclear energies with mass number should then be accounted for by $\mathcal{E}$; we shall see presently how this conception has been put to the test of observation. If we assume $J(r)$ to be positive for all values of r , and to vary sufficiently smoothly with r , both integrals $\mathcal{L}^\pm$ will be positive. The bulk of the integrals is contributed by an interval of distances $|\vec{x}^{(1)} - \vec{x}^{(2)}|$ of the order of the range of force, which for large nuclei is small compared with the nuclear radius. The g 's being in that case about inversely proportional to

the square of the nuclear volume, the $\mathcal{L}$'s will be inversely proportional to this volume: in other words, for large values of A , the $\mathcal{L}$'s become inversely proportional to A .

The explicit expression for $\mathcal{E}$ in terms of the quantum numbers P, P', P'' of the supermultiplet is, of course, immediately given by group theory. But it can also be derived in a quite elementary way. In the first place, we may treat the problem on the fictitious assumption that the range of the nuclear force is so large, compared with nuclear dimensions, that the variation of $J(r)$ over the volume of the nucleus can be neglected: we shall call this the *long range fiction*. In this case, $\mathcal{L}^+ = J$ and $\mathcal{L}^- = 0$, so that (9) becomes

$$\text{av } \mathcal{V}_{W, \text{ long range}} = [\tfrac{1}{2} A(A-1) W - \mathcal{E} M] J. \quad (10)$$

The interpretation of the first term of this formula is evident; the second term shows us that we shall get an expression for $\mathcal{E}$ by calculating in a more direct way the expectation value of the nuclear potential energy for a pure Majorana interaction of long range. In fact,

$$\mathcal{E} = - \text{av } \tfrac{1}{2} \sum_{i,k} P_x^{(ik)}, \quad (11)$$

and simple symmetry considerations (HUND [37]) lead to the following result: Let n_a be the number of antisymmetrically coupled pairs of coordinates in the antisymmetrized normal form of the spatial wave-functions, and n_s the number of symmetrically coupled pairs in their symmetrized normal form. We may then write:

$$\mathcal{E} = n_a - n_s. \quad (12)$$

We now go a step further in the schematization of the problem by assuming an individual model and choosing a specified stationary state of the given symmetry character; that corresponding to the configuration represented by fig. 10.14 is especially suited to our purpose, because it permits an easy evaluation of the numbers n_a, n_s : the former is the number of pairs of like nucleons with parallel spins, while the latter is the number of *bonds*, this term denoting a pair of nucleons which belong to the same spatial wave-function. This result can be verified directly from (11) (INGLIS and YOUNG [37]) by setting up the total wave-function which describes the stationary state in question as a Slater determinant (4.14) and by remembering that (4.341-32) $P_x = -P_\sigma P_\tau$. From fig. 10.14 we read off

$$\begin{aligned} n_a &= \sum_{l=1}^4 \tfrac{1}{2} A_l (A_l - 1) \\ n_s &= 6 k_4 + 3 k_3 + k_2 \\ &= 3 A_4 + 2 A_3 + A_2, \end{aligned} \quad (13)$$

by (10.14-5), so that

$$\mathcal{E} = \frac{1}{2} [A_1(A_1-1) + A_2(A_2-3) + A_3(A_3-5) + A_4(A_4-7)]. \quad (14)$$

Using (10.14-4, 8), we may also write

$$\mathcal{E} = \frac{1}{2} \left[\frac{1}{2} A \left(\frac{1}{2} A - 8 \right) + P(P+4) + P'(P'+2) + P''^2 \right] \quad (15)$$

or

$$\mathcal{E} = \frac{1}{8} A^2 - 2A - \frac{5}{2} + \mathcal{E}' \quad (16)$$

with

$$\mathcal{E}' = \frac{1}{2} [(P+2)^2 + (P'+1)^2 + P''^2].$$

10.311. *Supermultiplet quantum numbers of ground state.* As we shall see in the following Chapter (11.33), we have reason to suppose that the parameters W and M in formula (1) satisfy the inequalities

$$-M > W > 0 \quad (17)$$

Since, on the other hand, the $\mathcal{L}$'s are positive and $\mathcal{L}^+ > \mathcal{L}^-$, the coefficient $-(W\mathcal{L}^- + M\mathcal{L}^+)$ of $\mathcal{E}$ in formula (9) will be positive. The maximum binding energy will therefore correspond to the minimum value of $\mathcal{E}$ or, by (16), of $\mathcal{E}'$; i.e. the ground state will belong to the supermultiplet with the smallest possible quantum numbers $P, P', |P''|$.

The neutron excess n being given, we therefore take

$$P = \frac{1}{2} |n|; \quad (18)$$

the minimum value of P' and $|P''|$ then depend on the parity of A . If A is odd, they are both $\frac{1}{2}$ and the ground state supermultiplet is

$$\left(\frac{1}{2} |n|, \frac{1}{2}, \pm \frac{1}{2} \right) \quad (\text{odd } A). \quad (19)$$

To determine the sign of P'' , we notice that according to (10.14-4, 8) and (18), we have

$$\begin{aligned} A_1 + A_2 &= \{N \text{ or } Z\} \\ A_1 - A_2 &= P' + P'' \geq 0, \end{aligned} \quad (20)$$

the notation $\{N \text{ or } Z\}$ representing the larger of the two numbers N, Z of neutrons or protons of the nucleus. According as this number $\{N \text{ or } Z\}$ is even or odd, the minimum value of $P' + P''$ is thus 0 or 1. Hence the rule: for odd A , P'' is $+\frac{1}{2}$ or $-\frac{1}{2}$ according as $\{N \text{ or } Z\}$ is odd or even. This rule may be expressed in other equivalent ways as follows:

10.311. Sign of P'' for odd A					
	$P = \frac{1}{2} n =$			N is	
	$2t + \frac{1}{2}$	$2t - \frac{1}{2}$		even	odd
$A = 4a + 1$	+	-	$n > 0$	-	+
$A = 4a + 3$	-	+	$n < 0$	+	-

(21)

If A is even, the same argument immediately leads to the conclusion: the minimum values of (P', P'') are $(0, 0)$ or $(1, 0)$ according as we have to do with an even or an odd nucleus (3.1). This result cannot be valid, however, for odd nuclei with zero neutron excess, since it would imply a smaller value for P than for P' . In this exceptional case, we must therefore abandon (18) and (20); but since, according to (10.14–10), the minimum value of $P + P' + P''$ is then 1, we may take $(P, P', P'') = (1, 0, 0)$. Summarizing, we may state that the supermultiplet to which the ground state of an even mass nucleus belongs is fixed by the following rules:

	$n \neq 0$		$n = 0$
	$ n = 4\nu$	$ n = 4\nu + 2$	
$A = 4a$	$(\frac{1}{2} n , 0, 0)$	$(\frac{1}{2} n , 1, 0)$	$(0, 0, 0)$
$A = 4a + 2$	$(\frac{1}{2} n , 1, 0)$	$(\frac{1}{2} n , 0, 0)$	$(1, 0, 0)$

(22)

We can now draw a graph of $\mathcal{E}'$ against of $P (= \frac{1}{2}|n|)$ by joining by straight lines the successive points of coordinates $(P, \mathcal{E}')$, the value of A being kept fixed. For odd A , such graphs will, according to (19), show a smooth increase with increasing P , while for even A , as shown by (22), this increase will be regularly interrupted by a succession of breaks. It is this characteristic difference which, so far as the approximative formula (9) may be trusted, would be responsible for the different behaviours of isobars of odd and even mass numbers (3.2). Likewise, it would account for the kinks occurring with a periodicity of 4 mass units in the mass-defect curve of the lightest stable nuclei (3.221).

10.32. Mass-defects of light nuclei on Wigner's approximation. To get the total binding energy of the ground state of a nucleus, we have to add to the nuclear potential energy term $av \mathcal{V}_w$, given by (9), the contribution from the kinetic and Coulomb energies. The latter can be written in the simple form (2.21–1), with (2.1–1); i.e., since $Z = \frac{1}{2}A - T_3$,

$$G = G_0 \left[\frac{1}{2}A \left(\frac{1}{2}A - 1 \right) - T_3 (A - T_3 - 1) \right] \quad (23)$$

$$G_0 = \frac{3}{5} \cdot \frac{e^2}{r_0} \cdot \frac{1}{A^{\frac{1}{3}}} = \frac{0.6}{A^{\frac{1}{3}}} \text{ MeV,}$$

according to (2.21–6). For the kinetic energy, we shall adopt (for want of better) the estimate (9.41–7) based on the Fermi gas model approximation. This shows that the kinetic energy depends only on the mass number A , except for a relatively small correction term of the form kP^2/A ; although this result concerns only even nuclei of large mass number, we may extend it to light nuclei, the value of the co-factor k being suitably specified for the different types of such nuclei. Collecting all terms, we get

for the total binding energy an expression of the type

$$|\mathcal{E}| = |\mathcal{E}_0(A)| - \mathcal{E}' \mathcal{L}(A) + G_0(A - T_3 - 1) T_3 - k \frac{P^2}{A}; \quad (24)$$

the coefficient $\mathcal{L}(A)$ of $\mathcal{E}'$ is a positive, smoothly varying function of A ; the function $\mathcal{E}_0(A)$ represents the sum of all the contributions depending only (or primarily) on A , which make up the greater part of the binding energy. Formula (24), which is due to WIGNER [37b], is not expected to hold for heavy nuclei owing to the perturbing effect of the Coulomb interactions on the supermultiplet system; for $A \lesssim 8$, the kinetic energy term becomes too uncertain and the formula becomes unreliable in this region too.

For intermediate nuclei (up to $A \approx 50$), Wigner's formula has been extensively compared with experiment by BARKAS [39a]. The observed mass-defects are first combined with the calculated Coulomb and kinetic energy corrections. The values so obtained for all nuclei with the same value of $\mathcal{E}'$ yield a graph of $|\mathcal{E}_0(A)| - \mathcal{E}' \mathcal{L}(A)$ against A . Comparison of such graphs for different $\mathcal{E}'$ gives the function $\mathcal{L}(A)$. Once this is known, $\mathcal{E}_0(A)$ can also be determined. The function $\mathcal{L}(A)$ derived in this way exhibits a smooth decrease with increasing A , starting from a value of about 3.8 mMU for $A = 8$, and it asymptotically tends to the form $50/A$ mMU (valid for $A \gtrsim 15$), in conformity with expectation (10.31).

As regards the function $|\mathcal{E}_0(A)|$, it is found to increase with A in a roughly linear manner, corresponding in the interval $A = 20 \dots 40$ to a value of the order of 9 mMU per nucleon. This increase is not altogether smooth, however: it exhibits two important breaks, one in the region $A = 16 \dots 20$, the other at $A = 40$. On the quasi-atomic model, such breaks can be interpreted as revealing the completion of orbital shells (10.13). If the Coulomb forces are sufficiently small, the stablest configurations of even mass nuclei are those of zero neutron excess; as long as this is the case (i.e. up to $A = 40$), orbital shells can be completely filled up with sets of two protons and two neutrons with saturated spins. The $1s$ shell would be completed with ${}^4\text{He}$; then would begin the filling of a p shell, which should be called according to the usual nomenclature the $2p$ shell: this shell would be closed in ${}^{16}\text{O}$. According to the evidence under discussion on the variation of mass-defect with mass number, the next shell would again be an s one, the $2s$ shell, terminating with ${}^{20}\text{N}$. It would be followed by a d shell, the $3d$ shell, completely filled up in ${}^{40}\text{Ca}$.

It must also be mentioned that Barkas was able to estimate the order of magnitude of the neglected variations of the spin dependent interaction within a group of isobaric nuclei and to correct the calculated binding energies for this omission. For this purpose, he compared isobars of mass $4a + 2$ with respective neutron excesses 0 and 2: the former is an odd nucleus, the latter an even one, the ground states of which, as shown by (22), both belong to the supermultiplet $(1, 0, 0)$. In both, the two nucleons

in excess of a completely saturated configuration of $2a$ protons and $2a$ neutrons are symmetrically coupled as regards their space coordinates. But the nuclei will be expected to differ as to the spin multiplicity: the odd nucleus will be (11.41) in a triplet state, $S = 1$, while for the even one, $S = 0$. The effective spin dependent interactions will therefore be of the ${}^3S$ and ${}^1S$ types, respectively. Their difference can readily be derived from that between the empirical mass-defects, when account is further taken of the easily estimated difference in Coulomb energy. In this way, values are found for the spin energy difference ranging from 4.6 mMU for the pair ${}^6\text{He} - {}^6\text{Li}$ to 0.6 mMU for the pair ${}^{30}\text{Si} - {}^{30}\text{P}$. The sign is such that the ${}^3S$ force is a stronger attraction than the ${}^1S$ attractive force. The correction adopted by Barkas consists in adding half the difference to the binding energy $\mathcal{E}_0(A)$ of the odd isobar and subtracting the same amount from that of the even one. The effect of the spin dependent interactions on the binding energies of the ground state thus appears on the whole to be rather small; this is somewhat surprising in view of the fact that the proportion of these forces in the nuclear potential should by no means be small compared with the ordinary and Majorana forces. Also in the interpretation of another important group of empirical data, pertaining to β -active nuclei (A1.2), Wigner's approximation meets with a larger amount of success than one might expect on that account.

10.33. Rough estimate of spin dependent interactions. The contribution of the spin dependent forces to the expectation value of the potential energy in a supermultiplet state can easily be estimated in the crude approximation afforded by the long range fiction (10.31). For if we replace the potential function $J(r)$ by a constant J , we have just to compute the expectation values

$$\text{av } \frac{1}{2} \sum_{i,k} P_{\tau}^{(ik)} \quad , \quad \text{av } \frac{1}{2} \sum_{i,k} P_{\tau'}^{(ik)}$$

in a state with given spin and isotopic quantum numbers S, T . Following a well-known procedure due to Dirac, one starts from

$$\text{av } \sum_{i,k} \vec{\sigma}^{(i)} \vec{\sigma}^{(k)} = \text{av } [(\sum_i \vec{\sigma}^{(i)})^2 - \sum_i (\vec{\sigma}^{(i)})^2] = 4 S(S+1) - 3A, \quad (25)$$

whence

$$\text{av } \frac{1}{2} \sum_{i,k} P_{\sigma}^{(ik)} = \text{av } \frac{1}{2} \sum_{i,k} \frac{1 + \vec{\sigma}^{(i)} \vec{\sigma}^{(k)}}{2} = \frac{1}{4} A(A-4) + S(S+1). \quad (26)$$

Similarly,

$$\text{av } \frac{1}{2} \sum_{i,k} P_{\tau}^{(ik)} = \frac{1}{4} A(A-4) + T(T+1). \quad (27)$$

Combining these results with formula (10), we find for the expectation value of the general interaction operator $\mathcal{V} = J\mathcal{O}$, with $\mathcal{O}$ given by (8.2-4),

$$\begin{aligned} \text{av } \mathcal{V}_{\text{long range}} = & J \left\{ \frac{1}{2} A(A-1) a_0 + [S(S+1) - \frac{3}{4} A] 2a_{\sigma} + [T(T+1) - \frac{3}{4} A] 2a_{\tau} \right. \\ & \left. + [P(P+4) + P'(P'+2) + P''^2 - S(S+1) - T(T+1) - \frac{3}{4} A] 2a_{\sigma\tau} \right\}. \end{aligned} \quad (28)$$

In deriving this expression, use has also been made of the formulae (8.2-5) and (15).

10.34. Coulomb energy of light nuclei. It has been noted in 3.3 that a marked influence of exchange Coulomb interaction on the mass-defect can be perceived for the lightest nuclei. FEENBERG and GOERTZEL [46a] have recently analyzed this effect on more general assumptions; in particular, their discussion does not involve the Slater-determinant representation of the wave-function, on which Phillips and Feenberg's just recalled interpretation in terms of exchange Coulomb energy was based. The more fundamental distinction, which replaces that between ordinary and exchange interactions, is that between symmetrically and antisymmetrically coupled pairs (10.31) of protons. Such pairs are characterized by the operators

$$\begin{aligned}\Sigma_p^{(ik)} &= \frac{1}{2} (1 + P_x^{(ik)}) T_-^{(i)} T_-^{(k)} \\ A_p^{(ik)} &= \frac{1}{2} (1 - P_x^{(ik)}) T_-^{(i)} T_-^{(k)},\end{aligned}\quad (29)$$

the expectation values of which we shall denote by p_s, p_a :

$$p_s = \text{av } \Sigma_p^{(12)} \quad , \quad p_a = \text{av } A_p^{(12)}; \quad (30)$$

these quantities are, of course, the same for any pair (ik) , and $\frac{1}{2}A(A-1)p_s$, $\frac{1}{2}A(A-1)p_a$ represent the respective numbers of proton pairs for which the symmetrized normal form of the spatial wave-functions is symmetrical, and the antrisymmetrized normal form antisymmetrical.

The average contributions of the two kinds of proton pairs to the expectation value of the Coulomb energy will be denoted by

$$G_s = \frac{1}{p_s} \text{av } \frac{e^2}{r^{(12)}} \Sigma_p^{(12)} \quad , \quad G_a = \frac{1}{p_a} \text{av } \frac{e^2}{r^{(12)}} A_p^{(12)}. \quad (31)$$

The total Coulomb energy,

$$\begin{aligned}G &= \text{av } \frac{1}{2} \sum_{i,k} \frac{e^2}{r^{(ik)}} T_-^{(i)} T_-^{(k)} = \frac{1}{2} A(A-1) \text{av } \frac{e^2}{r^{(12)}} T_-^{(1)} T_-^{(2)} \\ &= \frac{1}{2} A(A-1) (p_s G_s + p_a G_a),\end{aligned}\quad (32)$$

can be decomposed into

$$G = \frac{1}{2} A(A-1) [(p_s + p_a) G^+ + (\frac{3}{4} p_s - \frac{1}{4} p_a) G^-], \quad (33)$$

with

$$\begin{aligned}G^+ &= \frac{1}{4} G_s + \frac{3}{4} G_a \\ G^- &= G_s - G_a.\end{aligned}\quad (34)$$

The meaning of this decomposition will become clear if the limiting case of heavy nuclei is considered: in this case, G_s and G_a tend to become equal (their common value being G_0), while (as we shall see presently) the

limiting value of the ratio p_s/p_a is $\frac{1}{3}$. This last circumstance justifies the choice of the combination G^+ to represent the main part of G ; the term in G^- is then a correction, only important for light nuclei and exhibiting, as we shall show, a strong dependence on the parity of Z .

In order to determine p_s and p_a , we have in the first place the obvious relation

$$\frac{1}{2} A(A-1)(p_s + p_a) = \frac{1}{2} Z(Z-1). \quad (35)$$

The difference $p_s - p_a$ can next be found from

$$\begin{aligned} \frac{1}{2} A(A-1)(p_s - p_a) &= \text{av} \frac{1}{2} \sum_{i,k} P_x^{(ik)} T_-^{(i)} T_-^{(k)} \\ &= -\text{av} \frac{1}{2} \sum_{i,k} P_\tau^{(ik)} T_-^{(i)} T_-^{(k)}, \end{aligned} \quad (36)$$

in virtue of $P_x = -P_\sigma P_\tau$ and

$$P_\tau^{(ik)} T_-^{(i)} T_-^{(k)} = T_-^{(i)} T_-^{(k)}. \quad (37)$$

In exactly the same way as (10.33-26), we derive

$$\text{av} \frac{1}{2} \sum_{i,k} P_\sigma^{(ik)} T_-^{(i)} T_-^{(k)} = \frac{1}{4} Z(Z-4) + S_p(S_p + 1), \quad (38)$$

S_p being the quantum number of the total proton spin, i.e. 0 for even Z and $\frac{1}{2}$ for odd Z , or

$$S_p = \frac{1}{4} [1 - (-1)^Z]. \quad (39)$$

Inserting (38) and (39) into (36), we get

$$\frac{1}{2} A(A-1)(p_s - p_a) = -\frac{1}{4} Z(Z-4) + \frac{3}{8} [1 - (-1)^Z]. \quad (40)$$

A first consequence of (35) and (40) is the property already mentioned: $\lim_{Z \rightarrow \infty} (p_s/p_a) = \frac{1}{3}$. More generally, the coefficients of formula (33) can be expressed entirely in terms of Z :

$$G = \frac{1}{2} Z(Z-1) G^+ + \frac{3}{8} [Z - \frac{1}{2} + \frac{1}{2} (-1)^Z] G^-. \quad (41)$$

If we now assume (just as we did for the functions $G^\pm$ in 10.31) that G^+ and G^- are smoothly varying functions of A , we see that the coefficient of G^- will give rise to fluctuations of the value of G according to the parity of Z , in qualitative agreement with the empirical data. The data on differences of Coulomb energy, obtained from isobaric pairs, would in fact allow us to determine empirically the variation of G^+ and G^- with A ; an element of uncertainty lies in the other possible cause of fluctuations in the value of Coulomb energies, viz. the variation of nuclear dimensions emphasized by Bethe (3.3). The detailed discussion, carried out by Feenberg and Goertzel, shows that no appreciable difference between G_s and G_a subsists for mass numbers larger than 17, and that the effect of this difference is considerable only for mass numbers up to 11.

CHAPTER XI

SATURATION PROPERTIES OF NUCLEAR FORCES

11.0. In this Chapter, we shall try to formulate necessary and sufficient conditions to be fulfilled by the nuclear potential (assumed to be a *static central* one of the same sign at all distances) in order that it should give rise to the saturation properties observed in heavy nuclei. For this purpose, the average interaction energy in a specified stationary state of such a nucleus must be set up, and this will necessarily involve the use of some approximate model. Since the conditions in question take the form of inequalities to be satisfied by the interaction parameters occurring in the general expression (8.2-1), a crude approximation such as that afforded by models of the individual type will do. It will turn out that it even suffices to fix the values of these parameters with considerable precision.

11.1. The saturation requirements

11.11. *Decomposition of mean nuclear energy into ordinary and exchange terms.* In any individual nuclear model, the eigenfunctions of the stationary states are to be built up of individual nucleon wave-functions of the form *

$$\psi_n(\mathbf{Q}) = \varphi_n(\vec{x}) v_{\pm}(\sigma'_z) u_{\pm}(\tau'_z), \quad (1)$$

the v 's and u 's being defined as in 4.32, 4.331. In order to satisfy the exclusion principle, we may write them as linear combinations of Slater-determinants

$$\frac{1}{\sqrt{A!}} \begin{vmatrix} \psi_{n_1}(\mathbf{Q}^{(1)}) & \dots & \psi_{n_1}(\mathbf{Q}^{(A)}) \\ \dots & \dots & \dots \\ \psi_{n_A}(\mathbf{Q}^{(1)}) & \dots & \psi_{n_A}(\mathbf{Q}^{(A)}) \end{vmatrix}, \quad (2)$$

each of which corresponds to a configuration of the kind represented by fig. 10.14. The nuclear interaction potential being of the general form (8.2-1) and, the Coulomb potential disregarded, any stationary state will belong to definite eigenvalues of the 3-component T_3 of the total isotopic vector (10.12-2) and of the z -component S_z of the total spin (10.14-6). But in general, it cannot be ascribed an eigenvalue of the quantity Y_z (10.14-7), since the latter commutes only with the $P_\sigma P_\tau$'s, but not generally with the P_σ 's and P_τ 's separately. Hence it is necessary* to combine several Slater-determinants, since any such determinant is an

* We do not consider any dependence on e'_3 , since we work in the non-relativistic approximation.

eigenfunction of Y_z also. Still, there are stationary states in which Y_z also is well-defined: they belong to a common eigenvalue 0 of all commutators $[P_\sigma, Y_z]$, $[P_\tau, Y_z]$. This is obviously the case, e.g., if all $\tau_{\frac{1}{2}}^{(i)'} are the same, i.e. if we consider a "nucleus" consisting solely of protons or of neutrons; again, if all $\sigma_z^{(i)'} are the same, corresponding to a nuclear state with an extreme value $\pm \frac{1}{2}A$ of the z -component of the total spin; more generally, if the configuration is such that — on account of the exclusion principle — no neutron-proton pair with opposite spins can be found which would allow an interchange (as an example of such a configuration, we may mention any one with saturated spins, as that of the ground state; a more general case is that represented by fig. 10.14; an instance of configurations *not* satisfying this condition is met with in 11.4 below). The eigenfunctions of such stationary states are single Slater-determinants of the type (2). Except in the last mentioned instance, we shall here exclusively have to do with states of this special class.$$

For the state represented by the Slater-determinant (2), the expectation value of the nuclear interaction energy

$$\mathcal{V}_{\text{nucl}} = \frac{1}{2} \sum_{i,k} \mathcal{V}_{\text{nucl}}^{(ik)} \quad (3)$$

is easily reduced to the following form

$$\begin{aligned} \overline{\mathcal{V}} &= \text{av } \mathcal{V}_{\text{nucl}} \\ &= \frac{1}{2} \sum_{n_1, n_2} \int \psi_{n_1}^*(Q^{(1)}) \psi_{n_2}^*(Q^{(2)}) \mathcal{V}_{\text{nucl}}^{(12)} [\psi_{n_1}(Q^{(1)}) \psi_{n_2}(Q^{(2)}) - \psi_{n_2}(Q^{(1)}) \psi_{n_1}(Q^{(2)})]; \end{aligned} \quad (4)$$

in this formula, the integral sign indicates summation over the spin and isotopic variables in addition to integration over space, while the summations over n_1 and n_2 independently extend over all the occupied individual nucleon states ψ_n . Owing to the special form (8.2-1) of $\mathcal{V}_{\text{nucl}}^{(12)}$, we can perform explicitly the summations over spin and isotopic variables and so express $\overline{\mathcal{V}}$ as a sum of space integrals of the ordinary and exchange types. In fact we observe that, according to (8.2-2),

$$\sum_{\tau_{\frac{1}{2}}^{(1)'}, \tau_{\frac{1}{2}}^{(2)'}} u_{\pm}(1) u_{\pm}(2) \tau^{(1)} \tau^{(2)} u_{\pm}(1) u_{\pm}(2) = \pm 1, \quad (5)$$

the sign being + or — according as the sign indices of the u 's are the same or opposite for the two particles, and that

$$\sum_{\tau_{\frac{1}{2}}^{(1)'}, \tau_{\frac{1}{2}}^{(2)'}} u_{+}(1) u_{-}(2) \tau^{(1)} \tau^{(2)} u_{-}(1) u_{+}(2) = 2; \quad (6)$$

similar formulae hold for $\vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$.

Beginning with the τ -summation, we get the separation of $\overline{\mathcal{V}}$ into neutron-neutron, proton-proton and proton-neutron energies:

$$\overline{\mathcal{V}} = \sum_{m_t} \overline{\mathcal{V}}_{m_t} = \overline{\mathcal{V}}_{+1} + \overline{\mathcal{V}}_{-1} + \overline{\mathcal{V}}_0. \quad (7)$$

One finds

$$\begin{aligned}\bar{\mathcal{V}}_{\pm 1} &= \frac{1}{2} \sum_{n_1}^{(\pm)} \sum_{n_2}^{(\pm)} \int \chi_{n_1}^*(1) \chi_{n_2}^*(2) \mathcal{V}_{\pm 1} [\chi_{n_1}(1) \chi_{n_2}(2) - \chi_{n_2}(1) \chi_{n_1}(2)] \\ \bar{\mathcal{V}}_0 &= \sum_{n_1}^{(+)} \sum_{n_2}^{(-)} \int \chi_{n_1}^*(1) \chi_{n_2}^*(2) \mathcal{V}_0^{\text{ord}} \chi_{n_1}(1) \chi_{n_2}(2) \\ &\quad - \sum_{n_1}^{(+)} \sum_{n_2}^{(-)} \int \chi_{n_1}^*(1) \chi_{n_2}^*(2) \mathcal{V}_0^{\text{exch}} \chi_{n_2}(1) \chi_{n_1}(2); \end{aligned} \quad (8)$$

in these formulae, one has put

$$\chi_n = \varphi_n(\vec{x}) v_{\pm}(\sigma_z) \quad (9)$$

and the symbols $\Sigma^{(+)}$, $\Sigma^{(-)}$ represent summations extending over the occupied neutron states and occupied proton states, respectively. Further, the effective potential operators occurring in (8) are defined as follows:

$$\begin{aligned}\mathcal{V}_{\pm 1} &= [a_0 + a_{\tau} + (a_{\sigma} + a_{\sigma\tau}) \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}] J(r) \\ \mathcal{V}_0^{\text{ord}} &= [a_0 - a_{\tau} + (a_{\sigma} - a_{\sigma\tau}) \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}] J(r) \\ \mathcal{V}_0^{\text{exch}} &= 2 [a_{\tau} + a_{\sigma\tau} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}] J(r). \end{aligned} \quad (10)$$

We see that both the like-nucleon energy $\bar{\mathcal{V}}_{\pm 1}$ and the proton-neutron energy $\bar{\mathcal{V}}_0$ consist of two parts of the ordinary and exchange types, respectively; moreover, in the proton-neutron case, the effective potentials of ordinary and exchange types are themselves different.

The spin-summation has now to be carried out in each of these various terms. By way of illustration, let us treat the exchange terms of $\bar{\mathcal{V}}_0$, which we denote by

$$\bar{\mathcal{V}}_0^{\text{exch}} = \sum_{n_1}^{(+)} \sum_{n_2}^{(-)} \int \chi_{n_1}^*(1) \chi_{n_2}^*(2) \mathcal{V}_0^{\text{exch}} \chi_{n_2}(1) \chi_{n_1}(2). \quad (11)$$

The summations $\Sigma^{(\pm)}$ must further be analysed into summations over all occupied (proton or neutron) states of the one or the other spin orientation, which we shall denote by $\uparrow \Sigma^{(\pm)}$ and $\downarrow \Sigma^{(\pm)}$. We may then group together the terms for which the spin orientations in the two independent summations $\Sigma_{n_1}^{(+)} \Sigma_{n_2}^{(-)}$ are the same, and those for which they are opposite. According to the formulae analogous to (5) and (6), the terms of the former group contain the factor $2(a_{\tau} + a_{\sigma\tau})$, those of the latter the factor $4a_{\sigma\tau}$. In order to write down the remaining space integrals, it is convenient to introduce quantities, called *mixed densities*, of the following type:

$$\uparrow \varrho_{+}(\vec{x}^{(1)}, \vec{x}^{(2)}) = \uparrow \sum_n^{(+)} \varphi_n^*(\vec{x}^{(1)}) \varphi_n(\vec{x}^{(2)}), \quad (12)$$

and similarly $\downarrow \varrho_{+}$, $\uparrow \varrho_{-}$, $\downarrow \varrho_{-}$; the summation in (12), e.g., extends over all occupied neutron states of the specified spin orientation. If we put

$\vec{x}^{(1)} = \vec{x}^{(2)} = \vec{x}$, we get the corresponding ordinary densities, which we may denote by the same symbols, but with a single argument $\vec{x}$, e.g.

$$\uparrow\varrho_+(\vec{x}) \equiv \uparrow\varrho_+(\vec{x}, \vec{x}) = \uparrow\sum_n^{(+)} \varphi_n^*(\vec{x}) \varphi_n(\vec{x}). \quad (13)$$

With this notation, the (exchange) integrals occurring in $\bar{\mathcal{V}}_0^{\text{exch}}$ may be symbolized in the following way:

$$\left(\begin{array}{c} \uparrow \\ + \end{array} \middle| \begin{array}{c} \downarrow \\ - \end{array}\right)_{\text{exch}} \equiv \Re \int \uparrow\varrho_+(\vec{x}^{(1)}, \vec{x}^{(2)}) J(|\vec{x}^{(1)} - \vec{x}^{(2)}|) \downarrow\varrho_-(\vec{x}^{(2)}, \vec{x}^{(1)}) d\nu^{(1)} d\nu^{(2)}, \quad (14)$$

and the result takes the form

$$\begin{aligned} \bar{\mathcal{V}}_0^{\text{exch}} &= \pi \bar{\mathcal{V}}_0^{\text{exch}} + \alpha \bar{\mathcal{V}}_0^{\text{exch}} \\ \pi \bar{\mathcal{V}}_0^{\text{exch}} &= 2(a_\tau + a_{\sigma\tau}) \left[\left(\begin{array}{c} \uparrow \\ + \end{array} \middle| \begin{array}{c} \uparrow \\ - \end{array}\right) + \left(\begin{array}{c} \downarrow \\ + \end{array} \middle| \begin{array}{c} \downarrow \\ - \end{array}\right) \right] \\ \alpha \bar{\mathcal{V}}_0^{\text{exch}} &= 4a_{\sigma\tau} \left[\left(\begin{array}{c} \uparrow \\ + \end{array} \middle| \begin{array}{c} \downarrow \\ - \end{array}\right) + \left(\begin{array}{c} \downarrow \\ + \end{array} \middle| \begin{array}{c} \uparrow \\ - \end{array}\right) \right]. \end{aligned} \quad (15)$$

If we had considered the ordinary term $\bar{\mathcal{V}}_0^{\text{ord}}$, e.g., the same procedure would have led to a expression of exactly the same type:

$$\bar{\mathcal{V}}_0^{\text{ord}} = \pi \bar{\mathcal{V}}_0^{\text{ord}} + \alpha \bar{\mathcal{V}}_0^{\text{ord}}, \quad (16)$$

but the co-factors in $\pi \bar{\mathcal{V}}_0^{\text{ord}}$ and $\alpha \bar{\mathcal{V}}_0^{\text{ord}}$ would have been different combinations of the a 's, and the symbols such as $\left(\begin{array}{c} \uparrow \\ + \end{array} \middle| \begin{array}{c} \downarrow \\ - \end{array}\right)$ would have had another meaning, viz.

$$\left(\begin{array}{c} \uparrow \\ + \end{array} \middle| \begin{array}{c} \downarrow \\ - \end{array}\right)_{\text{ord}} \equiv \int \uparrow\varrho_+(\vec{x}^{(1)}) J(|\vec{x}^{(1)} - \vec{x}^{(2)}|) \downarrow\varrho_-(\vec{x}^{(2)}) d\nu^{(1)} d\nu^{(2)}. \quad (17)$$

Quite generally, therefore, we get the expression

$$\bar{\mathcal{V}}_{m_t} = \pi \bar{\mathcal{V}}_{m_t}^{\text{ord}} + \alpha \bar{\mathcal{V}}_{m_t}^{\text{ord}} - \pi \bar{\mathcal{V}}_{m_t}^{\text{exch}} - \alpha \bar{\mathcal{V}}_{m_t}^{\text{exch}}, \quad (18)$$

in which each term appears as a combination of integrals of the form (14) or (17), with co-factors depending on the a 's. The various combinations of integrals and co-factors are summarized in the following table; the same combination of integrals has to be taken for the ordinary and exchange terms of the same type, the meaning of the symbols being given by (14) or (17) as the case may be:

11.11-1. Decomposition of mean nuclear energy			
Type of term	Combination of integrals	Co-factor	
		Ordinary term	Exchange term
$\pi \bar{\mathcal{V}}_{\pm 1}$	$\frac{1}{2} \left[\left(\begin{array}{c} \uparrow \\ \pm \end{array} \middle \begin{array}{c} \uparrow \\ \pm \end{array}\right) + \left(\begin{array}{c} \downarrow \\ \pm \end{array} \middle \begin{array}{c} \downarrow \\ \pm \end{array}\right) \right]$	$a_0 + a_\tau + a_\sigma + a_{\sigma\tau}$	$a_0 + a_\tau + a_\sigma + a_{\sigma\tau}$
$\alpha \bar{\mathcal{V}}_{\pm 1}$	$\left(\begin{array}{c} \uparrow \\ \pm \end{array} \middle \begin{array}{c} \downarrow \\ \pm \end{array}\right)$	$a_0 + a_\tau - a_\sigma - a_{\sigma\tau}$	$2(a_\sigma + a_{\sigma\tau})$
$\pi \bar{\mathcal{V}}_0$	$\left(\begin{array}{c} \uparrow \\ + \end{array} \middle \begin{array}{c} \uparrow \\ - \end{array}\right) + \left(\begin{array}{c} \downarrow \\ + \end{array} \middle \begin{array}{c} \downarrow \\ - \end{array}\right)$	$a_0 - a_\tau + a_\sigma - a_{\sigma\tau}$	$2(a_\tau + a_{\sigma\tau})$
$\alpha \bar{\mathcal{V}}_0$	$\left(\begin{array}{c} \uparrow \\ + \end{array} \middle \begin{array}{c} \downarrow \\ - \end{array}\right) + \left(\begin{array}{c} \downarrow \\ + \end{array} \middle \begin{array}{c} \uparrow \\ - \end{array}\right)$	$a_0 - a_\tau - a_\sigma + a_{\sigma\tau}$	$4a_{\sigma\tau}$

A considerable simplification occurs for configurations in which all the nucleon spins are saturated. In this case,

$$\uparrow \varrho_{\pm} = \downarrow \varrho_{\pm} = \frac{1}{2} \varrho_{\pm}, \quad (19)$$

$\varrho_{\pm}$ denoting the total (ordinary or mixed) density of the neutrons or protons; we are thus left with 4 different types of integrals only, which may be denoted by symbols such as

$$\langle + | - \rangle = \begin{cases} \int \varrho_+(\vec{x}^{(1)}) J(r^{(12)}) \varrho_-(\vec{x}^{(2)}) d\nu^{(1)} d\nu^{(2)} & \text{(ordinary)} \\ \Re \int \varrho_+(\vec{x}^{(1)}, \vec{x}^{(2)}) J(r^{(12)}) \varrho_-(\vec{x}^{(2)}, \vec{x}^{(1)}) d\nu^{(1)} d\nu^{(2)} & \text{(exchange).} \end{cases} \quad (20)$$

Instead of the expression (18), we have, more simply,

$$\bar{V}_{m_t} = \bar{V}_{m_t}^{\text{ord}} - \bar{V}_{m_t}^{\text{exch}}, \quad (21)$$

with the following integrals and co-factors:

11.11-2. Decomposition of mean nuclear energy in states with saturated spins			
Type of term	Integrals	Co-factor	
		Ordinary	Exchange
$\bar{V}_{\pm 1}$	$\frac{1}{2} (+ +) \text{ or } \frac{1}{2} (- -)$	$2(a_0 + a_{\tau})$	$a_0 + a_{\tau} + 3a_{\sigma} + 3a_{\sigma\tau} = 2(a_0 + a_{\tau}) + q$
$\bar{V}_0$	$(+ -)$	$a_0 - a_{\tau}$	$a_{\tau} + 3a_{\sigma\tau} = a_0 - a_{\tau} + \frac{1}{2}(q + 3)$

The equivalent expressions for the coefficients of the exchange integrals follow from (8.2-12).

In particular, consider the ground state of a nucleus containing as many neutrons as protons; there is then just one ordinary and one exchange integral

$$\begin{aligned} I &= \int \varrho(\vec{x}^{(1)}) J(r^{(12)}) \varrho(\vec{x}^{(2)}) d\nu^{(1)} d\nu^{(2)} \\ X &= \int \varrho(\vec{x}^{(1)}, \vec{x}^{(2)}) J(r^{(12)}) \varrho(\vec{x}^{(2)}, \vec{x}^{(1)}) d\nu^{(1)} d\nu^{(2)}, \end{aligned} \quad (22)$$

with

$$\varrho(\vec{x}^{(1)}, \vec{x}^{(2)}) = \sum_n \varphi_n^*(\vec{x}^{(1)}) \varphi_n(\vec{x}^{(2)}), \quad (23)$$

the summation extending over the occupied (lowest) states. The mean nuclear energy is then, by (7) and (21), since $\varrho_+ = \varrho_- = \varrho$,

$$\begin{aligned} \bar{V} &= 2a_0 I - \frac{1}{2}(a_0 + 3a_{\tau} + 3a_{\sigma} + 9a_{\sigma\tau}) X \\ &= 2a_0 I - [2a_0 + \frac{3}{4}(1 + q)] X; \end{aligned} \quad (24)$$

it depends only on a_0 (and q).

11.12. Saturation properties of exchange interaction terms. The asymptotic behaviours for large numbers of nucleons of the ordinary and exchange integrals are radically different. The former, depending on a product of two densities, vary, roughly, as the product of the two proton or neutron numbers, i.e. quadratically with A and consequently do not exhibit any saturation. The mixed densities, however, which enter into the definition of the exchange integrals, reduce asymptotically to a delta function $\delta(\vec{x}^{(1)} - \vec{x}^{(2)})$; indeed, for a very heavy nucleus, the summation in (12) may approximately be extended over all states φ_n , which by the completeness relation just yields the delta function. In general, the number of neutrons being larger than the number of protons, the mixed neutron densities will be nearer their asymptotic behaviour than the mixed proton densities. If, then, we replace in (14) the mixed neutron density $\uparrow \rho_+$ by the δ -function, we get

$$\left(\begin{array}{c} \uparrow \\ + \end{array} \middle| \begin{array}{c} \downarrow \\ - \end{array} \right)_{\text{exch}} \simeq J(0) \int \uparrow \rho_-(\vec{x}) d\vec{v}, \quad (25)$$

a quantity approximately proportional to the number of protons, i.e. linear in A , as required to account for the saturation of nuclear bonds. This conclusion is clearly valid for all exchange integrals.

According to this argument, a Wigner potential in the sense of 4.341 will give rise to an ordinary non-saturation interaction and to an exchange interaction exhibiting saturation: this is the well-known situation in the theory of chemical binding. If the potential is of the Majorana type, we have just the reverse situation: the ordinary interaction terms possess the saturation property.

11.121. Velocity dependent potentials. Another extensive class of energy operators with saturation properties has been pointed out by WHEELER [36]*. In general, an effective potential operator may depend both on the relative coordinates $\vec{x}$ and on the conjugate relative momentum operators $\vec{p} \equiv \frac{\hbar}{i} \text{grad}$ of the nucleon pair. It can then also be expressed as an integral operator, as follows:

$$\mathcal{V}_{\text{eff}} \left(\vec{x}, \frac{\hbar}{i} \text{grad} \right) \psi(\vec{x}) = \int J(\vec{x}, \vec{\xi}) \psi(\vec{\xi}) d\vec{v}_{\xi}, \quad (26)$$

with

$$J(\vec{x}, \vec{\xi}) = (2\pi\hbar)^{-3} \int \mathcal{V}_{\text{eff}}(\vec{x}, \vec{p}) e^{\frac{i}{\hbar}(\vec{x}-\vec{\xi})\vec{p}} d\vec{v}_p. \quad (27)$$

The equivalence of the two representations is easily verified by introducing in (27) the Taylor expansion of $\mathcal{V}_{\text{eff}}$ with respect to $\vec{p}$, substituting the result in the right-hand side of (26), transforming this by partial integrations with respect to $\vec{\xi}$, and using the Fourier integral theorem, or the equivalent symbolic formula (N.13) for $\delta(\vec{x} - \vec{\xi})$.

* See also WAY and WHEELER [36], WAY [37].

Now, the ordinary integrals for such a potential are, according to (17), (13), in which we go over to relative coordinates by means of (4.311-2, 3), of the general form

$$\begin{aligned}
 (\uparrow | \downarrow)_{\text{ord}} &= \Re \uparrow \sum_{n_1}^{(+)} \downarrow \sum_{n_2}^{(-)} \int \varphi_{n_1}^*(\vec{X} + \tfrac{1}{2} \vec{x}) \varphi_{n_2}^*(\vec{X} - \tfrac{1}{2} \vec{x}) J(\vec{x}, \vec{\xi}) \cdot \\
 &\quad \varphi_{n_1}(\vec{X} + \tfrac{1}{2} \vec{\xi}) \varphi_{n_2}(\vec{X} - \tfrac{1}{2} \vec{\xi}) d\nu_X d\nu_x d\nu_{\xi} \quad (28) \\
 &= \Re \int \uparrow \varrho_+(\vec{X} + \tfrac{1}{2} \vec{x}, \vec{X} + \tfrac{1}{2} \vec{\xi}) J(\vec{x}, \vec{\xi}) \downarrow \varrho_-(\vec{X} - \tfrac{1}{2} \vec{x}, \vec{X} - \tfrac{1}{2} \vec{\xi}) d\nu_X d\nu_x d\nu_{\xi}.
 \end{aligned}$$

They thus depend on *mixed* densities. In absolute value, a mixed density such as (12) presents a maximum at $\vec{x}^{(1)} = \vec{x}^{(2)}$, which becomes higher and narrower when more states are occupied. The behaviour of the integral (28) is essentially different according as $J(\vec{x}, \vec{\xi})$ varies more or less rapidly than the narrower mixed density factor when $\vec{x} - \vec{\xi}$ varies. In the first case, we get no saturation: an extreme example is that of an ordinary Wigner potential $J(\vec{x}, \vec{\xi}) = J(\vec{x}) \delta(\vec{x} - \vec{\xi})$. But if the variation of $J(\vec{x}, \vec{\xi})$ is sufficiently slow over the domain in which the narrower mixed density is appreciable, we may (as above) replace that mixed density ($\uparrow \varrho_+$, say) by $\delta(\frac{\vec{x} - \vec{\xi}}{2}) = 8 \delta(\vec{x} - \vec{\xi})$, and (28) becomes

$$(\uparrow | \downarrow)_{\text{ord}} \simeq \int 8 J(\vec{x}, \vec{x}) \downarrow \varrho_-(\vec{X} - \tfrac{1}{2} \vec{x}) d\nu_X d\nu_x. \quad (29)$$

On the Fermi gas model, the density being approximately constant over the whole nucleus, (29) may be written

$$(\uparrow | \downarrow)_{\text{ord}} \simeq \int 8 J(\vec{x}, \vec{x}) d\nu_x \cdot \int \downarrow \varrho_-(\vec{X}) d\nu_X, \quad (30)$$

which exhibits the saturation property.

The condition of slow variation of $J(\vec{x}, \vec{\xi})$ means that we have to do with an essential *velocity dependence* of the effective potential. In fact, according to (27), the function $\mathcal{V}_{\text{eff}}(\vec{x}, \vec{p})$ must then have appreciable values for all $\vec{p}$ comprised within a domain $|\vec{p}| < p_0$, if $\hbar/p_0$ represents the dimensions of the region in which the mixed density is $\neq 0$; on the Fermi gas model, this limiting momentum is of the order of the maximum momentum p_m (9.41-3). We shall not further discuss this class of nuclear potentials giving rise to saturation effects, referring the reader to the papers quoted above.

It may be observed that the Majorana potential might be regarded as belonging to the type (26), (27), viz. if

$$\mathcal{V}_{\text{eff}}(\vec{x}, \vec{p}) = J(\vec{x}) e^{-2i\vec{x}\vec{p}\vec{h}} \quad (31)$$

or

$$J(\vec{x}, \vec{\xi}) = J(\vec{x}) \delta(\vec{x} + \vec{\xi}). \quad (32)$$

Although (30) also formally holds in this case, — actually reducing to (25), — we have here to do with a rather singular case, in which the velocity dependence embodied in (31) is only apparent, on account of the exclusion principle (4.341–32).

11.13. The saturation requirements. The total energy of a stationary state is made up of three contributions: kinetic energy, nuclear energy and Coulomb energy:

$$\mathcal{E} = K + \bar{V} + G. \quad (33)$$

Since K and G are positive*, a state of binding ($\mathcal{E} < 0$) necessitates an attractive nuclear interaction stronger in absolute value than the sum of kinetic and Coulomb energy. The absolute value of the binding energy is given by

$$|\mathcal{E}| = |\bar{V}| - (K + G). \quad (34)$$

The expression (9.41–7) for the kinetic energy practically satisfies the condition of being proportional to A , so that we may confine the discussion to the potential energy $\bar{V} + G$. This energy is composed of terms linear in A and of terms approximately quadratic in A : these will be called the saturation and non-saturation terms, respectively.

In the first place, we have to require that the non-saturation part of the potential energy should in no case give rise to an attraction, since for heavy nuclei this attraction would become preponderant and give rise to stable configurations not exhibiting the saturation property. The Coulomb energy is always repulsive, but numerically so insignificant that it can safely be left out of consideration. The *first saturation requirement* has therefore to be formulated for the non-saturation part of the nuclear energy $\bar{V}$.

In the second place, we have to take care that, for stable states of actual nuclei, the saturation part of the nuclear energy yields an attraction, exceeding in absolute value the total positive contribution to the energy, arising from the non-saturation part of the potential energy and from the kinetic energy. This will be our *second saturation requirement*.

11.2. The first saturation requirement

11.21. Necessary saturation conditions. Let us consider the mean nuclear energy of some specified stationary state. In order to satisfy the first saturation requirement, we have to express that the non-saturation

* Strictly speaking, the Coulomb interaction also includes an attractive exchange term, roughly proportional to Z , but this is usually quite negligible.

terms of this quantity must be ≥ 0 . Since the expectation value calculated on an individual model represents an upper limit to the exact eigenvalue, the condition so obtained is in any case *necessary*. Now, all ordinary integrals (11.11-17) occurring in ${}^{\pi}\bar{V}_{m_t}^{\text{ord}} + {}^{\alpha}\bar{V}_{m_t}^{\text{ord}}$ (11.11-18) are positive, so that we finally get inequalities to be satisfied by the interaction parameters a . We take a nucleus consisting of equal numbers of neutrons and protons. If all spins are saturated, we get from formula (11.11-24), valid in such a case, the condition first enunciated by BREIT and FEENBERG [36b]

$$a_0 \geq 0. \quad (1)$$

But, as pointed out by KEMMER [37b], we must also prevent the configurations in which all spins are parallel from exhibiting non-saturation binding; from table 11.11-1, considering that $(\uparrow|\uparrow) = (\uparrow|_{+}) = (\uparrow|_{+})$ in our case, we read off the corresponding condition

$$a_0 + a_{\tau} \geq 0. \quad (2)$$

We might further consider "nuclei" containing only neutrons. For such "neutron clusters", table 11.11-1 yields the instability conditions:

(a) spins saturated (BREIT and FEENBERG [36b]):

$$a_0 + a_{\tau} \geq 0; \quad (3)$$

(b) spins parallel (FEENBERG [37c]):

$$a_0 + a_{\tau} + a_{\tau} + a_{\tau\tau} \geq 0. \quad (4)$$

But these conditions are weaker than (2). In the first place, from (2) and the first formula (8.2-12), we get the inequality

$$a_0 + a_{\tau} + a_{\tau} + a_{\tau\tau} \geq \frac{1}{3}, \quad (5)$$

which implies (4). The condition (5), first noticed by FEENBERG [37c], gets a simple interpretation when compared with the definition (8.2-9) of ${}^3p$: it means that the effective interaction in states of the type ${}^3P$ is necessarily a repulsion, amounting to at least $\frac{1}{3}$ of the absolute value of the ${}^3S$ attraction. As we have seen (8.1, 8.33), the proton-proton and proton-neutron scattering experiments are as yet inconclusive as regards Feenberg's prediction, although the latest results (7.131) would seem to fit it quite nicely. As to inequality (3), it follows from (2) and from the relation

$$a_{\tau} - a_{\tau} = \frac{1-q}{4} > 0, \quad (6)$$

derived from (8.2-12) together with $q < 1$.

In this connexion, it may be observed that the instability of neutron clusters, even when the like-particle forces are assumed to be equal to the proton-neutron forces in all common configurations, is essentially a consequence of (a) the exclusion principle, (b) the fact that the attraction between two nucleons is smaller in the ${}^1S$ than in the ${}^3S$ configuration ($q < 1$)*, (c) the saturation character of the forces. Indeed, since the

* It is not necessary that (as is actually the case) there should be no bound state in the ${}^1S$ configuration: the "dineutron" might be stable, provided its binding energy be smaller than that of the deuteron.

exclusion principle prevents an accumulation of the neutrons in the lowest nucleon states to the same degree as in a "normal" nucleus, the lowest configuration of the neutron cluster (fig. 11.21, (a)) has rather to be compared with a more loosely bound configuration, such as (b) in fig. 11.21, of the normal nucleus. The comparison is simplest when the latter configuration is one in which all spins are parallel: for the only difference

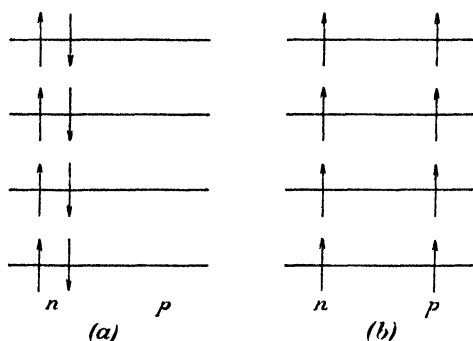


Fig. 11.21. Comparison of neutron cluster (a) with normal nucleus (b). Symbolical representation as in fig. 10.14.

in the interactions between pairs of nucleons is then that those of the singlet type (necessarily even) in (a) are replaced in (b) by interactions of the triplet type. But because of $q < 1$, this means that when going over from (b) to (a), a repulsion between neutrons with opposite spins is superposed on the forces present in configuration (b). This repulsion, then, being quadratic in the neutron number, becomes preponderant, and cannot be compensated by any attraction, since the latter would also be present in configuration (b) and imply a non-saturation binding in this configuration.

Another important consequence of Kemmer's condition (2) is to exclude any nuclear potential energy consisting solely of ordinary central forces, even with spin dependence: such a potential would correspond to (8.2-1, 4, 5)

$$a_{\tau} = a_{\tau\tau} = 0, \quad \text{or} \quad M = H = 0,$$

so that (8.2-12) would give

$$a_0 + a_{\tau} = -1,$$

in contradiction with (2). This means that forces yielding the type of saturation binding exactly similar to chemical binding are not sufficient, because they would not lead to an attraction in a triplet configuration: it is a well-known feature of the theory of chemical valency that the largest binding is obtained just between two particles with antiparallel spins. It is therefore necessary to introduce forces of the exchange type involving (at least implicitly) a dependence on the isotopic variables, as was done for the first time by Heisenberg and by Majorana. *In contrast with chemical bonds, brought about by spin exchange between the valency electrons, nuclear bonds also depend essentially on an exchange of charge between protons and neutrons.* From the point of view of a field theory of nuclear

interaction, it is interesting to note that this conclusion from a very general, qualitative argument entails the existence of charged nuclear fields, and rules out any theory involving *only* neutral fields (8.31).

As is well-known, — and can be verified by means of (8.2–12) together with (1), — a pure Heisenberg force ($a_0 = a_\tau$, $a_\sigma = a_{\sigma\tau} = 0$) leads to an attraction in the ^3S configuration and to a repulsion in the ^1S configuration: a situation ruled out by the fact that the complete saturation of the nuclear bonds is first attained in the α -particle, and not in the deuteron. It was just this argument that led Majorana to suggest his type of potential ($a_0 = a_\tau = a_\sigma = a_{\sigma\tau}$). But again, a pure Majorana force will not do, because it does not give any splitting of the ^3S and ^1S levels ($q = 1$).

It is not obvious that the necessary conditions (1), (2) should also be sufficient for the fulfilment of the first saturation requirement. Actually, it can be proved, under certain conditions, that this is the case when the additional inequality

$$a_{\sigma\tau} \geq 0 \quad (7)$$

is assumed. We shall now proceed to the proof of this statement. So far as the determination of the a 's is concerned, however, we shall not need the sufficiency theorem: the second saturation requirements, combined with the necessary conditions (1) and (2), will lead to a complete determination of the a 's.

11.22. Sufficient saturation conditions. Conditions (1), (2) and (7) are, at least for an extensive class of short-range potential functions $J(r)$, sufficient for the fulfilment of the first saturation requirement. We have thus to show that in virtue of these conditions the non-saturation part of the nuclear potential energy $\bar{V}$ will not be negative. The proof will be carried out in two steps: we shall first prove the inequality just mentioned on the long range fiction (10.31), i.e. treating J as a constant; since, however, the long range approximation of $\bar{V}$ is larger than its true value, this result must be completed by showing that an equivalent inequality still holds when the distance dependence of $J(r)$ is taken into account.

The first step is quite straightforward*. The long range approximation of $\bar{V}$ is given by (10.33–28) in terms of the multiplet quantum numbers. Let us put

$$s = S/A, \quad \tau = T/A, \quad p = P/A, \quad p' = \dots; \quad (8)$$

we have thus to show that

$$\frac{1}{2} a_0 + 2s^2 a_\sigma + 2\tau^2 a_\tau + 2(p^2 + p'^2 + p''^2 - s^2 - \tau^2) a_{\sigma\tau} \geq 0. \quad (9)$$

We begin by looking for the values of s , τ , p , ... which minimize the

* We follow here essentially the discussion given by BREIT and WIGNER [38b]; it is, however, much shortened due to the use of the parameters a instead of the W, B, H, M .

expression on the left: it will then suffice to prove the inequality for the minimum value so obtained. Because of condition (7) and of the inequality (6), we have only to distinguish between the following three possibilities:

$$(a) a_\tau > 0, \quad a_\sigma > 0; \quad (b) a_\tau > 0, \quad a_\sigma < 0; \quad (c) a_\tau < 0, \quad a_\sigma < 0.$$

In case (a), the minimum is reached for *

$$s = \tau = p = p' = p'' = 0$$

and condition (1) is sufficient for the validity of (9). In case (b), the supermultiplet quantum numbers being first kept fixed, we have to make s as large as possible, i.e. $s = p$, and τ as small as possible, i.e. $\tau = 0$; then we must take for p its largest value, $p = \frac{1}{2}(10.14-9)$, and for the other quantum numbers the smallest value $p' = p'' = 0$. The left-hand side of (9) becomes $\frac{1}{2}(a_0 + a_\tau)$, which is ≥ 0 by condition (2). It will be observed that the configurations of minimum potential energy considered in the two preceding cases are just those from which the inequalities (1) and (2), respectively, were derived as necessary saturation conditions.

Case (c) requires a little more attention: the supermultiplet being fixed, we must, since $|a_\sigma| > |a_\tau|$, above all choose s as large as possible: $s = p$; the largest value of τ compatible with this is then $\tau = p'$, so that the left-hand side of (9) reduces to

$$\frac{1}{2}a_0 + 2p^2a_\sigma + 2p'^2a_\tau + 2p''^2a_{\sigma\tau}.$$

The choice $p = p' = \frac{1}{2}$, $p'' = 0$, which would correspond to independent variations of p , p' , p'' is not acceptable, however, because it is not always compatible with the restriction (10.14-10) imposed on the parity of $P + P' + P''$. But, turning back to formulae (10.14-4, 8), we see that by decreasing A_4 and increasing A_1 by the same amount, we increase both P and P' without modifying either A or P'' ; we may therefore restrict ourselves to the cases for which $A_4 = 0$, i.e. $p'' = p + p' - \frac{1}{2}$. Taking then the largest value $p = \frac{1}{2}$, we have $p'' = p'$. For the latter, we must take the largest or smallest possible value according as $a_\tau + a_{\sigma\tau}$ is < 0 or > 0 . Since condition (2), combined with the first formula (8.2-12), implies the last inequality, we have to choose $p' = p'' = 0$ and find that condition (2) is sufficient, as in case (b).

The result of the preceding discussion can be formulated by saying that, provided conditions (1), (2) and (7) are fulfilled, the mean long range potential energy in any multiplet state satisfies an inequality of the form

$$\text{av } \mathcal{V}_{\text{long range}} \geq -A\alpha J, \quad (10)$$

in which the numerical factor α is positive; according to (10.33-28), we may in fact take

$$\alpha = \begin{cases} \frac{1}{2}[a_0 + 3a_\sigma + 3(a_\tau + 3a_{\sigma\tau})] & \text{if } a_\sigma > 0 \\ \frac{1}{2}[a_0 + a_\sigma + a_\tau + a_{\sigma\tau}] & \text{if } a_\sigma < 0: \end{cases} \quad (11)$$

* We may, of course, assume for simplicity A to be even.

for it is clear that α can be computed, in each of the cases (a), (b), (c), for the special configuration which also minimizes the quadratic terms*. The inequality (10) means that the average interaction $\text{av } \mathcal{V}_{\text{long range}}$ does not yield a larger binding than the saturation effect represented by the right-hand side. The second step of the proof of the sufficiency theorem, indicated by WIGNER [36], will accordingly consist in showing that a similar inequality:

$$\bar{\mathcal{V}} \geq -A\alpha J^* \quad (12)$$

with a suitable constant J^* , still holds for the expectation value $\bar{\mathcal{V}}$ of the nuclear energy when the constant potential J is replaced by a function of distance $J(r)$ representing a short-range interaction. Actually, the theorem will be established for a potential of the general form

$$J(r) = \int_0^\infty g(\varrho) f(r/\varrho) d\varrho$$

$$\text{with } f(x) = \begin{cases} \frac{4\pi}{3} (1 - \frac{3}{4}x + \frac{1}{16}x^3) & \text{for } x \leq 2 \\ 0 & \text{for } x \geq 2. \end{cases} \quad (13)$$

In this formula $f(x)$ represents the volume of the space common to two spheres of unit radius, whose centres are at a distance x from each other, while $g(\varrho)$ is an arbitrary, nowhere negative, integrable function. It is clear that the function $f(r/\varrho)$, with fixed ϱ , describes a special kind of limited range interaction; a superposition such as (13) of such potentials of various ranges 2ϱ and strengths $g(\varrho)$ should at any rate represent an extensive class of short-range interactions.

Let us first consider an interaction of the special form $g(\varrho) f(r/\varrho)$ with a fixed value of ϱ . We have to compute

$$\bar{\mathcal{V}}_\varrho = \text{av } \frac{1}{2} g(\varrho) \sum_{i,k} f\left(\frac{r^{(ik)}}{\varrho}\right) \mathcal{O}^{(ik)}, \quad (14)$$

$\mathcal{O}^{(ik)}$ being the operator (8.2-1). Owing to the geometrical meaning of $f(x)$, we get, for any given configuration of the space coordinates $\vec{x}^{(i)}$ of the nucleons, exactly the operator occurring in (14) by assuming that all nucleons within any sphere of radius ϱ have the potential energy

$$\frac{1}{2} g(\varrho) \frac{1}{\varrho^3} \sum_{i,k}^{(e)} \mathcal{O}^{(ik)} \quad (15)$$

(to be summed over all nucleons within the sphere), and integrating with respect to the coordinates of the centre of the sphere over all space: for the pair (i, k) will then contribute the interaction $g(\varrho)/\varrho^3 \cdot \mathcal{O}^{(ik)}$ when, and

* The value of α found for case (b) may be replaced by that corresponding to case (c); it is positive in virtue of (5), which is a consequence of (2).

only when, the centre of the sweeping sphere is situated in the space common to the spheres of radius ϱ and centres $\vec{x}^{(l)}, \vec{x}^{(k)}$, the volume of which space is just $\varrho^3 f(r^{(lk)}/\varrho)$. In order now to perform the integration with respect to the centre of the sweeping sphere, it is convenient to divide

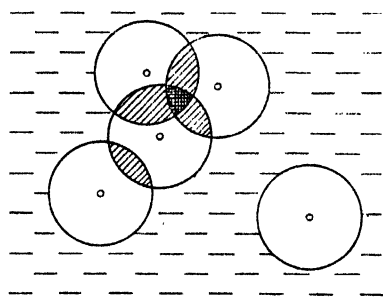


Fig. 11.22. Division of space in regions V_s (see text).



all space into $A + 1$ regions, called V_s ($s = 0, 1, 2, \dots, A$), such that when the centre of the sweeping sphere is in V_s , this sphere contains s nucleons (see fig. 11.22); obviously

$$\sum_{s=1}^A s V_s = A \cdot \frac{4\pi}{3} \varrho^3. \quad (16)$$

But for any position of the centre of the sphere within V_s , we are dealing with an interaction (15) of the "long range" type between the s nucleons, the expectation value of which, according to (10)*, is not smaller than

$$-s a \frac{g(\varrho)}{\varrho^3}.$$

The total interaction, averaged over spin and isotopic variables, is therefore $\geq -a \frac{g(\varrho)}{\varrho^3} \sum_{s=1}^A s V_s$, i.e., by (16), $\geq -\frac{4\pi}{3} a g(\varrho) A$. Averaging over the space coordinates $\vec{x}^{(l)}$ does not alter this result, so that

$$\bar{V}_e \geq -\frac{4\pi}{3} a g(\varrho) A. \quad (17)$$

Integrating (17) over ϱ , we indeed get (12), with

$$J^* \equiv \frac{4\pi}{3} \int_0^\infty g(\varrho) d\varrho = J(0), \quad (18)$$

according to (13).

* At this stage of the argument, we are dealing with the expectation value in a state corresponding to fixed values of the $\vec{x}^{(l)}$; but inequality (10) also holds for such a state, since it can be represented by a superposition of multiplet states.

11.3. The second saturation requirement

11.30. The second saturation requirement will be more difficult to formulate by our approximate methods of calculation of the energy of nuclear states, since its fulfilment depends on a balance between the various contributions to this energy, which will partly depend on the choice of the nuclear potential, but partly also on the accuracy of the method. We must therefore not expect as clear-cut conditions as those expressing the first requirement. Nevertheless, we owe to VOLZ [37] a formulation of the second requirement which permits a rather accurate determination of the interaction parameters a .

11.31. Light α -nuclei. Let us first consider some light " α -nucleus" (9.22), such as $^{16}_8\text{O}$. If we calculate its binding energy by the quasi-atomic method (9.21), the average nuclear energy will be of the form (11.11–24), so that we can expect from the formulation of the second saturation requirement in this case further information on the parameter a_0 . If one chooses a nuclear potential of the Gauss type and takes as trial wavefunctions of the individual nucleon states the eigenfunctions of the spatial harmonic oscillator, the interaction integrals I and X , as well as the mean kinetic and Coulomb energies, can be calculated explicitly. Adopting definite numerical values for the range of force, the potential strength and the ratio q , the variational procedure can be carried out for different values of a_0 . Since $I > X$, the calculated binding energy will decrease in absolute value as a_0 increases; by requiring that the quasi-atomic approximation should at any rate be sufficiently good, for this nucleus, to give the right sign of the binding energy, one therefore obtains an *upper limit* a_0 for a_0 .

Volz took $\kappa^{-3} = 2,17 \cdot 10^{-13}$ cm, $J = 37,9$ MeV, $q = 0,5$; but the use of better values (6.432) should not alter the order of magnitude of his result. He finds $a_0 = 0,15$ and states that even a slight sharpening of the condition imposed on the calculated binding energy leads to a considerable smaller value of a_0 . We thus get a condition of the form

$$a_0 \leq a_0, \quad a_0 \ll 1. \quad (1)$$

The comparison of the limitations (11.21–1) and (1) undoubtedly points to the determination

$$a_0 = 0 \quad (2)$$

as the most likely one; a small positive value of a_0 can, however, not be definitely excluded.

11.32. Heavy nuclei. We next turn to a study of heavy isobaric nuclei (of mass number 200, say) with varying neutron excess; more specifically, we consider even nuclei in states with saturated nucleon spins. From table 11.11–2, applicable to this case, we see that we have to do

with the combinations of parameters $a_0 + a_\tau$ and $a_0 - a_\tau$, or more simply, if we adopt (2), with the parameter a_τ only. It is indicated here to use the Fermi gas model, the nuclear radius being regarded as a variational parameter; all integrals occurring in the table 11.11-2 can again be calculated explicitly if the potential is of the Gauss type. But we cannot be confident that the calculated binding energy $\mathcal{E}$ will give a sufficiently exact idea of the value $\mathcal{E}'$ which should actually correspond to the nuclear potential adopted. To remedy this situation, Volz introduces the ratio

$$\epsilon = \frac{|\mathcal{E}'| + K + G}{|\mathcal{E}| + K + G} \quad (3)$$

of actual to calculated nuclear potential energy (11.13-34) and tries in each case so to adjust the variational parameter R as to get the minimum value ϵ_0 of this ratio, which he calls the "error coefficient". The solution so obtained is the best one of the type considered which yields the value $|\mathcal{E}'|$ for the binding energy when the nuclear potential is ϵ_0 times the assumed value. Since we always have $|\mathcal{E}| < |\mathcal{E}'|$, the error coefficient ϵ_0 will be ≥ 1 . But if we now impose the value $|\mathcal{E}'|$ which we wish to obtain, the ϵ_0 resulting from a certain choice of potential could become < 1 : this would mean, then, that this choice is inadequate. So ϵ_0 furnishes some measure, not only of the accuracy of the method of approximation, but also of the adequacy of the nuclear potential adopted*.

Coming back to the even isobars of mass number 200, the error coefficient can be calculated for different values of the neutron excess and various choices of the parameter a_τ ; for $|\mathcal{E}'|$, one may take throughout the value 1.7 MU, corresponding to the (probable) mass-defect of the stable nucleus $^{200}_{80}\text{Hg}$. For each value of a_τ , the stablest nucleus is that with the smallest error coefficient, so that we get an idea of the variation with a_τ of the neutron excess of the stablest nucleus of mass number 200. Volz' result is represented by fig. 11.32, the main features of which should

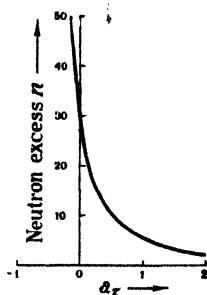


Fig. 11.32. Neutron excess n of stablest nucleus of mass number 200 in terms of interaction parameter a_τ .

remain valid if better potential constants were used. As expected, negative a_τ 's favour large neutron excesses, due to predominance of non-saturation

* The calculation of the error coefficient in suitably chosen cases may therefore also serve to derive limitations of the a 's imposed by the first saturation requirement. Thus Volz deduced in his paper conditions practically equivalent to (11.21-1,3).

attraction: but with increasing a_τ , the neutron excess diminishes rapidly. The empirical value ($n \approx 40$) would correspond, according to fig. 11.32, to a negative a_τ of very small absolute value, but we cannot expect the method to yield such a precise determination of a_τ . We can at most exclude such a_τ 's as would lead to neutron excesses smaller than, say, half the empirical value. This gives an upper limitation of a_τ :

$$a_\tau \leq a_\tau, \quad a_\tau \approx 0,1 \quad (4)$$

(Volz gets $a_\tau = 0,12$). According to (11.21-6), this condition implies an analogous limitation for a_σ :

$$a_\sigma \leq a_\sigma, \quad a_\sigma \approx 0 \quad (5)$$

(in fact, with $q \approx 0,6$, the upper limit $a_\sigma = a_\tau - \frac{1}{4}(1-q)$ becomes nearly 0). It would seem that a natural choice of the parameter a_σ , just compatible with (5) and with Kemmer's condition (11.21-2), combined with (2), would be, as proposed by KEMMER [37b],

$$a_\sigma = 0. \quad (6)$$

On the contrary, the determination $a_\tau = 0$, originally proposed by Volz, is in contradiction with Kemmer's condition, on account of (2) and (11.21-6). If, however, we combine Kemmer's condition with the inequality (1) rather than the equality (2), we cannot exclude the possibility of a small positive value for a_σ .

11.33. Determination of interaction parameters. Once we adopt, in accordance with (2) and (6),

$$a_0 = a_\tau = 0, \quad (7)$$

the values of a_τ and $a_{\sigma\tau}$ follow from (8.2-12):

$$\begin{aligned} a_\tau &= \frac{1}{4}(1-q) \approx 0,1 \\ a_{\sigma\tau} &= \frac{1}{1\frac{1}{2}}(1+3q) \approx 0,23. \end{aligned} \quad (8)$$

The final form of the nuclear potential operator is thus

$$\mathcal{V}_{\text{nuc}} = \tau^{(1)} \tau^{(2)} [a_\tau + a_{\sigma\tau} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}] J(r^{(12)}), \quad (9)$$

the constants a_τ , $a_{\sigma\tau}$ being given by (8), whereas the analytical form of $J(r)$ remains unsettled; it must be remembered that a small additional potential of the form

$$\mathcal{V}'_{\text{nuc}} = [a_0 + a_\sigma \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}] J(r^{(12)}), \quad (10)$$

likewise with positive a_0 , a_σ , would still be admissible. It will be recalled that the form (9) is just that corresponding to a symmetrical meson theory (8.32-17), whereas (10) corresponds to a neutral theory (8.32-16).

It is certainly remarkable that in spite of the inaccuracies affecting the preceding discussion, its outcome should receive such a simple interpretation on meson field theory.

According to (8.2-5), the potential (9) actually corresponds to a mixture of the four types of exchange potentials (4.341-34) in proportions given by

$$\begin{aligned} W &= \frac{1}{3}(3q-1) \approx 0,13 & M &= -\frac{1}{3}(3q+1) \approx -0,93 \\ B &= \frac{1}{3}M \approx -0,46 & H &= 2W \approx 0,26 : \end{aligned} \quad (11)$$

the most important term is thus a Majorana potential, but the Bartlett term is also of considerable weight. The effective potentials for odd states, derived from the operator (9), have already been discussed in 8.32; they are repulsive both for singlet and for triplet configurations. In fact, according to (8.2-9), we find

$$-^1p = 3q \approx 1,8, \quad (12)$$

while $-^3p$ takes its lowest possible value (11.21)

$$-^3p = \frac{1}{3}. \quad (13)$$

11.4. Stability of isobars

Quite independently of the saturation properties of the nuclear bindings, another set of conditions limiting the permissible values of the interaction parameters a can be derived from the consideration of isobaric nuclei. However, these conditions are obtained only in the very rough approximation corresponding to the long range fiction (10.31) and are consequently not as reliable as the saturation conditions just discussed. Still, it will be worth while to examine them too in order to verify their compatibility with the results arrived at in the preceding section.

We have seen (3.21) that odd nuclei heavier than ^{14}N are unstable, whereas the neighbouring even isobars can both be stable. According to the stability condition 1.131, this situation is expressed by the inequalities

$$W[N, Z] < W[N+1, Z-1], \quad W[N+1, Z-1] > W[N+2, Z-2], \quad (1)$$

if $[N, Z]$ represents the composition of one of the even isobars. The inequalities (1) imply the further one

$$W[N, Z] - W[N+1, Z-1] < \frac{1}{2}\{W[N, Z] - W[N+2, Z-2]\}, \quad (2)$$

which, if we neglect unimportant terms arising from the variations of kinetic and Coulomb energies, reduces to an inequality involving only nuclear potential energies:

$$\bar{V}[N, Z] - \bar{V}[N+1, Z-1] < \frac{1}{2}\{\bar{V}[N, Z] - \bar{V}[N+2, Z-2]\}. \quad (3)$$

The average nuclear energies of the ground states of the even isobars are immediately given, on the individual model approximation, by the formulae (11.11-7, 21) with the integrals and coefficients given by table 11.11-2. Introducing the long range approximation in the definition

(11.11–20) of the integrals, and taking account of the orthogonality of the spatial wave-functions φ_n , one finds that these integrals reduce to integral multiples of the constant potential J , specified as follows:

m_t	$\overline{V}_{m_t}^{\text{ord}}$	$\overline{V}_{m_t}^{\text{exch}}$
+1	number of neutron pairs	number of neutron bonds *
-1	number of proton pairs	number of proton bonds
0	number of neutron-proton pairs	number of neutron-proton bonds
* The concept of <i>bond</i> is defined in 10.31.		

(4)

The calculation of the different terms of $\bar{V}$ for a given configuration thus becomes a simple matter of enumeration (INGLIS and YOUNG [37]) *.

The treatment of the odd isobar requires some attention, because the supermultiplet of the ground state (10.331) splits under the influence of the spin dependent forces into a singlet and a (degenerate) triplet term, and we have to calculate the energies of these two terms. For this purpose, we consider the two distinct configurations ((a), (b) in fig. 11.4) for

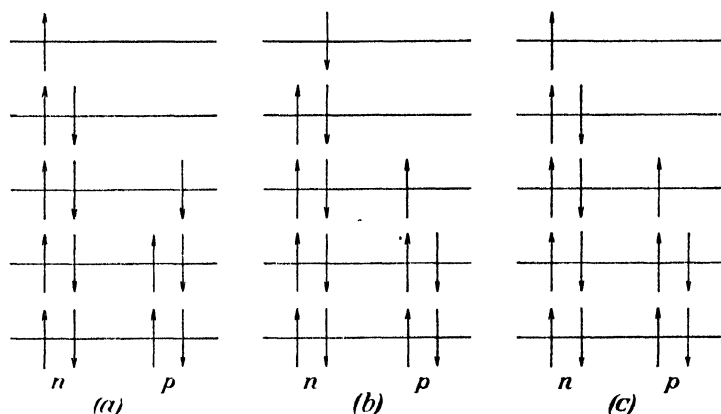


Fig. 11.4. Lowest configurations of odd isobars:

(a), (b) resonating configurations with $S_z = 0$;

(c) configuration $S_z = 1$.

Symbolic representation explained in fig. 10.14.

which the z -component of the spin $S_z = 0$: between these two configurations, which differ by the orientations of the spins of the odd proton and neutron, there will be a "resonance" corresponding to the separation of singlet and triplet stationary states. In other words, the eigenfunctions of these two states will be linear combinations of the Slater-determinants describing the resonating configurations, and we have to solve a secular

* The integrals of table 11.11-1 can be expressed in the same way; one has simply to add to the numbers occurring in (4) the qualification "with parallel spins" or "with antiparallel spins", according as one has to do with a $\pi\bar{V}$ or a $\alpha\bar{V}$.

equation of the second order to find the eigenvalues of $\bar{V}$. The diagonal elements are immediately given by the same argument as above, and have obviously the same value $\bar{V}_{\text{diag}}$ in both configurations; as to the non-diagonal elements, an easy calculation yields, on the long range approximation, the common value $2J(a_\sigma - a_{\sigma\tau})$. Therefore

$$\bar{V}[N+1, Z-1] = \bar{V}_{\text{diag}}[N+1, Z-1] \pm 2J(a_\sigma - a_{\sigma\tau}). \quad (5)$$

The corresponding eigenfunctions are proportional (as expected) to the sum and difference, respectively, of the Slater-determinants; the difference pertains to the singlet, the sum to the triplet state. It is readily verified that the direct calculation of the mean nuclear energy in the other triplet sub-states, corresponding to $S_z = \pm 1$, yields the same result as for the triplet sub-state with $S_z = 0$: in fact, the state with $S_z = 1$, e.g., is represented by a single Slater-determinant (configuration (c) in fig. 11.4), and the comparison of configurations (a) and (c), keeping in mind the rules contained in (4) and the accompanying footnote, immediately shows that the difference between $\bar{V}(\text{config. (c)})$ and $\bar{V}_{\text{diag}}(\text{config. (a)})$ is just $2J(a_\sigma - a_{\sigma\tau})$.

For the differences occurring on both sides of inequality (3), one thus finally gets, if one further makes the natural assumption that the neutron excess n of the isobar $[N, Z]$ is positive,

$$\frac{1}{2} \{ \bar{V}[N, Z] - \bar{V}[N+2, Z-2] \} = -2J \{ a_\tau(n+3) + 3a_{\sigma\tau} \} \quad (6)$$

$$\bar{V}[N, Z] - \bar{V}[N+1, Z-1] = -2J \{ [a_\tau(n+2) + a_\sigma + 5a_{\sigma\tau}] \pm (a_\sigma - a_{\sigma\tau}) \},$$

the $\pm$ -sign referring to the triplet and singlet state of the odd isobar, respectively. Expressing that condition (3) must hold as well for the singlet as for the triplet state of the odd isobar, we thus get the inequalities (BREIT and WIGNER [38b]).

$$3a_{\sigma\tau} - a_\tau > 0, \quad (7)$$

$$2a_\sigma - a_\tau + a_{\sigma\tau} > 0. \quad (8)$$

Since, by (8.2-12),

$$2a_\sigma - a_\tau + a_{\sigma\tau} = \frac{1}{2}(a_0 + a_\tau) + \frac{1}{6}(3q - 1), \quad (9)$$

condition (8) can be deduced from Kemmer's condition (11.21-2), provided only $q < \frac{1}{2}$.

Inequality (7), on the contrary, is not reducible to our saturation conditions (11.21-1, 2). With the help of (8.2-12) (by forming the expression $3 + q$), it may be seen that, together with (11.21-1), it implies (11.21-7), so that (11.21-1, 2) and (7) together can be considered (11.22) as a set of necessary and sufficient conditions for the fulfilment of both the first saturation and odd isobar instability requirements. Condition (7) is, of course, fulfilled by our final choice (11.33-8) of the parameters a_τ , $a_{\sigma\tau}$; in terms of the exchange potential parameters (8.2-5), it takes the simple form

$$H > M. \quad (10)$$

Combining (7) with the second formula (8.2-12), one derives the further inequality

$$a_0 - 3a_\sigma + q > 0; \quad (11)$$

inserting (7) and (11) into the expression (8.2-9) of ${}^1p$, one finds

$${}^1p < q; \quad (12)$$

the ${}^1P$ effective potential is either a repulsion or an attraction smaller in absolute value than that in ${}^1S$ states. If one uses the equality (11.31-2) instead of the inequality (11.21-1), one gets further limitations on the p 's: from (8.2-9, 12) and (11.21-2) one then derives that (FEENBERG [37c])

$${}^1p \geq -3q; \quad (13)$$

from (8.2-12) and (7), that (BREIT and WIGNER [38b])

$$-{}^3p < \frac{1}{2}(4q + 3). \quad (14)$$

Grouping together (12) and (13), (14) and (11.21-5), we have then, if $a_0 = 0$,

$$\begin{aligned} \frac{1}{2} &\leq -{}^3p < \frac{1}{2}(4q + 3) \\ -3q &\leq {}^1p < q. \end{aligned} \quad (15)$$

If further (11.32-6) $a_\sigma = 0$, $-{}^3p$ and ${}^1p$ reach, as we have seen (11.33-12, 13), their lower limits.

11.41. Ground state of odd isobar. With our choice of interaction parameters, $a_\sigma - a_{\sigma\tau} < 0$, so that, according to the estimate (5), the triplet state of the isobar lies lower than the singlet state. For the lightest odd nuclei (the only stable ones), calculations based on the quasi-atomic model (HUND [37]) confirm this conclusion, actually indicating the ground state to be a ${}^{13}S$ one. In the case of ${}^2\text{H}$, ${}^6\text{Li}$ and ${}^{14}\text{N}$, it has been ascertained that, in agreement with this result, the total angular momentum has the quantum number $J = 1$.

CHAPTER XII

PROPERTIES OF HEAVY NUCLEI

12.0. This Chapter contains a somewhat closer discussion of the Fermi gas model for heavy nuclei. It will be seen that in spite of its inadequacy with respect to quantitative determinations, this model yields a fairly comprehensive picture of the ground states of such nuclei. In the first approximation, various effects can be treated with a view to deriving a general expression for the binding energy, including the correction for surface tension and Coulomb interaction. The next approximation will be carried out only neglecting the latter effects, i.e. for the "standard heavy nucleus" (9.41), and besides furnishing a more accurate expression for the volume energy, will disclose in an interesting way the tendency to " α -clustering" already referred to above (9.12). Finally, the penetration of fast nucleons into a heavy nucleus will be investigated; in particular, a derivation will be given of the formula used in 7.14.

12.1. Mass-defects and nuclear radii

12.11. *The volume energy.* In first approximation, we get the total volume energy of the nucleus $[N, Z]$ simply by adding to the kinetic energy given by (9.41-7) (with $g_s = 2$, corresponding to complete spin saturation) the average value of the nuclear energy, which can be calculated with the help of formulae (11.11-20, 21) and the accompanying table 11.11-2. The individual wave-functions $\varphi_n(\vec{x})$ being plane waves:

$$\varphi_n(\vec{x}) = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar} \vec{p}_n \vec{x}}, \quad (1)$$

the integrals (11.11-20) occurring in the nuclear energy can ultimately be expressed in terms of certain Fourier coefficients of the nuclear potential $Jw(\kappa r)$. For this purpose, let us first introduce the Fourier expansion of the law of force

$$w(\kappa r) = \int w_p(p) e^{-\frac{i}{\hbar} \vec{p} \vec{x}} \frac{V}{(2\pi\hbar)^3} dv_p; \quad (2)$$

the Fourier coefficients are of the form

$$w_p(p) = \frac{1}{V} \int w(\kappa r) e^{\frac{i}{\hbar} \vec{p} \vec{x}} dv = \frac{2\pi}{V\kappa^3} F\left(\frac{p}{\hbar\kappa}\right) \quad (3)$$

with

$$F(u) \equiv \frac{2}{u} \int_0^\infty w(\xi) \xi \sin u\xi d\xi.$$

Next, we calculate the matrix-elements of the nuclear potential:

$$\begin{aligned} \langle \vec{p}_i^{(1)} \vec{p}_i^{(2)} | J(r) | \vec{p}_f^{(1)} \vec{p}_f^{(2)} \rangle &= \int \varphi_{\vec{p}_i^{(1)}}^*(\vec{x}^{(1)}) \varphi_{\vec{p}_i^{(2)}}^*(\vec{x}^{(2)}) J(r^{(12)}) \varphi_{\vec{p}_f^{(1)}}(\vec{x}^{(1)}) \varphi_{\vec{p}_f^{(2)}}(\vec{x}^{(2)}) d\vec{v}^{(1)} d\vec{v}^{(2)} \\ &= \delta(\vec{p}_i^{(1)} + \vec{p}_i^{(2)} - \vec{p}_f^{(1)} - \vec{p}_f^{(2)}) J w_p(p_0), \end{aligned} \quad (4)$$

with

$$\vec{p}_0 \equiv \vec{p}_i^{(1)} - \vec{p}_f^{(1)} (= \vec{p}_f^{(2)} - \vec{p}_i^{(2)}).$$

With this notation *, the formulae (11.11–20) take the form

$$(+|-) = \begin{cases} \sum_{\vec{p}_i^{(1)} \vec{p}_i^{(2)}}^{(+)} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(-)} \langle \vec{p}_i^{(1)} \vec{p}_i^{(2)} | J | \vec{p}_f^{(1)} \vec{p}_f^{(2)} \rangle & \text{(ordinary)} \\ \sum_{\vec{p}_i^{(1)} \vec{p}_i^{(2)}}^{(+)} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(-)} \langle \vec{p}_i^{(1)} \vec{p}_i^{(2)} | J | \vec{p}_f^{(2)} \vec{p}_f^{(1)} \rangle & \text{(exchange).} \end{cases} \quad (5)$$

Accordingly, the ordinary integrals reduce to $J w_p(0)$ times N^2 , Z^2 or NZ , as the case may be, so that their total contribution to the energy, according to table 11.11–2 in which (11.31–2) $a_0 = 0$, is proportional to the square of the neutron excess $(N-Z)^2$; and since $w_p(0)$ is inversely proportional to A , this contribution is, more precisely, proportional to $A \cdot (n/A)^2$, just like the analogous term in the kinetic energy. As regards the exchange integrals, which involve summations (or integrations) over momenta up to the maximum values $p_m^{(N)}$, $p_m^{(Z)}$ for neutron or proton states (9.41–1), we arrive at the same conclusion: for these integrals will ultimately be functions of the $p_m^{(N)}$, $p_m^{(Z)}$, which only depend on the ratios N/A , Z/A , i.e. on n/A . If, therefore, assuming that we have to do with an even nucleus of relatively small neutron excess, we expand such expressions in powers of n/A , the lowest power occurring (besides a term independent of n) will be the second, because the volume energy, as long as the Coulomb interaction is disregarded, is symmetrical with respect to its dependence on N and Z . We thus arrive, for the volume energy, at an expression of the form suggested by the empirical distribution of mass-defects, i.e. a mean energy per nucleon $-\varepsilon_1 [1 - \gamma(n/A)^2]$ (2.21–3).

12.12. The surface energy. The model hitherto considered does not give rise to any surface effects, since it corresponds to considering the nucleon assembly under the influence of a fictitious constant (and negative) potential abruptly rising to zero at the boundary of the domain occupied by the nucleons. In order to account for the surface energy, one may (B & B, § 29) replace the sharp rise of the potential by a gradual one. More definitely, let us call x a length measured along the normal to the

* The indices i, f refer to the "initial" and "final" state, respectively, of the transition to which the matrix-element is related.

boundary, and varying from a value 0 outside (i.e. where the potential takes the value 0) to a value l , at which the potential assumes its constant negative value; let us assume the potential to vary linearly with x throughout the surface layer. If the thickness l is large in comparison with the wave-lengths of the nucleons, i.e. $l \gtrsim d$, we can apply to any volume element of the layer the statistical arguments leading to a relation between the maximum value of the momentum at that point (depending on the fictitious potential) and the particle density: the latter quantity being proportional to the cube of the maximum momentum will therefore vary as $(x/l)^{3/2}$; calling ϱ_0 the constant value of the density inside the nucleus, we have thus, within the layer,

$$\varrho = \varrho_0 (x/l)^{3/2}. \quad (6)$$

The mean kinetic energy at x will be proportional to $\varrho^{2/3}$, so that the mean kinetic energy per unit volume will be proportional to $\varrho^{5/3}$. If the density were ϱ_0 throughout, without boundary layer, the volume of the nucleus would have a value V such that $\varrho_0 V$ is equal to the number of nucleons, and the kinetic energy would have a corresponding value K . From the density law (6), we easily derive the change of kinetic energy

$$\delta K = -\frac{4}{35} K \cdot \frac{lS}{V}, \quad (7)$$

S denoting the surface of the nucleus. We have just seen how the mean nuclear energy can be expressed in terms of the maximum momenta; we therefore know its expression as a function of x within the layer and are thus able to calculate the variation due to this layer. In this way, the total contribution from surface effects to the average energy can in principle be estimated. It would be pointless, however, to go into further details of such calculations, since we cannot expect them to yield reliable quantitative results.

It must still be observed that there is, besides the term (7), another surface contribution to the kinetic energy, which has nothing to do with the variation of the potential energy at the boundary: it simply arises from the discreteness of the eigenvalues of the momenta of the individual nucleons and can be estimated by performing the summations over the momenta to a greater accuracy than that obtained by replacing them simply by integrations. Such estimates, performed by FEENBERG [41c], yield, besides a main term of the form $(9.41-7)$, $K^{(0)}A \left[1 + \frac{5}{9} \left(\frac{n}{A} \right)^2 \right]$, with $K^{(0)} = K(\frac{1}{2}d/r_0)^2$, a surface term $K^{(0)}A^{2/3} \cdot 1.92 \left[1 + \frac{2}{9} \left(\frac{n}{A} \right)^2 \right]$ and an additional term $c_0 K^{(0)} A^{1/3}$ (with a numerical constant $c_0 \approx 1$), which apparently cannot admit of any simple geometrical interpretation*.

* See further a paper by VOLKOFF [42a], in which the influence of the shape of the nucleus on the kinetic energy is also discussed.

12.13. Nuclear radii. The expression finally obtained by adding the volume and surface contributions from kinetic and nuclear energy, as well as the Coulomb interaction between the protons, shows the expected dependence on the proton and neutron numbers. It remains to be seen whether at least the order of magnitude of the different terms comes out about right. If one assumes a definite distance dependence for the nuclear potential, with given numerical values of the strengths and range, the only remaining parameter is the nucleon radius r_0 (it also occurs in the potential energy terms through the maximum momenta). We may then look for the "best" value of r_0 , i.e. that which minimizes the total energy.

It proves convenient to take as variational parameter the dimensionless quantity

$$r = \frac{1}{2} (9\pi)^{1/3} (\kappa r_0)^{-1} = \frac{1.523}{\kappa r_0}; \quad (8)$$

the maximum momenta can then simply be expressed as

$$p_m^{(N) \text{ or } (Z)} = p_m^{(0)} \left(1 \pm \frac{n}{A} \right)^{1/3}, \quad (9)$$

with

$$p_m^{(0)} = \hbar \kappa r, \quad (10)$$

so that all terms arising from the nuclear interaction immediately appear as functions of r . The mean total energy per nucleon has therefore the general form

$$\varepsilon(r) = -\varepsilon_1(r) [1 - \gamma(r) (n/A)^2] + \sigma(r) A^{-1/3} + \varepsilon_0 r A^{-4/3} Z^2, \quad (11)$$

ε_0 being a constant. Strictly speaking, the best value r should be found by solving the equation $\varepsilon'(r) = 0$, i.e.

$$\varepsilon'_1(r) = [\varepsilon_1 \gamma]' (n/A)^2 + \sigma' A^{-1/3} + \varepsilon_0 A^{-4/3} Z^2. \quad (12)$$

As a first approximation, we may take the value r_0 for which

$$\varepsilon'_1(r_0) = 0: \quad (13)$$

this corresponds to a constant density of nuclear matter, irrespective of the mass number. Since, on account of (13),

$$\varepsilon'_1(r) = (r - r_0) \varepsilon''_1(r_0), \quad (14)$$

we get the next approximation by inserting in (14) the value (12) of $\varepsilon'_1(r)$, in which we may replace r by r_0 . This gives

$$r_1 = r_0 \left\{ 1 - \frac{1}{9MC^2} \left[(\varepsilon_1 \gamma)'_{r_0} \left(\frac{n}{A} \right)^2 + \sigma'(r_0) r_0 A^{-1/3} + G(r_0) \right] \right\}; \quad (15)$$

in this formula, $G(r_0)$ denotes the value of the Coulomb energy for $r = r_0$ while C represents the velocity of sound in nuclear matter, referred to the

velocity of light as unit. This quantity is indeed defined, ϱ being the density, by

$$C^2 = \frac{1}{M} \varrho^2 \frac{d^2(-\varepsilon_1)}{d\varrho^2} = \frac{1}{9M} r_0^2 \frac{d^2(-\varepsilon_1)}{dr_0^2}; \quad (16)$$

the last equality follows from the preceding one because of $\varrho \sim r_0^{-3}$ and (13). Relation (15), which has been derived in a less complete form by FEENBERG [41a] and as above by PRESENT [41], shows that one would expect a slight increase of the nucleon radius r_0 with increasing mass number. In fact, the three correction terms in (15) all work in the same sense: the increase of the relative neutron excess with increasing A means looser binding and smaller density; the contraction of the nucleus due to surface forces becomes smaller in heavier nuclei, again resulting in a decrease of the density; and the Coulomb repulsion of the protons clearly produces the same effect.

Explicit calculations according to the above scheme have been carried out most extensively with the Gauss potential. On the determination of r_0 from (13) and of $\varepsilon_1(r_0)$ we shall say more in the next section, after we have calculated the next approximation to ε_1 : for the first approximation turns out to be completely inadequate. Even then, it will appear that a set of values of J and κ consistent with the empirical values of r_0 and ε_1 would correspond to a much tighter binding than indicated by the analysis of the two-nucleon system. Adopting such values for J and κ , we find that to account for the empirical value of the surface tension (2.21-5), we should have to take $l \approx 4.8 r_0$, an improbably large value. Finally, Present's estimate* of the correction terms in formula (15) leads to the expression for the radius

$$r'_0 = r_0 \left[1 + 0.8 \left(\frac{n}{A} \right)^2 - \frac{0.3}{A^{1/3}} + 0.010 \frac{Z^2}{A^{4/3}} \right]; \quad (17)$$

the factor between brackets increases by 9 % from ^{56}Fe to ^{207}Pb .

12.14. Variation of density within a nucleus. The Coulomb repulsion between protons must obviously give rise to a tendency for the proton density to vary within a nucleus from a minimum value at the centre to a maximum near the boundary. A non-uniform proton distribution may be expected to create forces which distort the neutron distribution and tend to make the two particle densities vary in the same manner. This density variation has been analysed by FEENBERG [41b] neglecting surface effects. It is then permissible to assume that the deviations from uniformity of the proton and neutron distributions are small. The only further assumption that must be made is that the total volume energy may be written as an integral $\int \mathcal{V}(\varrho^{(n)}, \varrho^{(p)}) dv$, the integrand being a function of

* Present's treatment differs slightly from the above in some particulars; but it is not worth while working out the resulting modifications in formula (17).

the neutron and proton densities $\rho^{(n)}$, $\rho^{(p)}$ only. Besides this volume energy, there occurs a correction term of kinetic energy, which is easily found to be

$$\frac{\hbar^2}{2M} \int \left[\frac{(\text{grad } \rho^{(n)})^2}{\rho^{(n)}} + \frac{(\text{grad } \rho^{(p)})^2}{\rho^{(p)}} \right] dv, \quad (18)$$

and the Coulomb energy, which, including the exchange term calculated by the statistical method, becomes

$$G = \frac{1}{2} \cdot \frac{Z-1}{Z} e^2 \int \frac{\rho^{(p)}(\vec{x}) \rho^{(p)}(\vec{x}')}{|\vec{x} - \vec{x}'|} dv dv' - \left(\frac{81}{64\pi} \right)^{1/3} e^2 \int (\rho^{(p)})^{4/3} dv. \quad (19)$$

The variational method may now be applied, using for $\rho^{(n)}$, $\rho^{(p)}$ trial functions which are chosen in the form of simple polynomials in the radius vector r , satisfying the boundary conditions

$$\frac{d\rho^{(n) \text{ and } (p)}}{dr} = 0 \quad \text{for } r = R \quad (20)$$

as well as the normalization conditions

$$\int_{(\text{nuc})} \rho^{(n)} dv = N, \quad \int_{(\text{nuc})} \rho^{(p)} dv = Z. \quad (21)$$

The final result is that both densities are found to vary in a parallel manner, tending toward the formation of a hollow centre within the nucleus. For the nucleus ${}^{200}_{80}\text{Hg}$, e.g., one finds

$$\frac{\rho^{(n)}(R)}{\rho^{(n)}(0)} = 1.20, \quad \frac{\rho^{(p)}(R)}{\rho^{(p)}(0)} = 1.43. \quad (22)$$

The resulting changes in energy and nuclear radius are quite insignificant: the change in total energy of ${}^{200}_{80}\text{Hg}$ is only about -7 MeV.

12.2. Standard heavy nucleus to second approximation

12.20. In carrying the approximation now a step further, we shall disregard all surface effects as well as Coulomb energy, which means that we consider the ground state of the "standard heavy nucleus", as defined in 9.41 (except for the fact that we provisionally leave the value of the nucleon radius r_0 undetermined). We shall begin by deriving a more accurate value of the mean volume energy per nucleon ε_1 ; we shall then discuss more closely the distribution of the different nucleons in this state, first as regards their momenta, then as regards their positions and spins. Especially in the latter respect interesting results are obtained.

12.21. The volume energy. In the calculation of the energy, we have been applying the usual perturbation method, using as a perturbation parameter the strength of the nuclear potential. In the expansion

$$\mathcal{E} = \mathcal{E}^{(0)} + \mathcal{E}^{(1)} + \mathcal{E}^{(2)} + \dots \quad (1)$$

in powers of this parameter, the initial approximation $\mathcal{G}^{(0)}$ is simply the unperturbed kinetic energy; the next one, $\mathcal{G}^{(1)}$, is the first approximation (the only one hitherto considered) to the average value $\bar{\mathcal{V}}^{(1)}$ of the potential energy, while the second, $\mathcal{G}^{(2)}$, consists of contributions to kinetic as well as to potential energy, $\mathcal{K}^{(2)}$ and $\bar{\mathcal{V}}^{(2)}$, which are very simply related to each other:

$$\begin{aligned}\mathcal{G}^{(0)} &= \mathcal{K}^{(0)} = K^{(0)} A, & \mathcal{G}^{(1)} &= \bar{\mathcal{V}}^{(1)} \\ \mathcal{G}^{(2)} &= \mathcal{K}^{(2)} + \bar{\mathcal{V}}^{(2)}, & \text{with } \bar{\mathcal{V}}^{(2)} &= -2\mathcal{K}^{(2)}.\end{aligned}\quad (2)$$

In the ground state of the system, the second order correction

$$\mathcal{G}^{(2)} = -\mathcal{K}^{(2)} = \frac{1}{2} \bar{\mathcal{V}}^{(2)} \quad (3)$$

is always negative, i.e. is equivalent to a tightening of the binding.

In order, therefore, to obtain the energy $\mathcal{G}$ up to the second approximation, it is sufficient to calculate the potential energy up to that approximation, and since the potential energy is itself of the first order, it suffices to use the first order approximation for the wave-function of the ground state. The latter can be written in the form of an expansion

$$\Psi = \frac{1}{\sqrt{1 + \sum_{\mu} |a_{\mu}|^2}} (\Psi_0 + \sum_{\mu} a_{\mu} \Psi_{\mu}) \quad (4)$$

in terms of the Slater-determinants Ψ_0, Ψ_{μ} , corresponding to the initial approximation for ground state and excited configurations. The only excited states Ψ_{μ} which combine with the ground state are those in which two individual states, say $n_i^{(1)}, n_i^{(2)}$, occupied in the ground state configuration, are replaced by two others, $n_f^{(1)}, n_f^{(2)}$, unoccupied in that configuration. The corresponding expansion coefficient a_{μ} is, in first approximation, for a nuclear potential of the general form (8.2-1),

$$a_{\mu} = \frac{(n_i^{(1)} n_i^{(2)} | \odot J(r) | n_f^{(1)} n_f^{(2)}) - (n_i^{(1)} n_i^{(2)} | \odot J(r) | n_f^{(2)} n_f^{(1)})}{E_i^{(1)} + E_i^{(2)} - (E_f^{(1)} + E_f^{(2)})}; \quad (5)$$

$E = p^2/2M$ denotes the energy of the individual state of momentum $\vec{p}$. Using the abbreviations i, f for the two sets of values of the spin and isotopic coordinates $\sigma_z^{(1)'}, \sigma_z^{(2)'}, \tau_z^{(1)'}, \tau_z^{(2)'}$, this may be written, more explicitly,

$$\begin{aligned}(i | a_{\mu} | f) &= \\ &= \frac{(i | \odot | f) (\vec{p}_i^{(1)} \vec{p}_i^{(2)} | J(r) | \vec{p}_f^{(1)} \vec{p}_f^{(2)}) - (i | \odot P_{\sigma} P_{\tau} | f) (\vec{p}_i^{(1)} \vec{p}_i^{(2)} | J(r) | \vec{p}_f^{(2)} \vec{p}_f^{(1)})}{E_i^{(1)} + E_i^{(2)} - (E_f^{(1)} + E_f^{(2)})}.\end{aligned}\quad (6)$$

The expectation value of any operator (such as the nuclear interaction energy) of the form

$$\mathcal{W} = \frac{1}{2} \sum_{i,k} \mathcal{W}^{(ik)}(Q^{(i)}, Q^{(k)}) \quad (7)$$

in the state (4) can be written (this particular form will presently be useful for another purpose)

$$\begin{aligned} \text{av } \mathcal{W} &= \frac{1}{2} A(A-1) \int_{Q^{(1)}} \int_{Q^{(2)}} g(Q^{(1)}, Q^{(2)}) \mathcal{W}^{(12)}(Q^{(1)}, Q^{(2)}) \\ g(Q^{(1)}, Q^{(2)}) &= \int_{Q^{(3)}} \dots \int_{Q^{(A)}} \Psi^*(Q^{(1)}, \dots, Q^{(A)}) \Psi(Q^{(1)}, \dots, Q^{(A)}), \end{aligned} \quad (8)$$

or, using again the same notation i, f for the spin and isotopic coordinates,

$$\begin{aligned} \text{av } \mathcal{W} &= \frac{1}{2} A(A-1) \int d\nu^{(1)} d\nu^{(2)} \sum_{i,f} (i | g(\vec{x}^{(1)}, \vec{x}^{(2)}) | f) (f | \mathcal{W}^{(12)}(\vec{x}^{(1)}, \vec{x}^{(2)}) | i) \\ (i | g(\vec{x}^{(1)}, \vec{x}^{(2)}) | f) &= \int_{Q^{(3)}} \dots \int_{Q^{(A)}} \Psi^*(\vec{x}^{(1)}, \vec{x}^{(2)}; i; Q^{(3)}, \dots, Q^{(A)}) \Psi(\vec{x}^{(1)}, \vec{x}^{(2)}; f; Q^{(3)}, \dots, Q^{(A)}). \end{aligned} \quad (9)$$

For reasons which will soon be apparent, we shall call g the *correlation operator*. In the state (4), its two first approximations are

$$\begin{aligned} (i | g | f)^{(0)} &= \frac{1}{A(A-1)} \sum_{\vec{p}^{(1)} \vec{p}^{(2)}}^{(\text{occ})} \varphi_{\vec{p}^{(1)}}^*(\vec{x}^{(1)}) \varphi_{\vec{p}^{(2)}}^*(\vec{x}^{(2)}) [(i | 1 | f) \varphi_{\vec{p}^{(1)}}(\vec{x}^{(1)}) \varphi_{\vec{p}^{(2)}}(\vec{x}^{(2)}) \\ &\quad - (i | P_\sigma P_\tau | f) \varphi_{\vec{p}^{(2)}}(\vec{x}^{(1)}) \varphi_{\vec{p}^{(1)}}(\vec{x}^{(2)})] \\ (i | g | f)^{(1)} &= \end{aligned} \quad (10)$$

$$\frac{1}{A(A-1)} \cdot 2\Re \sum_{\vec{p}_i^{(1)} \vec{p}_i^{(2)}}^{(\text{occ})} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(\text{unocc})} \varphi_{\vec{p}_i^{(1)}}^*(\vec{x}^{(1)}) \varphi_{\vec{p}_i^{(2)}}^*(\vec{x}^{(2)}) \varphi_{\vec{p}_f^{(1)}}(\vec{x}^{(1)}) \varphi_{\vec{p}_f^{(2)}}(\vec{x}^{(2)}) (i | a_\mu | f);$$

in these formulae, the summations are to be extended, according to the indication ^(occ) or ^(unocc), over all occupied or all unoccupied individual states of momentum $\vec{p}$ in the configuration Ψ_0 , each state being counted only once (such summations can be replaced, as usual, by integrations over the corresponding domains of phase space).

From the formulae (9) and (10) we derive for the total energy the well-known expressions

$$\begin{aligned} \mathcal{G}^{(1)} = \overline{\mathcal{V}}^{(1)} &= \frac{1}{2} \sum_{\vec{p}^{(1)} \vec{p}^{(2)}}^{(\text{occ})} [(\vec{p}^{(1)} \vec{p}^{(2)} | J(r) | \vec{p}^{(1)} \vec{p}^{(2)}) \text{tr } \mathcal{O} \\ &\quad - (\vec{p}^{(1)} \vec{p}^{(2)} | J(r) | \vec{p}^{(2)} \vec{p}^{(1)}) \text{tr } \mathcal{O} P_\sigma P_\tau] \end{aligned} \quad (11)$$

$$\mathcal{G}^{(2)} = \frac{1}{2} \overline{\mathcal{V}}^{(2)} = -\frac{1}{2} \sum_{\vec{p}_i^{(1)} \vec{p}_i^{(2)}}^{(\text{occ})} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(\text{unocc})} \frac{(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \mathcal{H} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})}{E_f^{(1)} + E_f^{(2)} - (E_i^{(1)} + E_i^{(2)})},$$

with the abbreviation

$$\begin{aligned} (\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \mathcal{H} | \vec{p}_f^{(1)} \vec{p}_f^{(2)}) &\equiv (\vec{p}_i^{(1)} \vec{p}_i^{(2)} | J(r) | \vec{p}_f^{(1)} \vec{p}_f^{(2)})^2 \text{tr } \mathcal{O}^2 \\ &\quad - (\vec{p}_i^{(1)} \vec{p}_i^{(2)} | J(r) | \vec{p}_f^{(1)} \vec{p}_f^{(2)}) (\vec{p}_i^{(1)} \vec{p}_i^{(2)} | J(r) | \vec{p}_f^{(2)} \vec{p}_f^{(1)})^* \text{tr } \mathcal{O}^2 P_\sigma P_\tau. \end{aligned} \quad (12)$$

In $\mathcal{G}^{(1)}$ as well as $\mathcal{G}^{(2)}$, there occur an ordinary and an exchange term; the formula for $\mathcal{G}^{(1)}$ is actually the same as (11.11-24), in which the coefficients $\text{tr} \mathcal{O}$ and $\text{tr} \mathcal{O} P_\sigma P_\tau$, as given by (8.21-13), had been deduced by a more elaborate analysis. Let us use again the notations (12.13-8, 10) and express all momenta in terms of $p_m^{(0)}$ by

$$\vec{u} = \frac{\vec{p}}{p_m^{(0)}} \quad (13)$$

or

$$\frac{\vec{p}}{\hbar \kappa} = \vec{u} \, r. \quad (14)$$

The cofactor $w_p(p_0)$ occurring in the expression (12.11-4) for the matrix-elements of the nuclear potential then takes, according to (12.11-3) and (14), the form

$$w_p(p_0) = \frac{1}{A} \cdot \frac{4}{3\pi} r^3 F(u_0 r). \quad (15)$$

The formulae (11) accordingly reduce to

$$\mathcal{G}^{(1)} = \frac{F(0)}{4\pi} A J \left[\text{tr} \mathcal{O} \cdot \frac{r^3}{6} - \text{tr} \mathcal{O} P_\sigma P_\tau \cdot f_{\text{exch}}^{(1)}(r) \right] \quad (16)$$

$$f_{\text{exch}}^{(1)}(r) = \frac{r^3}{6} \left(\frac{3}{4\pi} \right)^2 \frac{1}{F(0)} \int dv_u \int dv_{u'} F(r | \vec{u} - \vec{u}' |)$$

(both integrals to be taken independently over the spheres $u \leq 1, u' \leq 1$).

$$\mathcal{G}^{(2)} = - \frac{1}{320\pi} A J^2 \frac{M}{\hbar^2 \kappa^2} \left[\text{tr} \mathcal{O}^2 \cdot f_{\text{ord}}^{(2)}(r) - \text{tr} \mathcal{O}^2 P_\sigma P_\tau \cdot f_{\text{exch}}^{(2)}(r) \right]$$

$$f_{\text{ord}}^{(2)}(r) = \frac{15 r^4}{8\pi^4} \int dv_u dv_{u'} dv_{u''} \frac{[F(r u')]^2}{\vec{u}' (\vec{u}' + \vec{u}'' - \vec{u})} \quad (17)$$

$$f_{\text{exch}}^{(2)}(r) = \frac{15 r^4}{8\pi^4} \int dv_u dv_{u'} dv_{u''} \frac{F(r u') F(r |\vec{u}' + \vec{u}'' - \vec{u}|)}{\vec{u}' (\vec{u}' + \vec{u}'' - \vec{u})}$$

(domain of integration defined by the inequalities *

$$u \leq 1, \quad u'' \leq 1, \quad |\vec{u} - \vec{u}'| \geq 1, \quad |\vec{u}' + \vec{u}''| \geq 1).$$

In this form they have been derived by EULER [37] for the special case of the Gauss potential **.

* We have put $\vec{u} = \vec{p}_i^{(1)}/p_m^{(0)}$, $\vec{u}' = (\vec{p}_i^{(1)} - \vec{p}_j^{(1)})/p_m^{(0)}$, $\vec{u}'' = \vec{p}_i^{(2)}/p_m^{(0)}$.

** A factor 5 has been introduced in our definition of the functions $f_{\text{ord}}^{(2)}$ and $f_{\text{exch}}^{(2)}$ only in order to conform with Euler's definitions.

Volz' saturation condition (11.31-2) implies, by (8.21-13), $\text{tr } \mathcal{O} = 0$ and so causes the non-saturation term in $\mathcal{G}^{(1)}$ to disappear; moreover, it gives, on account of (8.21-13) and (8.2-12),

$$\text{tr } \mathcal{O} P_\sigma P_\tau = 6(1+q) \approx 9.6, \quad (18)$$

thus completely fixing the constant factor in $\mathcal{G}^{(1)}$. At small densities of the nuclear matter, the factor $f_{\text{exch}}^{(1)}$ becomes $\approx r^3/6$, i.e. increases proportionally to this density. But for large densities, it tends asymptotically towards a constant value. This behaviour, which is in conformity with the saturation property of nuclear binding, is conditioned by the fact that, according to the definition (12.11-3) of $F(u)$, the main contributions to the energy come from transitions in which the transfer of momentum p_0 is smaller than $\hbar\kappa$, i.e. $p_0/p_m^{(0)} \leq r^{-1}$. At small densities, the occupied states are few and their momenta therefore not widely different, so that the required transitions are possible from most of them; at high densities, on the contrary, only a small fraction of the nucleons can take part in such transitions: any nucleon can exchange its momentum with only $(p_0/p_m^{(0)})^3 \approx r^{-3}$ of the others, which just compensates the increase ($\sim r^3$) of the binding energy due to the diminution of the average distance between the nucleons. The general properties of $f_{\text{exch}}^{(1)}$ are clearly illustrated by the explicit expression which can be obtained for a nuclear potential of the Gauss type. This corresponds to

$$F(u) = \frac{1}{2} \sqrt{\pi} e^{-u^2/4}. \quad (19)$$

The evaluation of $f_{\text{exch}}^{(1)}$ is then elementary and yields

$$f_{\text{exch}}^{(1)}(r) = 2 \int_0^r e^{-t^2} dt + \frac{1}{r^3} [2 - 3r^2 + (r^2 - 2)e^{-r^2}]; \quad (20)$$

one has $f_{\text{exch}}^{(1)}(\infty) = \sqrt{\pi}$.

Going over to $\mathcal{G}^{(2)}$, we first determine the coefficients $\text{tr } \mathcal{O}^2$ and $\text{tr } \mathcal{O}^2 P_\sigma P_\tau$ with the help of the formulae (8.21-16) and (11.33-12, 13), the latter depending on both saturation conditions (11.33-7); we get

$$\begin{aligned} \text{tr } \mathcal{O}^2 &= 4(1 + 3q^2) = 4 \cdot 2.153 \\ \text{tr } \mathcal{O}^2 P_\sigma P_\tau &= 2(-1 + 3q^2) = 2 \cdot 0.153. \end{aligned} \quad (21)$$

The numerical values correspond to $q = 0.62$, the value which, according to table 6.432, fits with the two-nucleon data in the case of the Gauss potential. This means that $\mathcal{G}^{(2)}$ depends on the combination

$$f_{\text{ord}}^{(2)} - 0.036 f_{\text{exch}}^{(2)}, \quad (22)$$

so that the exchange contribution is of little significance.

It has been observed by WATANABE [39] that a very slight modification of Kemmer's saturation condition $a_\sigma = 0$ (while maintaining $a_0 = 0$) would suffice to get rid entirely of second order exchange effects. In fact, one can make $\text{tr } \mathcal{O}^2 P_\sigma P_\tau$ exactly zero, i.e. satisfy the condition

$$9 \cdot 3p^2 + 1p^2 = 3(1 + q^2) \quad (23)$$

by

$$a_0 = \frac{1}{4} [9q - 1 - \sqrt{3(7q^2 - 6q + 7)}] = 0,0087, \quad (24)$$

as is easily found by inserting in (23) the values of ${}^3p$ and ${}^1p$ in terms of a_0 and q deduced from (8.2-9, 12), with $a_0 = 0$. Moreover, the condition (23) implies only a slight change of $\text{tr } \mathcal{O}^2$:

$$\frac{1}{4} \text{tr } \mathcal{O}^2 = \frac{3}{2} (1 + q^2) = 2,076. \quad (25)$$

The discussion of the $f^{(2)}$'s has only been performed for the case of the Gauss potential. Their calculation requires a lengthy procedure of approximation, for the details of which the reader is referred to Euler's paper. The behaviour of $f_{\text{ord}}^{(2)}$ is found to be quite similar to that of $f_{\text{exch}}^{(1)}$:

$$\begin{aligned} f_{\text{ord}}^{(2)}(r) &\approx \frac{5}{3} \sqrt{2\pi} r^3 \quad [\text{for } r \ll 1], \\ f_{\text{ord}}^{(2)}(r) &\simeq 10(1 - \log 2) - \frac{5}{2} r^{-2} + \frac{1}{2} r^{-4} + 0,16 r^{-6} \quad (26) \\ &[\text{for } r \geq 1,6, \text{ with an error } \leq 20\%]. \end{aligned}$$

The saturation feature is again due to the limitation of the possible transitions resulting from the short range of the nuclear force and the exclusion principle. As to $f_{\text{exch}}^{(2)}(r)$, it is the same as $f_{\text{ord}}^{(2)}$ at small r ; then it becomes smaller than $f_{\text{ord}}^{(2)}$, reaches a maximum and tends asymptotically to zero for $r \rightarrow \infty$. It can further be verified that the higher approximations of the energy all show the same behaviour as $f_{\text{exch}}^{(2)}(r)$.

Summing up, we are now in a position to write down the complete expression for the volume energy per nucleon in explicit form for a Gauss potential. It will be convenient to revert to our notation (5.11-9), involving the quantity $\hbar^2 \kappa^2 / M$ as unit of energy. Remembering that the mean kinetic energy per nucleon is just $K^{(0)} = 0,3 (\hbar^2 \kappa^2 / M) r^2$, we have*, by (16), (17), (18), (21), (22),

$$- \eta_1 = - \frac{M}{\hbar^2 \kappa^2} \varepsilon_1 \quad (27)$$

$$= 0,3 r^2 - 0,685 b f_{\text{exch}}^{(1)}(r) - 0,00857 b^2 [f_{\text{ord}}^{(2)}(r) - 0,036 f_{\text{exch}}^{(2)}(r)]:$$

this is (apart from the numerical values of the coefficients) the formula established by Euler; $f_{\text{exch}}^{(1)}$ and $f_{\text{ord}}^{(2)}$ are given by (20) and (26), respectively.

For a given nuclear potential, i.e. for given b , the best values of η_1 and r (and consequently of ε_1 and r_0 , κ being also given) are determined by (27) together with the condition of stationary energy

$$\frac{d\eta_1}{dr} = 0. \quad (28)$$

* With Watanabe's modification (23), (25), the last term would be $-0,00826 b^2 f_{\text{ord}}^{(2)}(r)$.

The data resulting from the analysis of the two-nucleon system (table 6.432),

$$\kappa^{-1} = 1,9 \cdot 10^{-13} \text{ cm} \quad , \quad J = 43,7 \text{ MeV}, \quad (29)$$

correspond to $\hbar^2 \kappa^2 / M = 11,5 \text{ MeV}$, and therefore

$$b = 3,80. \quad (30)$$

However, if we adopt this value for b , and draw the graph $\eta_1(r)$, we find for the extreme

$$r_0 \approx 1,84 \quad , \quad \eta_1 \approx 0,464 \quad (31)$$

$$\text{or} \quad r_0 \approx 0,56 d \quad , \quad \varepsilon_1 \approx 5,3 \text{ MeV},$$

i.e. too large a nucleon radius r_0 and, above all, much too small an average binding energy. The relative contributions of $\mathcal{G}^{(0)}$, $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$ to this result* are, in units $(\hbar^2 \kappa^2 / M) \cdot A$,

$$\mathcal{G}^{(0)} = 0,97, \quad \mathcal{G}^{(1)} = -1,14, \quad \mathcal{G}^{(2)} = -0,29: \quad (32)$$

Such figures illustrate at the same time the considerable improvement effected by the second approximation (without it, we would get a maximum value of $\varepsilon_1 \approx 1,95 \text{ MeV}$) and the very poor convergence of the method.

Conversely, we may try to determine the nuclear potential for which Euler's formula would be in agreement with the empirical values of r_0 and ε_1 . For this purpose, it is more convenient to take as energy unit the maximum zero-point energy $E_m^{(0)}$, which depends only on r_0 ; we put accordingly, taking account of (12.13-10),

$$\eta'_1 = \frac{\varepsilon_1}{E_m^{(0)}} = \frac{2}{r^2} \eta_1 \quad , \quad b' = \frac{J}{E_m^{(0)}} = \frac{2}{r^2} b \quad (33)$$

and rewrite Euler's formula and the condition of stationary energy in the form

$$\begin{aligned} \eta'_1 &= -0,6 + 0,685 b' f_{\text{exch}}^{(1)}(r_0) + 0,00428 b'^2 r_0^2 [f_{\text{ord}}^{(2)} - 0,036 f_{\text{exch}}^{(2)}] \\ 0 &= -1,2 + 0,685 b' r_0 \frac{d}{dr_0} f_{\text{exch}}^{(1)} + 0,00428 b'^2 r_0^3 \frac{d}{dr_0} [f_{\text{ord}}^{(2)} - 0,036 f_{\text{exch}}^{(2)}]. \end{aligned} \quad (34)$$

The second equation (34) allows us to determine b' in terms of r_0 , the first one subsequently to calculate the corresponding η'_1 . The empirical value $\eta'_1 = 0,60$ is found to correspond roughly to

$$r_0 \approx 2,6 \quad , \quad b' \approx 2,05 \quad (b \approx 6,93) \quad (35)$$

i.e.

$$\kappa^{-1} \approx 2,4 \cdot 10^{-13} \text{ cm}, \quad J \approx 49,6 \text{ MeV}: \quad (36)$$

both range and strength of potential ought thus to be chosen appreciably larger than is indicated by the experiments on two-nucleon systems. As a

* More exactly, we quote the values for $r = 1,8$.

kind of compromise, one might adopt for the purpose of numerical estimates, the set

$$r_0 = 2.5, \quad b = 4.2 \quad (37)$$

used by WATANABE [39]. Assuming $r_0 = \frac{1}{2}d$, the value $r_0 = 2.5$ corresponds to a range $\kappa^{-1} = 2.33 \cdot 10^{-13}$ cm. The value $b = 4.2$ is that which, for such a range, would yield about the right binding energy for the deuteron*. But, of course, this range is much too large to be compatible with the scattering experiments, and the average binding energy ε_1 comes out much too low (≈ 4.1 MeV).

From (34) we may derive an explicit formula for the velocity of sound in nuclear matter, as defined by (12.13–16):

$$C^2 = \frac{2E_m}{9M} \left\{ 0.6 + 2.055 b' r_0^{-3} [r_0^2 - 4 + e^{-\frac{4}{r_0}} (r_0^4 + 3r_0^2 + 4)] \right. \\ \left. + 0.00214 b'^2 \left[15 + \frac{4.4}{r_0^2} - \frac{30.91}{r_0^4} + \dots \right] \right\}. \quad (38)$$

With the set of parameters (35), this gives

$$C^2 \approx 8.1 \cdot 10^{-3}. \quad (39)$$

12.22. The momentum distribution. In the initial approximation, corresponding to the absolute zero of temperature for the degenerate Fermi gas of nucleons, the distribution of the momenta of these nucleons is simply described by saying that all the lowest states are occupied up to a limit fixed by the total number of nucleons. The effect of the nuclear interactions, by causing transitions from these states into unoccupied ones, will be to spread out this distribution in the neighbourhood of the limiting momentum $p_m^{(0)}$, lowering the density in momentum space below $p_m^{(0)}$ and giving rise to a "tail" in the distribution above this value. A quantitative study of this effect is due to WATANABE [39].

We again restrict ourselves to the consideration of a standard heavy nucleus, in the stationary state described by the formula (4) above. Let $N_\mu(n)$ denote the number (0 or 1) of nucleons occupying, in the configuration μ , any one of the four substates of momentum $\vec{p}$, characterized by an index n which stands for the spin and isotopic coordinates. The probability $F(\vec{p}) \cdot 4V(2\pi\hbar)^{-3} d\nu_p$ for a nucleon to have its momentum in the interval $(\vec{p}, \vec{p} + d\vec{p})$ is then clearly defined by

$$F(\vec{p}) = \frac{1}{A} \cdot \frac{\text{av}_n [\varepsilon + \sum_\mu |\alpha_\mu|^2 N_\mu(n)]}{1 + \sum_\mu |\alpha_\mu|^2}. \quad (40)$$

with

$$\varepsilon = 1 \quad \text{if } p < p_m^{(0)}, \quad \varepsilon = 0 \quad \text{if } p > p_m^{(0)}.$$

* From a table (B & B, table III, p. 111) calculated for a deuteron binding energy of 2.15 MeV, it can indeed be inferred that when the range of the Gauss potential increases from 1.9 to 2.3 (10^{-13} cm), the b -value will become about 10% larger.

the average having to be taken over the substates n . Assuming

$$\sum_{\mu} |\alpha_{\mu}|^2 \ll 1, \quad (41)$$

a condition necessary for the rapid convergence of the approximation procedure embodied in formula (4), we may write (40) more simply

$$\begin{aligned} F(\vec{p}) &= \frac{1}{A} \{1 - \text{av}_n \sum_{\mu} |\alpha_{\mu}|^2 [1 - N_{\mu}(n)]\} \quad (p < p_m^{(0)}) \\ F(\vec{p}) &= \frac{1}{A} \text{av}_n \sum_{\mu} |\alpha_{\mu}|^2 N_{\mu}(n) \quad (p > p_m^{(0)}), \end{aligned} \quad (42)$$

or, according to (6),

$$\begin{aligned} F(\vec{p}) &= \frac{1}{A} \left\{ 1 - \frac{1}{4} \sum_{\vec{p}_i^{(2)}}^{(\text{occ})} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(\text{unocc})} \frac{(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \hat{n} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})}{[E_i^{(1)} + E_i^{(2)} - (E_f^{(1)} + E_f^{(2)})]^2} \right\}_{\vec{p}_i^{(1)} = \vec{p}} \quad (p < p_m^{(0)}) \\ F(\vec{p}) &= \frac{1}{A} \left\{ \frac{1}{4} \sum_{\vec{p}_i^{(1)} \vec{p}_i^{(2)}}^{(\text{occ})} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(\text{unocc})} \frac{(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \hat{n} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})}{[E_i^{(1)} + E_i^{(2)} - (E_f^{(1)} + E_f^{(2)})]^2} \right\}_{\vec{p}_f^{(1)} = \vec{p}} \quad (p > p_m^{(0)}). \end{aligned} \quad (43)$$

In these formulae use has been made of the notation (12); the factors $\frac{1}{4}$ account for the fact that the indicated summations over momenta are to be carried out independently of each other, whereas each pair of momenta $\vec{p}_i^{(1)}, \vec{p}_i^{(2)}$ or $\vec{p}_f^{(1)}, \vec{p}_f^{(2)}$ ought to be counted only once when performing the summation over the configurations μ in (42). The traces occurring in the numerators $(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \hat{n} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})$ given by (12) arise from the averaging over the spin and charge substates; as above, we shall neglect the exchange term, with the coefficient $\text{tr } O^2 P_{\sigma} P_{\tau}$. Performing again the variable substitution (13) or (14), with the resulting formula (15), we finally put the expressions (43) into the form

$$\begin{aligned} F(\vec{p}) &= \frac{1}{A} [1 - \beta \varphi_{\text{occ}}(u; r)] \quad (p < p_m^{(0)}) \\ F(\vec{p}) &= \frac{1}{A} \cdot \beta \varphi_{\text{unocc}}(u; r) \quad (p > p_m^{(0)}), \end{aligned} \quad (44)$$

with

$$\beta \equiv \frac{1}{4} \text{tr } O^2 \cdot \frac{b^2 r^2}{16 \pi^4} [F(0)]^2 \quad (45)$$

and

$$\varphi(u; r) = \int d\mathbf{v}_{u'} \int d\mathbf{v}_{u''} \frac{[F(r\mathbf{u}')/F(0)]^2}{[\vec{u}'(\vec{u}' + \vec{u}'' - \vec{u})]^2}. \quad (46)$$

the domains of integration being defined by the inequalities

$$u'' \leq 1 \quad ; \quad |\vec{u}' + \vec{u}''| \geq 1$$

and

$$\begin{cases} |\vec{u} - \vec{u}'| \geq 1 & \text{for } \varphi_{\text{occ}} \\ |\vec{u} + \vec{u}'| \leq 1 & \text{for } \varphi_{\text{unocc}}. \end{cases} \quad (47)$$

The evaluation of these intricate integrals, carried out by Watanabe in the case of a Gauss potential, yields the expected results: he finds that, for any fixed value of r , φ_{occ} increases when u increases from 0 to 1, while for u increasing further from 1 upwards, φ_{unocc} decreases rather steeply, tending to zero roughly as $e^{-r^2 u^2/2}$ for $u \rightarrow \infty$. For $u = 1$, there is in general a discontinuity of $F(\vec{p})$; e.g. for the set of parameters (37), one has

$$\varphi_{\text{occ}}(0) = 0,19 \quad , \quad \varphi_{\text{occ}}(1) = 4,51 \quad , \quad \varphi_{\text{unocc}}(1) = 3,90, \quad (48)$$

while according to (45), (19) and (25) $\beta = 0,116$: the corresponding distribution function $F(\vec{p})$ is represented in fig. 12.22.

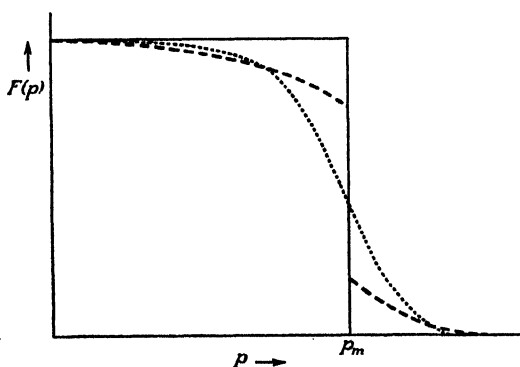


Fig. 12.22. Momentum distribution in standard heavy nucleus;
 ——— zero-point Fermi distribution,
 - - - - - distribution due to nuclear interaction,
 Fermi distribution at „equivalent” temperature.

By allowing us to calculate a more accurate value of the kinetic energy, the determination of $F(\vec{p})$ affords an alternative method of estimating the second order approximation to the nuclear energy. In fact, we get

$$\begin{aligned} K/A &= \int_0^\infty \frac{p^2}{2M} F(\vec{p}) \frac{4V}{(2\pi\hbar)^3} dv_p \\ &= K \left\{ 1 + 5\beta \left[\int_1^\infty u^4 du \varphi_{\text{unocc}}(u; r_0) - \int_0^1 u^4 du \varphi_{\text{occ}}(u; r_0) \right] \right\}; \end{aligned} \quad (49)$$

the correction term is just $K^{(2)}/A$, i.e., by (3), $-\mathcal{E}^{(2)}/A$. For the parameter values (37), Watanabe gives

$$K^{(2)}/K^{(0)} = 2,95\beta = 0,34, \quad (50)$$

but there is a serious discrepancy between this figure and that ($\approx 0,21$)

resulting from (27) for the same choice of parameters. As, however, all numerical data of this section can have only an illustrative significance, we shall not try to remove this discrepancy.

The modification of the momentum distribution brought about by the nuclear interactions is qualitatively of the same kind as that due to thermal agitation. It is interesting to compare the distribution (44) with that pertaining to an *ideal* Fermi gas (without nuclear interactions), the temperature of which is chosen so as to correspond to the same average kinetic energy (49). On account of the formulae (9.42–8, 10) and (50) with (9.41–4), this condition fixes the “equivalent” temperature T_0 by

$$\frac{5\pi^2}{12} \left(\frac{T_0}{E_m} \right)^2 = 2,95\beta = 0,34, \quad (51)$$

whence

$$T_0 = 0,29 E_m \approx 7 \text{ MeV} \quad (52)$$

by (9.41–5). It appears from fig. 12.22 that the graphs of the two momentum distributions offer indeed a striking similarity.

We can finally estimate in how far the convergence criterion (41) is fulfilled. We have

$$\begin{aligned} \sum_{\mu} |a_{\mu}|^2 &= \frac{1}{2} \sum_{\vec{p}_i^{(1)} \vec{p}_i^{(2)}}^{(\text{occ})} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(\text{unocc})} \frac{(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \hat{H} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})}{[E_i^{(1)} + E_i^{(2)} - (E_f^{(1)} + E_f^{(2)})]^2} \\ &= \frac{3}{2} A \beta \int_0^1 u^2 du \varphi_{\text{occ}}(u; r_0) = \frac{3}{2} A \beta \int_1^{\infty} u^2 du \varphi_{\text{unocc}}(u; r_0), \end{aligned} \quad (53)$$

which yields, according to Watanabe,

$$\sum_{\mu} |a_{\mu}|^2 = 1,18 A \beta = 0,137 A \quad (54)$$

for the parameter values (37). The convergence is thus quite bad, even for small integral values of A , and gets worse for heavier nuclei: the individual models become in fact more and more inadequate as the number of nucleons increases, and the saturation properties of the nuclear interactions come more and more to the foreground. Still, as observed by Watanabe, the relation (54) shows that results derived by the perturbation method on the Fermi gas model should *formally* be valid for $A \ll 1$, and this remark may possibly lead to less pessimistic conclusions with respect to certain quantities depending in some simple way on the mass number.

Thus, it follows from the fact that the probability $F(\vec{p}) \cdot 4V(2\pi\hbar)^{-3} \cdot d\nu_{\vec{p}}$, according to its definition (40), is independent of A , that the same property holds for the expectation value of any quantity (e.g. the kinetic energy) additive with respect to the different nucleons: we would therefore expect the perturbation method to yield a fair approximation for such expectation values.

In particular, this conclusion should apply to the average binding energy per nucleon, which, as we have seen, is also independent of A , in conformity with the saturation character of the nuclear forces. If in spite of that no satisfactory quantitative agreement with the empirical data can be reached, it is because the model does not allow sufficiently for the markedly local character of the interaction of any particular nucleon with the others. It must be emphasized, however, that even Euler's approximation means a considerable step towards the due recognition of the importance of local fluctuations in the distribution of the nucleons, arising from their mutual interaction: this will appear from a closer discussion of the spatial correlations between the individual nucleons, which are found in this approximation.

12.23. The spatial correlations. In order to study the spatial distribution of the nucleons, we have to turn back to the correlation operator introduced in formula (8) or (9): in fact, the diagonal matrix-elements of this operator (which is obviously spherically symmetrical) give the probability for two nucleons of specified charge and spin to be at a distance $r = |\vec{x}| = |\vec{x}^{(1)} - \vec{x}^{(2)}|$ apart. We shall distinguish two cases, according as the nucleons considered have or have not the same charge and spin orientation, i.e. do or do not belong to the same isotopic and spin substates. Two nucleons with the same charge and spin we will call *congruent**; if they differ either in charge or spin or both, they will be called *non-congruent*. The *spatial correlation* of a pair of congruent or non-congruent nucleons will then be measured by the total probability

$$g(r) = \sum_i (i | g | i). \quad (55)$$

the sum extending over the spin and isotopic states which correspond to congruence or non-congruence of the nucleon pair: congruent nucleons can be in any of the four states ${}^3(\tau)_{\pm 1} {}^3(\sigma)_{\pm 1}$, non-congruent nucleons in any of the other 12 possible states. Quantities pertaining to a pair of congruent or non-congruent nucleons will be distinguished by the respective suffixes $=, \neq$.

In terms of the ordinary and mixed densities defined by (11.11–12, 13, 23), the initial approximation to the correlation operator, given by the first formula (10), may be written

$$(i | g | f)^{(0)} = \frac{1}{A(A-1)} \cdot \frac{1}{16} [(i | 1 | f) \varrho(\vec{x}^{(1)}) \varrho(\vec{x}^{(2)}) - (i | P_\sigma P_\tau | f) |\varrho(\vec{x}^{(1)}, \vec{x}^{(2)})|^2]. \quad (56)$$

Now, one readily finds, with the eigenfunctions (1),

$$\begin{aligned} \varrho(\vec{x}) &= \frac{A}{V} \\ \varrho(\vec{x}^{(1)}, \vec{x}^{(2)}) &= \frac{A}{V} G(r). \end{aligned} \quad (57)$$

Like nucleons are nucleons of like charge, irrespective of the spin orientation (0.2).

with

$$G(r) = \frac{3}{(r \times r)^3} (\sin r \times r - r \times r \cos r \times r),$$

and therefore

$$(i|g|f)^{(0)} = \frac{1}{16 V^2} [(i|1|f) - (i|P_\sigma P_\tau|f) G^2(r)]. \quad (58)$$

The spatial correlations are, in this approximation,

$$\begin{aligned} g_{=}(r) &= \frac{1}{4 V^2} [1 - G^2(r)] \\ g_{\neq}(r) &= \frac{3}{4 V^2}. \end{aligned} \quad (59)$$

We see that, even in the absence of any nuclear interaction, there is some correlation between congruent nucleons: while non-congruent nucleons are uniformly distributed, congruent ones tend to repel each other (fig. 12.23): it is this well-known effect of the exclusion principle which, as shown by (58) in connexion with (9), gives rise, in the next approximation, to the exchange energy and thereby to the saturation property of the nuclear forces.

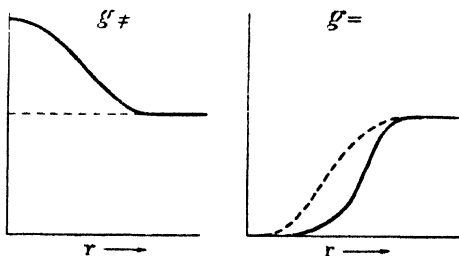


Fig. 12.23. Spatial correlations between congruent and non-congruent nucleons. The dotted lines represent the initial approximations $g^{(0)}$, in which no account is taken of nuclear interactions.

The next approximation *

$$(i|g|f)^{(1)} = -\frac{1}{16 V^2} \cdot \frac{b r}{4 \pi^2} [(i|\mathcal{O}|f) G_{\text{ord}}^{(1)}(r) - (i|\mathcal{O} P_\sigma P_\tau|f) G_{\text{exch}}^{(1)}(r)]$$

$$G_{\text{ord}}^{(1)}(r) = \frac{1}{2} \left(\frac{3}{2\pi} \right)^2 \int dv_u dv_{u'} dv_{u''} \frac{\cos r \times \vec{u}' \times \vec{x}}{\vec{u}' (\vec{u}' + \vec{u}'' - \vec{u})} F(r u') \quad (60)$$

$$G_{\text{exch}}^{(1)}(r) = \frac{1}{2} \left(\frac{3}{2\pi} \right)^2 \int dv_u dv_{u'} dv_{u''} \frac{\cos r \times \vec{u}' \times \vec{x}}{\vec{u}' (\vec{u}' + \vec{u}'' - \vec{u})} F(r |\vec{u}' + \vec{u}'' - \vec{u}|)$$

(domain of integration defined by

$$u \leq 1, u'' \leq 1, |\vec{u} - \vec{u}'| \geq 1, |\vec{u}' + \vec{u}''| \geq 1),$$

shows that the nuclear interaction, besides modifying the spatial correlation

* This approximation has been investigated by VOLZ, but only a short account of his results, including fig. 12.23, has been published (V. WEIZSÄCKER [38], p. 216).

between congruent nucleons, also brings about a spatial correlation between non-congruent nucleons; in fact, using the expression (8.2-7) for $\mathcal{O}$, we find for the relevant coefficients

$$\begin{aligned}\Sigma_{=}(i|\mathcal{O}|i) &= \Sigma_{=}(i|\mathcal{O}P_{\sigma}P_{\tau}|i) = -4 \cdot {}^3p = -\frac{8}{3} \\ \Sigma_{\neq}(i|\mathcal{O}|i) &= -3({}^3s + {}^1s) - ({}^3p + {}^1p) = -\frac{8}{3} \\ \Sigma_{\neq}(i|\mathcal{O}P_{\sigma}P_{\tau}|i) &= +3({}^3s + {}^1s) - ({}^3p + {}^1p) = 6(1+q) - \frac{8}{3},\end{aligned}\quad (61)$$

the numerical values following from (11.33-12, 13). Accordingly,

$$\begin{aligned}g_{=}(r) &= -\frac{1}{4V^2} \cdot \frac{br_0}{4\pi^2} \cdot \frac{1}{3} (G_{\text{ord}}^{(1)} - G_{\text{exch}}^{(1)}) \\ g_{\neq}(r) &= +\frac{1}{4V^2} \cdot \frac{br_0}{4\pi^2} \cdot \left[\frac{2}{3} (G_{\text{ord}}^{(1)} - G_{\text{exch}}^{(1)}) + \frac{3}{2} (1+q) G_{\text{exch}}^{(1)} \right].\end{aligned}\quad (62)$$

Clearly, $G_{\text{ord}}^{(1)} > G_{\text{exch}}^{(1)} > 0$; since, moreover, $b > 0$ and $q > 0$, we see that the effect of nuclear interaction is to widen the "hole" in the spatial distribution of congruent nucleons around a given one, while non-congruent nucleons tend to accumulate in the vicinity of this nucleon (fig. 12.23). In other words, the density fluctuations resulting from the nuclear interactions favour the formation of *clusters*, the simplest (and therefore most frequent) of which consist of four nucleons in the four different spin and isotopic states; such clusters, which have a structure akin to that of α -particles, we shall call α -clusters.

The second order contribution $\mathcal{G}^{(2)}$ to the nuclear energy is intimately connected with the spatial correlations between the constituent nucleons. By inserting (60) into (9) and expressing the $G^{(1)}$'s in terms of the $g^{(1)}$'s with the help of (62), we can indeed put $\mathcal{G}^{(2)}$ into the form

$$\mathcal{G}^{(2)} = \frac{1}{2} \cdot \frac{A(A-1)}{2} \int J(r) dv^{(1)} dv^{(2)} [a_{=} g_{=}(r) + a_{\neq} g_{\neq}(r)]; \quad (63)$$

the coefficients $a_{=}$, $a_{\neq}$ are easily calculated, taking account of (21):

$$\begin{aligned}a_{=} &= \frac{1 + 3q + 7q^2 + 9q^3}{1 + q} \\ a_{\neq} &= -\frac{1 + q^2}{1 + q}.\end{aligned}\quad (64)$$

The negative correlation function $g_{=}(r)$ has a positive coefficient, and vice versa. The fact that $\mathcal{G}^{(2)}$ is thus always negative receives a simple interpretation in that the formation of clusters has obviously the effect of increasing the binding energy (in absolute value). As a matter of fact, one may say that the reason why the above estimate of $\mathcal{G}^{(2)}$ leads to such a poor quantitative result is that the individual nuclear model used as a starting point, although implying a tendency to α -clustering, does not

allow one (so to speak) to take advantage of this feature in calculating the binding energy: the strong binding of the particles forming an α -cluster is not properly taken into account. In fact, it is just this clustering which allows the full saturation of the nuclear bonds to come into play and thereby accounts for by far the greatest part of the binding energy. In the nucleus ^{16}O , for instance, from a total mass-defect of 127.16 MeV an amount of 112.80 MeV could be attributed to four α -clusters with the same binding as normal α -particles. The question therefore naturally arises whether a model in which the existence of α -clusters would be recognized from the outset would not yield better results. This leads us to an examination of the α -particle model and other models of the collective type (9.22), which will be dealt with in the next Chapter.

On account of its connexion with the density fluctuations or spatial correlations of the nuclear system, the second order energy $\mathcal{E}^{(2)}$ has been termed by Euler *fluctuation energy* and by Wigner *correlation energy*.

The correlation energy has especially been discussed by WIGNER [34a, b] in connexion with the problem, formally very similar, of the metallic state. In this case, one has simply (for N electrons)

$$\mathcal{E} = \mathcal{K} + \frac{N(N-1)}{2} \int \frac{e^2}{r} dv^{(1)} dv^{(2)} \{ [g_{\equiv}^{(0)}(r) + g_{\neq}^{(0)}(r)] + \frac{1}{2} [g_{\equiv}^{(1)}(r) + g_{\neq}^{(1)}(r)] \}$$

and the correlation turns out to be negative for electrons with opposite spins as well as for those of like spins, i.e. any two electrons tend to keep as far apart from each other as possible. As a result, the electrostatic repulsion between them is decreased, and the correlation energy accordingly means a tightening of the binding. The effect of correlation is also of some importance, as pointed out by GOMBÁS [43], in calculations of the energy of the stationary states of heavy atoms and ions by means of the Thomas-Fermi method: the correlation energy here appears as a generalization of the polarization effect met with in the treatment of the Helium atom (BETHE [33], p. 339).

12.24. The virial theorem. As in atomic theory, the virial theorem yields an exact relation between the mean kinetic energy $\mathcal{K}$ and the potential energy of a nuclear system; its usefulness in this field has first been emphasized by HILL [37]. For a potential of the type

$$\mathcal{V}_{\text{nuc}} = \frac{1}{2} \sum_{i,k} \mathcal{V}^{(ik)} (\kappa r^{(ik)}),$$

the virial theorem can be expressed in the form (WATANABE [39])

$$\mathcal{K} = \frac{1}{2} \text{av} \sum_i \vec{x}^{(i)} \text{grad}^{(i)} \left(\frac{1}{2} \sum_{i,k} \mathcal{V}^{(ik)} \right) = \frac{1}{2} \text{av} \kappa \frac{d}{d\kappa} \mathcal{V}_{\text{nuc}}. \quad (65)$$

Now, we may certainly assume that the distance dependence of the attractive nuclear potential will be such as to imply, quite generally,

$$\text{av} \frac{d}{d\kappa} (\kappa |\mathcal{V}_{\text{nuc}}|) < 0; \quad (66)$$

therefore

$$\mathcal{K} > \frac{1}{2} |\overline{\mathcal{V}}|. \quad (67)$$

This inequality seems indeed to hold generally for actual nuclei. Since the mean kinetic energy per nucleon K , in the initial approximation of the Fermi gas model, is about equal to the empirical value of the total energy ϵ_1 per nucleon, the ratio $K/|\bar{V}|$ should be about $\frac{1}{2}$ in this approximation in order to yield the correct binding energy; and any improvement on this approximation will tend to increase it (9.41).

12.3. Penetration of fast nucleons into heavy nuclei

12.30. With the view of interpreting the disintegration "stars" observed in photographic emulsions exposed to cosmic radiation (7.14), HEISENBERG [37] has performed a detailed analysis of the collision between a heavy (Ag or Br) nucleus and a very fast nucleon (with an energy of a few 100 MeV, say). In passing through the nucleus, the impinging particle will communicate part of its energy to several constituent nucleons, which may be knocked out of the nucleus and thus produce in the photographic emulsion a "star" of tracks diverging from the spot occupied by the nucleus. Owing to the strong coupling between the nuclear particles, only those nucleons will actually be able to leave the nucleus which have acquired a sufficient energy in a collision taking place near the surface. The energy distribution of these nucleons, which can be derived empirically from a detailed study of the tracks (WAMBACHER [40]), will primarily depend on the cross-sections for the collisions between the impinging fast nucleon and any one of the constituent nucleons, the states of motion of which may be assumed to form a Fermi distribution at the absolute zero of temperature. Our first task will therefore be to calculate this cross-section for a given transfer of momentum.

12.31. Differential collision cross-section. In computing the differential cross-section, we shall use the non-relativistic expression (8.2-1) for the nuclear potential; this is permissible as long as the momentum transfer is small compared with M , — a condition not very well fulfilled for the largest momentum transfers possible with an impinging nucleon of a few 100 MeV energy. For such large momentum transfers it would be desirable to carry out a more refined calculation with the help of the meson field theory of the nuclear interaction; however, we shall presently restrict ourselves to cases in which large momentum transfers occur with negligible probability. We shall further assume that we have to deal with a standard heavy nucleus, and we shall neglect any electrostatic interaction with the impinging nucleon. Since this nucleon is a very fast one, Born's approximation may be used, and the differential cross-section takes the form

$$dS = \frac{c}{v_i^{(1)}} \cdot V \cdot \frac{2\pi}{\hbar} |(i|\mathcal{V}_{\text{nuc}}|f)|^2 dn_f; \quad (1)$$

V is the volume of the nucleus, $v_i^{(1)}$ the initial velocity of the impinging particle, dn_f the number of stationary states per unit energy of the system

after the collision, and $(i | \mathcal{V}_{\text{nucl}} | f)$ the matrix-element of the nuclear potential, which, according to (12.11-4), can be written

$$(i | \mathcal{V}_{\text{nucl}} | f) = J w_p(p) (i | \odot | f) - J w_p(p') (i | \odot P_\sigma P_\tau | f) \quad (2)$$

with
$$\vec{p} = \vec{p}_i^{(1)} - \vec{p}_f^{(1)} \quad , \quad \vec{p}' = \vec{p}_i^{(1)} - \vec{p}_f^{(2)} . \quad (3)$$

the momenta further satisfying the conservation law

$$\vec{p}_i^{(1)} + \vec{p}_i^{(2)} = \vec{p}_f^{(1)} + \vec{p}_f^{(2)} . \quad (4)$$

We are at liberty to take either of the quantities $\vec{p}$ or $\vec{p}'$ as representing the transfer of momentum involved in the particular collision we are considering; the initial momenta being given, either $\vec{p}$ or $\vec{p}'$ (together with the spin and charge variables) completely specifies the final state. In fact, by (3) and (4), we have, e.g.,

$$\vec{p}_f^{(1)} = \vec{p}_i^{(1)} - \vec{p} \quad , \quad \vec{p}_f^{(2)} = \vec{p}_i^{(2)} + \vec{p} . \quad (5)$$

Let us introduce polar coordinates in momentum space, choosing $\vec{p}_i^{(1)}$ as polar axis and measuring the azimuths from the plane $(\vec{p}_i^{(1)}, \vec{p}_i^{(2)})$. The angular coordinates of the vectors $\vec{p}_i^{(2)}, \vec{p}$ may be denoted by $(\Theta, 0)$ and (ϑ, φ) , respectively; we put

$$\zeta = \cos \vartheta, \quad z = \cos \Theta \quad (6)$$

and get from (5)

$$\begin{aligned} p_f^{(1)2} &= p_i^{(1)2} + p^2 - 2p p_i^{(1)} \zeta \\ p_f^{(2)2} &= p_i^{(2)2} + p^2 + 2p p_i^{(2)} [z \zeta + \sqrt{1-z^2} \sqrt{1-\zeta^2} \cos \varphi] . \end{aligned} \quad (7)$$

On account of (7), the equation of energy conservation *

$$E_i^{(1)} + E_i^{(2)} = E_f^{(1)} + E_f^{(2)} \quad (8)$$

establishes a relation between p, φ, ζ and the initial momenta; consequently, p' can in principle be expressed in terms of the initial momenta and of two of the variables p, φ, ζ . In particular, for $\vec{p}_i^{(2)} = 0$, the right-hand side of (8) depends only on p and ζ , so that, for a given momentum transfer p , the number dn_f of final states per unit energy is simply

$$\begin{aligned} dn_f &= \frac{V}{(2\pi\hbar)^3} p^2 dp d\varphi \left| \frac{d\zeta}{d(E_f^{(1)} + E_f^{(2)})} \right| \\ &= \frac{V}{(2\pi\hbar)^3} p^2 dp d\varphi \cdot \frac{E_f^{(1)}}{p_i^{(1)}} \\ &= \frac{V}{(2\pi\hbar)^3} p dp d\varphi \cdot \frac{c}{v_i^{(1)}} \cdot \left(1 - \frac{\sqrt{M^2 + p^2} - M}{E_i^{(1)}} \right) , \end{aligned} \quad (9)$$

* It will be convenient here to use relativistic energies, denoted by the letter E .

on account of (8) and (7); for small momentum transfers, $p \ll p_i^{(1)}$, the factor between brackets reduces to unity. The value of ζ in terms of p and $p_i^{(1)}$, derived from (8), shows that in the case of small p ,

$$\zeta \ll 1 \quad (\text{for } p \ll p_i^{(1)}); \quad (10)$$

this means that in most collisions (for small momentum transfers are the most frequent), while the deviation of the incident fast particle is only very small, a struck nucleon initially at rest recoils in a direction almost at right angles to the path of the fast particle. Since in our case all $p_i^{(2)} \ll p_i^{(1)}$,

this conclusion (10) remains valid when the restriction $\vec{p}_i^{(2)} = 0$ is dropped; likewise, the expression (9) for dn_f remains approximately true.

The absolute square of the matrix-element (2) contains the absolute squares of its two terms, with coefficients $|(i|\mathcal{O}|f)|^2$ and $|(i|\mathcal{O}P_\sigma P_\tau|f)|^2$, and two cross-terms, with coefficients $(i|\mathcal{O}|f)(f|\mathcal{O}P_\sigma P_\tau|i)$ and $(i|\mathcal{O}P_\sigma P_\tau|f)(f|\mathcal{O}|i)$. If we are only interested in the average cross-section for any spin or charge state of the interacting nucleons, we have to sum over the different final spin and charge states and to average over the initial states. This amounts to replacing the coefficients of the absolute squares just mentioned by the trace $\frac{1}{16} \text{tr } \mathcal{O}^2$ and those of the cross-terms by $\frac{1}{16} \text{tr } \mathcal{O}^2 P_\sigma P_\tau$. A considerable simplification is obtained by adopting Watanabe's device (12.21–23) of putting the latter quantity equal to zero, and thus getting rid of the cross-terms. The value of $\frac{1}{16} \text{tr } \mathcal{O}^2$ is then given by (12.21–25). Using the notation (12.11–3), we may therefore write the average differential cross-section in the general form

$$d\bar{S} = 0,52 \mathcal{D}^2 \left\{ \left[\frac{F(p/\hbar\kappa)}{\hbar\kappa} \right]^2 + \left[\frac{F(p'/\hbar\kappa)}{\hbar\kappa} \right]^2 \right\} \cdot \left(1 - \frac{M^2 + p^2 - M}{E_i^{(1)}} \right) p dp d\varphi, \quad (11)$$

with

$$\mathcal{D} = \frac{c}{v_i^{(1)}} \cdot \frac{J}{\hbar\kappa} \cdot \frac{1}{\kappa}. \quad (12)$$

For small momentum transfers, i.e. $p \ll p_i^{(1)}$, the contribution from the second term, involving p' , will be much smaller than that from the first term, since p' will be larger than p and the Fourier amplitude $F(p/\hbar\kappa)$ is a decreasing function of its argument. Formula (11) therefore takes in this case the simpler form

$$d\bar{S} \approx 0,52 \mathcal{D}^2 \left[\frac{F(p/\hbar\kappa)}{\hbar\kappa} \right]^2 p dp d\varphi \quad (p \ll p_i^{(1)}). \quad (13)$$

For a collision of small momentum transfer involving a definite spin and charge transition, the differential cross-section is obtained simply by replacing in (13) the average factor 0,52 by $|(i|\mathcal{O}|f)|^2$. The collision process can then be analyzed somewhat more closely. If account is taken of the saturation conditions (11.33–7), the factor $|(i|\mathcal{O}|f)|^2$ can be split into a function of the spins only and a further factor $|(i|\tau^{(1)}\tau^{(2)}|f)|^2$

depending only on the charges of the colliding nucleons. If we average this last factor over the initial isotopic states of the struck nucleon, we find $\frac{1}{2} \sum |(i|\tau_i^{(1)}|f)|^2$ independently of the final isotopic state of the nucleon.

This means that after the collision, the struck nucleon is as likely to be in a proton as in a neutron state. On the other hand, the relative probabilities of the impinging nucleon remaining or not in the same isotopic state after the collision are as 1 to 2.

12.32. Total collision cross-section. Let us now calculate the total cross-section $\bar{S}(E)$ for all collisions in which the final kinetic energy of the *slower* nucleon is $\geq E$. The condition $p_f^{(2)} \geq P$, P being the momentum corresponding to the energy E , fixes the lower limit p_0 of integration over p ; if P is assumed to be small compared with $p_i^{(1)}$, we may use (10) and get from (7)

$$p_0^2 = P^2 - p_i^{(2)2} - 2p_0 p_i^{(2)} \sqrt{1-z^2} \cos \varphi \quad (14)$$

i.e.

$$p_0 \approx -p_i^{(2)} \sqrt{1-z^2} \cos \varphi + \sqrt{P^2 - p_i^{(2)2} (\sin^2 \varphi + z^2 \cos^2 \varphi)}. \quad (15)$$

The upper limit of integration over p is the absolute value of the momentum transfer $\vec{p}_1$ corresponding to an equal distribution of momentum in the final state, $\vec{p}_f^{(1)} = \vec{p}_f^{(2)}$. We have thus to integrate $d\bar{S}$ over p from p_0 to p_1 , and average the result over the Fermi distribution of the initial momentum $p_i^{(2)}$.

The calculation can most easily be performed upon the assumption that large momentum transfers play only a negligible role; for this means that one may use the reduced form (13) for $d\bar{S}$ and integrate over p up to infinity instead of p_1 , which affords considerable simplification. On the other hand, this assumption is only fulfilled for types of nuclear potentials involving a sufficiently fast decrease with increasing distance, i.e. a sufficiently fast decrease of the Fourier coefficients $F(p/\hbar\kappa)$ with increasing p . We shall presently see that this condition means a considerable restriction in the application of the theory, and it would certainly be desirable to extend the calculations beyond their present scope. Starting from formula (13), we write

$$\int_{p_0}^{\infty} d\bar{S} \equiv 0.52 D^2 \mathcal{G}(p_0/\hbar\kappa) d\varphi, \quad (16)$$

$$\mathcal{G}(x) \equiv \int_x^{\infty} F^2(u) u du;$$

accordingly,

$$\bar{S}(E) = 0.52 D^2 \cdot \frac{1}{2} \int_{-1}^{+1} dz \cdot \frac{3}{p_m^{(0)3}} \int_{p_i^{(2)}}^{p_m^{(0)}} p_i^{(2)2} dp_i^{(2)} \int_0^{2\pi} \mathcal{G}(p_0/\hbar\kappa) d\varphi. \quad (17)$$

p_0 being given by (15), and $p_m^{(0)}$ denoting the maximum momentum of the

zero-point distribution, as given by (12.13-10, 8); the corresponding maximum kinetic energy will be called $E_m^{(0)}$. Expressing momenta in terms of $p_m^{(0)}$, we put

$$y \equiv \frac{p_i^{(2)}}{p_m^{(0)}} \quad \text{and} \quad \eta \equiv \frac{P}{p_m^{(0)}} = \sqrt{\frac{E}{E_m^{(0)}}}. \quad (18)$$

Formula (17) then takes the form

$$\bar{S}(E) = 0.52 \cdot \frac{3}{2} \mathcal{D}^2 \int_{-1}^{+1} dz \int_0^1 y^2 dy \int_0^{2\pi} d\varphi \mathcal{G}(r u_0), \quad (19)$$

with $u_0 \equiv p_0/p_m^{(0)}$ given, according to (15) and (18), by

$$u_0 = -y \sqrt{1-z^2} \cos \varphi + \sqrt{\eta^2 - y^2 (\sin^2 \varphi + z^2 \cos^2 \varphi)}. \quad (20)$$

This last formula, as well as (19), can be put into a much simpler form by a change of variables indicated by BAGGE [39]: it consists in going over from the polar coordinates (y, Θ, φ) to cylindrical coordinates (λ, ϱ, ψ) , the axis of which is the perpendicular to the axis of the polar coordinates in the plane of origin of the azimuths φ : λ is the distance from the origin measured along this axis, ϱ the normal distance from the axis, and ψ the azimuth around it. Therefore,

$$\lambda = y \sin \Theta \cos \varphi, \quad \varrho = \sqrt{y^2 - \lambda^2}, \quad \cos \psi = \frac{y}{\varrho} \cos \Theta \quad (21)$$

and

$$u_0 = -\lambda + \sqrt{\eta^2 - \varrho^2}; \quad (22)$$

further, the integrations in (19) must be replaced by

$$\int_0^{2\pi} d\psi \int_0^1 \varrho d\varrho \int_{-\sqrt{1-\varrho^2}}^{+\sqrt{1-\varrho^2}} d\lambda \dots$$

Thus, with the new substitutions

$$x = r u_0 = r(-\lambda + \sqrt{\eta^2 - \varrho^2}) \\ u = r \sqrt{1 - \varrho^2},$$

(19) is reduced to the form

$$\bar{S}(E) = 0.52 \cdot \frac{3}{2} \mathcal{D}^2 f(r, \eta), \quad (23)$$

with

$$f(r, \eta) = 2\pi r^{-3} \int_0^r u du \frac{\sqrt{r^2(\eta^2-1) + u^2 + x}}{\sqrt{r^2(\eta^2-1) + u^2 - x}} \mathcal{G}(x) dx \\ = \frac{\pi}{4} r^{-3} \int_{r(\eta-1)}^{r(\eta+1)} [2r^2(\eta^2+1) - x^2 - \frac{r^4}{x^2}(\eta^2-1)^2] \mathcal{G}(x) dx. \quad (24)$$

the second expression for $f(r, \eta)$ following from the first by elementary transformations.

The cross-section $\bar{S}(E)$ directly gives the number $\bar{n}(E)$ of secondary particles with a kinetic energy $\geq E$ produced on the average within the nucleus by a single impinging nucleon:

$$\bar{n}(E) = \frac{A}{\pi R^2} \bar{S}(E). \quad (25)$$

However, only a fraction of these secondaries will be able to escape from the nucleus, the rest dissipating their energy in the form of "heat" by collisions with other constituent nucleons. We shall now try to get an idea of the efficiency of such collisions in slowing down a nucleon penetrating into nuclear matter.

12.33. Energy loss of a fast nucleon in nuclear matter. The mean energy loss of a fast nucleon of initial velocity $v_i^{(1)}$ per unit of path in nuclear matter is

$$\frac{dE_i^{(1)}}{ds} = -\frac{A}{V} \cdot \frac{1}{2} \int_{-1}^{+1} dz \int_0^{p_m^{(0)}} \frac{3 p_i^{(2)2} dp_i^{(2)}}{p_m^{(0)3}} \int (E_f^{(2)} - E_i^{(2)}) d\bar{S}, \quad (26)$$

$d\bar{S}$ being given by (11), and the integration over p extending from a lower limit defined (on account of the exclusion principle) by $p_f^{(2)} = p_m^{(0)}$ up to the upper limit p_i . With the same restriction as above, we may replace $d\bar{S}$ by its reduced expression (13) and the upper limit of momentum transfer by infinity; using the unrelativistic expressions for the energies $E_f^{(2)}$ and $E_i^{(2)}$, and taking account of (10), we can then put (26) into the form

$$\frac{dE_i^{(1)}}{ds} = -\frac{A}{V} E_m^{(0)} \cdot \frac{3}{2} \cdot 0.52 \mathcal{D}^2 \int_{-1}^{+1} dz \int_0^1 y^2 dy \int_0^{2\pi} d\varphi \cdot \int_{y^2+u^2+2uy\sqrt{1-z^2}\cos\varphi}^{\infty} F^2(ru) r^2 u du [u^2 + 2uy\sqrt{1-z^2}\cos\varphi]; \quad (27)$$

going over to Bagge's variables (21), we get

$$\frac{dE_i^{(1)}}{ds} = -\frac{A}{V} E_m^{(0)} \cdot \frac{3}{2} \cdot 0.52 \mathcal{D}^2 \cdot \frac{2\pi}{r^2} \int_0^1 \varrho d\varrho \int_{-\sqrt{1-\varrho^2}}^{+\sqrt{1-\varrho^2}} d\lambda \cdot \int_{r(-\lambda+\sqrt{1-\varrho^2})}^{\infty} F^2(u) u du (u^2 + 2r\lambda u). \quad (28)$$

After performing the variable substitution

$$x' = r(-\lambda + \sqrt{1-\varrho^2}), \quad u' = r\sqrt{1-\varrho^2}$$

and some obvious partial integrations, and using (12.13-10), this finally becomes

$$\frac{dE_i^{(1)}}{ds} = -\frac{A}{V} \cdot 0.52 \mathcal{D}^2 \cdot \frac{2\pi (\hbar \kappa)^2}{M} \cdot \mathcal{H} \quad (29)$$

with

$$\mathcal{H} \equiv \frac{1}{2} \int_0^{\infty} F^2(u) u^3 du. \quad (30)$$

It is remarkable that we would have obtained exactly the same result if we had assumed all struck nucleons to be initially at rest. In fact, we should then write

$$\begin{aligned} \frac{dE_i^{(1)}}{ds} &= -\frac{A}{V} \cdot \int_{E_f^{(2)}=M}^{E_f^{(2)}=\infty} (E_f^{(2)} - M) d\bar{S} \\ &= -\frac{A}{V} \cdot 0.52 \mathcal{D}^2 \cdot 2\pi \int_0^{\infty} \frac{p^2}{2M} \left[\frac{F(p/\hbar\kappa)}{\hbar\kappa} \right]^2 p dp, \end{aligned}$$

which is just (29), (30) *. This result is in conformity with a general property first enunciated by WILLIAMS [38]: *the mean energy loss per unit path of a fast nucleon within a nucleus is the same as though the constituent nucleons were at rest and independent of each other, irrespective of their actual interactions and states of motion.* The physical reason for this behaviour is primarily the short range of nuclear interaction. When considering the passage of a fast nucleon of velocity βc at a distance r from a constituent nucleon, one has to compare the "time of collision" $(r/\beta c) \sqrt{1-\beta^2}$ with the "natural period" $2r_0/\bar{v}$ of the bound nucleon; the latter time is that necessary for the nucleon of mean velocity $\bar{v}$ to cross the "cell" of dimensions $\approx 2r_0$ within which it is on the average confined by its binding to the other nucleons. If the time of collision is long compared to the natural period, we have "adiabatic" conditions and the course of the collision is essentially affected by the state of binding of the constituent nucleon; in the opposite case, when we have a "sudden impulse", the collision takes place as if the colliding particles were free. This last behaviour thus prevails as long as the distance of closest approach r is smaller than some critical distance of the order of magnitude $\frac{\beta}{\sqrt{1-\beta^2}} \cdot \frac{c}{\bar{v}} \cdot 2r_0$.

But if the impinging nucleon is very fast ($\beta \approx 1$), this critical distance is many times larger than the range of the interaction between the colliding particles. As the distance of approach increases, any interaction will therefore altogether cease long before adiabatic conditions could at all set in: inside the range of interaction, we have just the conditions of sudden impulse. This argument makes it clear that Williams' theorem should be of very wide scope indeed.

It will be observed that the integral $\mathcal{H}$ exists only if $F(u)$ decreases faster than u^{-2} for large u ; this limitation on the admissible types of nuclear potentials is a necessary condition for the validity of our simplified

* BAGGE [39] verifies the same property for the special case of a Gauss potential in an unnecessarily complicated manner.

treatment (ROSENFELD [45a]). In fact the breakdown of the theory caused by the divergence of $\mathcal{H}$ cannot, in view of Williams' theorem, be attributed to the use of the Fermi gas model; nor is there any cogent reason to doubt the legitimacy of Born's approximation in calculating the collision cross-section. The divergence of $\mathcal{H}$ is an indication that we have to take a more careful account of the contributions from collisions with large momentum transfers. The table 12.33, which contains the explicit expressions for the quantities $F(u)$, $G(x)$ and $\mathcal{H}$ for various types of nuclear potentials, shows that the restriction imposed upon the choice of the distance dependence of the potential by the convergence of $\mathcal{H}$ is a very severe one; in particular, it is unfortunate that the meson potential has to be excluded.

In the case of the well and meson potentials, the logarithmic divergence of $\mathcal{H}$ can be traced to the occurrence of an infinite field intensity at some point (at the boundary of the well or at the centre of the meson field of force), as a result of which the frequency of large momentum transfers is increased. To be sure, it would be possible to avoid this by a slight modification of the potential: in order to make $\mathcal{H}$ convergent, we have just to "round off" the shape of the well:

$$\begin{aligned} w(\xi) &= 1 - \frac{1}{2} e^{\alpha(\xi-1)} & \text{for } 0 \leq \xi \leq 1 \\ w(\xi) &= \frac{1}{2} e^{-\alpha(\xi-1)} & \text{for } 1 \leq \xi, \end{aligned} \quad (31)$$

or to "cut off" the singularity of the meson potential:

$$\begin{aligned} w(\xi) &= \text{const} & \text{for } \xi \leq \xi_0 \\ w(\xi) &= e^{-\xi/\xi_0} & \text{for } \xi > \xi_0. \end{aligned} \quad (32)$$

But, of course, the convergence of $\mathcal{H}$ alone is not sufficient to justify the application of our approximate treatment. Actually, the Gauss and exponential potentials are about the only important instances of nuclear potentials for which this treatment can be trusted.

12.33. The quantities $F(u)$, $G(x)$ and $\mathcal{H}$ for some types of nuclear potentials

Potential	$w(\xi)$	$F(u)$	$G(x)$	$\mathcal{H}$
Well	—	$\frac{2}{u^3} [\sin u - u \cos u]$	$\frac{1}{x^4} [\sin^2 x - x \sin 2x + x^2]$	∞
Gauss	$e^{-\xi^2}$	$\frac{\sqrt{\pi}}{2} e^{-u^2/4}$	$\frac{\pi}{4} e^{-x^2/2}$	$\frac{\pi}{4}$
Meson	$e^{-\xi/\xi_0}$	$\frac{2}{1+u^2}$	$\frac{2}{1+x^2}$	∞
Exponential	$e^{-2\xi}$	$\frac{1}{2} \cdot \frac{1}{(1+\frac{1}{2}u^2)^2}$	$\frac{1}{6} \cdot \frac{1}{(1+\frac{1}{2}x^2)^3}$	$\frac{1}{6}$

12.34. Range of fast nucleon in nuclear matter. The right-hand side of equation (29) contains the energy $E_l^{(1)}$ through the factor

$$\left(\frac{c}{v_l^{(1)}}\right)^2 = \frac{E_l^{(1)2}}{E_l^{(1)2} - M^2},$$

which occurs in $\mathcal{D}^2$ according to (12). The integration of (29) consequently yields for the range $R^{(1)}$ of the nucleon of initial energy $E_l^{(1)}$ in nuclear matter

$$R^{(1)} = \frac{(E_l^{(1)} - M)^2}{ME_l^{(1)}} \cdot \frac{1}{\frac{3}{2} \cdot 0.52 \cdot \frac{\hbar}{(\pi r_0)^2} \left(\frac{J}{M}\right)^2} \cdot r_0. \quad (33)$$

Working out this formula for a Gauss potential, with the numerical data (12.21–29) and $r_0 = \frac{1}{2}d$, we find

$$R^{(1)} = \frac{(E_l^{(1)} - M)^2}{ME_l^{(1)}} \cdot \frac{1}{2.64 \cdot 10^{-3}} \cdot r_0. \quad (34)$$

The exponential potential (table 6.432) leads to a very similar result, the factor 2.64 in the denominator being replaced by 2.97. In order to traverse a ^{80}Br nucleus a nucleon must, according to formula (34), have an initial kinetic energy of at least 155 MeV. On the other hand, for a kinetic energy of about 50 MeV, the range amounts to only 0.1 of the diameter of the Br nucleus.

12.35. Theory of disintegration stars. In view of the last estimates, we shall assume, following Heisenberg, that only the secondaries released in a superficial spherical shell of thickness $(4\pi/3)^{\frac{1}{2}} r_0 \approx 1.6 r_0$ with a momentum directed outward are able to escape from the nucleus, provided their total kinetic energy E is larger than the mean potential energy $|\bar{V}|$ of the nucleus. In the nuclear model here adopted, the latter quantity is $E_m^{(0)} + \varepsilon$, ε being the absolute value of the mean binding energy of a nucleon (2.21); the ejected nucleons of energy $\geq E$ are therefore observed outside the nucleus to have a kinetic energy $\geq E_0$, with

$$E_0 \approx E - (E_m^{(0)} + \varepsilon). \quad (35)$$

Their mean number per impinging nucleon is

$$\begin{aligned} \bar{n}_0(E_0) &= \bar{n}(E) \cdot \frac{1}{2} \cdot \frac{\text{volume of shell}}{\text{volume of nucleus}} \\ &= \bar{n}(E) \cdot \frac{3}{2A^{1/3}} \left[1.6 - \frac{(1.6)^2}{A^{1/3}} \right] \\ &= \frac{3}{2\pi r_0^2} \cdot \mathcal{S}(E) \left[1.6 - \frac{2.56}{A^{1/3}} \right]. \end{aligned} \quad (36)$$

owing to (25). Inserting (23) with the expression (12) for $\mathcal{D}$, we get, for the Gauss potential, and $r_0 = \frac{1}{2}d$,

$$\bar{n}_0(E_0) = 0.09 \left(\frac{c}{v_l^{(1)}} \right)^2 \left(1.6 - \frac{2.56}{A^{1/3}} \right) f_G(r_G, \eta) \quad (37)$$

$$r_G \approx 2.07.$$

The distribution function $f_G(r_G, \eta)$ can in this case be evaluated explicitly; it was first derived by BAGGE [39]. Putting $\beta_G = \frac{1}{2} r_G^2$, it may be written as

$$f_G(r_G, \eta) = g[\sqrt{\beta_G}(\eta-1)] - g[\sqrt{\beta_G}(\eta+1)]$$

with

$$g(x) \equiv \frac{\pi^{5/2}}{64 \beta_G^{3/2}} [4 \beta_G^2 (\eta^2 - 1)^2 + 4 \beta_G (\eta^2 + 1) - 1] [1 - \Phi(x)] \quad (38)$$

$$- \frac{\pi^2}{32 \beta_G^{3/2}} \left[1 + \frac{2 \beta_G^2}{x^2} (\eta^2 - 1)^2 \right] x e^{-x^2}$$

and

$$\Phi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

For $\eta \geq 1$, $\beta_G > 2$, it reduces approximately to $g[\sqrt{\beta_G}(\eta-1)]$, while for $\eta \gg 1$, one gets, with the help of the asymptotic expansion of $\Phi(x)$,

$$f_G(r_G, \eta) \simeq \frac{\pi^2 \eta}{8 \beta_G^2 (\eta-1)^3} e^{-\beta_G (\eta-1)^2}. \quad (39)$$

Take, e.g., a nucleon of initial kinetic energy ≈ 465 MeV (i. e. $E_l^{(1)}/M = 3/2$) colliding with a $^{80}_{35}\text{Br}$ nucleus; the minimum kinetic energy of the expelled secondaries is, according to (35), $E_m + \varepsilon$ with $\varepsilon \approx 8.6$ MeV, corresponding to $\eta_{\min} = \sqrt{1 + \frac{\varepsilon}{E_m}} = 1.16$ (9.41-5). Since, moreover,

$$\left(\frac{c}{v_l^{(1)}} \right)^2 = 1.8, \quad 1.6 - \frac{2.56}{A^{1/3}} = 1.01, \quad f_G(2.07; 1.16) \approx 0.88,$$

we get from (37) for the total average number of secondaries per collision

$$\bar{n}_0(0) \approx 0.14; \quad (40)$$

this represents $0.35 \cdot f_G(2.07; 1.16)/f_G(2.07; 1) = 0.35 \cdot 0.88/1.44 = 0.22$ of the total number of secondaries generated within the nucleus.

The experimental material has been carefully analyzed (BAGGE [39], ORTNER [40]) with a view to obtaining statistics of the numbers of particles of definite energy composing the disintegration stars. The order of magnitude (40) derived for the total number of particles seems rather too low to account for the results of observation. This discrepancy led BAGGE [39] to suggest a somewhat less direct mechanism of production

of the stars: the impinging cosmic ray particle would, by an "explosive" process, give rise to several nucleons of high energy, which would then release by intranuclear collisions the particles constituting the star. This hypothesis would in any case uphold the *proportionality* between the mean number of star particles of energy $\geq E_0$ and the function $f(r, \eta)$, which characterizes the underlying collision process and in particular embodies the dependence of its cross-section on the *range* of the nuclear potential. Such a proportionality

$$\bar{n}_0(E_0) \sim f(r, \eta) \quad (41)$$

is all that is required to derive from the observed numbers of particles $\bar{n}_0(E_0)$ a determination of the range.

12.36. Determination of range of nuclear force. The available statistics (ORTNER [40]) extend from $E_0 = 10$ MeV to $E_0 = 50$ MeV; taking again $\varepsilon = 8.6$ MeV and allowing for the variation of E_m^0 around E_m for varying r_0 , we are interested in values of η lying in the interval $1.1 \lesssim \eta \lesssim 1.85$, while we expect r_G to lie between 2 and 2.2. Now, when η and r_G vary within these intervals, it appears that the function $f_G(r_G, \eta)$ can to a fair approximation be represented by

$$f_G \sim 10^{-(0.233 + 0.15\beta_G)\eta^2}. \quad (42)$$

By (12.13-10), (18), (35), we have

$$\beta_G = \frac{1}{2} r_G^2 = \frac{M}{\hbar^2 \kappa^2} E_m^{(0)}, \quad \eta^2 = \frac{E_0}{E_m^{(0)}} + \text{const},$$

so that *

$$\log_{10} f_G = \text{const} - 0.233 \frac{E_0}{E_m} \left(\frac{2r_0}{d} \right)^2 - 0.15 \frac{1}{(\kappa d)^2} \cdot \frac{E_0}{\hbar^2/d^2 M},$$

or, expressing E_0 in MeV,

$$\log_{10} f_G = \text{const} - E_0 \left[0.0288 \frac{1}{(\kappa d)^2} + 0.0096 \left(\frac{2r_0}{d} \right)^2 \right]. \quad (43)$$

It will be observed that, although f_G strictly speaking involves not only κ but also r_0 , it is so insensitive to variations of the latter parameter as to provide a practically unique determination of the former; at any rate, any uncertainty as to the proper choice of the numerical value of r_0 is by far outweighed by the margin of error of the experimental data **.

The range determination for the Gauss potential with the help of (43) has already been discussed in 7.14.

* It seems as though the difference (35) between E_0 and E had disappeared from the expression for $\log_{10} f_G$; but actually, the numerical coefficients would be different for another range of variation of η .

** This point is not sufficiently stressed by Heisenberg nor by Bagge.

CHAPTER XIII

COLLECTIVE NUCLEAR MODELS

13.1. The α -particle model

13.10. The occurrence of α -clusters (12.23) in a nucleus is, of course, something quite different from a structure built up of α -particles keeping their individuality: the α -clusters lack the permanence of such α -particles; the nucleons which at some time have joined to form an α -cluster will soon separate and enter other transient clusters. Even so, one might presume that the life-time of an α -cluster would be sufficiently long to cause the nuclear structure to bring out at least some features of a model consisting of an assembly of permanent α -particles (and additional loose nucleons) bound together by suitably defined forces: compared with the individual models, such an idealisation would approximate the true nuclear constitution so to speak from an opposite direction. In the early work along these lines, the interaction between the α -particles was assumed to have the properties of the van der Waals attraction between the molecules of a liquid; in particular, most of the argument was based on the assumption that the α -particle forces were additive, i.e. that the total interaction of a system of α -particles was the sum of the interactions of adjacent pairs. A closer investigation, however, shows that this will not be the case unless the elementary law of interaction between two nucleons involves a sufficiently strong residual attraction of the "ordinary" (non-exchange) type and of longer range than the bulk of the force. If this rather artificial assumption is not made, it is found that the van der Waals force between two α -particles, in sharp contrast with that between molecules, has a *shorter* range than the first order "exchange" forces; and this fundamental difference makes the analogy of the assembly of α -particles with a liquid droplet quite untenable. The resulting situation would rather suggest a comparison of α -particle interactions with the chemical binding forces between atoms: just as the latter arise from exchange of electrons between atomic systems, the former are primarily conditioned by the exchange of nucleons between the α -clusters. The properties of a nucleus envisaged as a quasi-molecular aggregate of α -particles are, then, not so rigidly dependent on the conception of these particles as permanent units: its state of motion, e.g., will be describable in terms of rotations and vibrations, and for a legitimate use of such a description, it will suffice that the periods of these motions be short in comparison with the life-time of an α -cluster: this condition will allow us to determine the validity of the conclusions which can be derived from the model. In this section, we shall in turn discuss the different points of the preceding argument.

13.11. Empirical evidence on α -particle interaction. We shall begin by an examination of the empirical evidence bearing on the problem of the interaction between two α -particles. This material, which has been subjected to a critical analysis by WHEELER [41a, b], consists of data on the scattering of α -particles in Helium [41a] and on the ${}^8\text{Be}$ nucleus [41b]. On the conception under discussion, the latter nucleus would just consist of two α -clusters and thus be comparable to a symmetric diatomic molecule. However, its ground state is not a state of binding, but a ${}^1S$ virtual level of about 0.125 MeV energy. Further, the existence of an excited state of about 2.8 MeV above the ground state, first recognized by Dee and Gilbert, has been well established by subsequent experiments. Additional information on this and other excited states of the system of two α -clusters may be derived from the analysis of the scattering experiments.

The scattering of α -particles by Helium can be treated quite similarly to the proton-proton scattering discussed in 7.11, with the simplifying circumstance that the identical scattering partners in this case have no intrinsic angular momentum and obey Bose statistics (1.22). Accordingly, only the waves of even orbital momentum contribute to the scattering cross-section, and from formula (7.11-14), — in which a now means $4e^2c/\hbar v$ owing to the double charge of the α -particle — we readily find that the ratio of observed to purely classical (Rutherford) scattering cross-section can be written as $|\Re|^2$, $\Re$ being the complex quantity

$$\Re = \Re_c(\theta) + \Re_c\left(\frac{\pi}{2} - \theta\right) + \sum_{\text{even } l} \Re_l(\theta)(1 - e^{2i\delta(l)})$$

$$\Re_c(\theta) = \frac{\cos^2 \theta}{\sqrt{\sin^4 \theta + \cos^4 \theta}} e^{-i\alpha \log \sin^2 \theta} \quad (1)$$

$$\Re_l(\theta) = -\frac{2i}{a} \sqrt{\frac{1}{\sin^4 \theta} + \frac{1}{\cos^4 \theta}} \sqrt{4\pi(2l+1)} Y_l^0(\cos 2\theta) e^{2il \sum_{k=1}^l \arctg a/k} ;$$

the two first terms represent pure Coulomb scattering, the interference between them giving rise to the characteristic deviation from Rutherford's formula, first predicted by Mott; the last sum is the contribution from the anomalous scattering due to the proper nuclear interaction between the α -particles*.

In an Argand diagram, $\Re$ is represented by a vector, the length of which is known from observation in terms of scattering angle θ and energy E of the impinging particles. On the other hand, it appears, according to (1), as the resultant of a number of vectors, of which those representing Coulomb scattering are completely fixed for any given θ and E , while those representing anomalous scattering are only known in

* One has $\gamma^{(l)} - \gamma^{(0)} = \sum_{k=1}^l \arctg a/k$.

length, their directions depending on the phases $\delta^{(l)}$. Provided we can restrict ourselves to the consideration of the few first values of l (in fact, $l = 0, 2, 4$), it is then possible, by an elegant graphical procedure described by Wheeler, to determine for each energy value one or several sets of possible values of the corresponding phases $\delta^{(0)}$, $\delta^{(2)}$, $\delta^{(4)}$. The analysis of the available experiments, which for a number of scattering angles in the interval $5^\circ \dots 45^\circ$, extend over a range of energies $E \approx 1.5 \dots 8.5$ MeV, finally leads to two * acceptable sets of phase functions $\delta^{(0)}(E)$, $\delta^{(2)}(E)$, $\delta^{(4)}(E)$.

In order to make a choice between these two, account has to be taken of the excited state of the ^8Be -nucleus at about 2.8 MeV: this leads us indeed to expect a resonance phenomenon in the scattering of α -particles of about 5.6 MeV kinetic energy. According to general dispersion theory (for details, cf. WHEELER [41a]), the phase of the partial wave having the same orbital momentum as the state in question will increase by about π in the vicinity of the resonance energy. Only one of the two solutions of the scattering problem satisfies this requirement, and is thus definitively accepted: the phase exhibiting the resonance behaviour is $\delta^{(0)}$, indicating that the Dee-Gilbert level of ^8Be , like the ground state, must be a ^1S level; this conclusion, as shown by WHEELER [41b], is in agreement with other evidence concerning the yields of nuclear reactions which involve this excited state. Moreover, its half-width can be estimated, by a closer study of the variation of $\delta^{(0)}$ with energy, to be $\Gamma \approx 0.8$ MeV (corresponding to a mean life $\hbar/c\Gamma \approx 0.8 \cdot 10^{-21}$ sec), also in agreement with previous inferences from nuclear reactions.

The behaviour of $\delta^{(2)}(E)$ points to the existence of a much broader excited ^1D state of ^8Be with excitation energy of about 4 to 5 MeV. From the fact that the phase $\delta^{(4)}$ takes appreciable values only for $E > 6$ MeV, i.e. [by formula (5.14-28), applied to the Coulomb potential] for a distance of closest approach shorter than $9 \cdot 10^{-13}$ cm, one can finally infer that this last figure represents an upper limit to the range of interaction between two α -particles.

13.12. Theory of the interaction between two α -particles. Unfortunately, there is as yet no complete theoretical treatment of the interaction between two α -particles, and therefore no quite definite conclusion about the character of this interaction can be reached; in fact, on the basis of the fragmentary discussion hitherto performed by various authors, either of two sharply opposed standpoints can still be maintained. The procedure followed in all the published ** investigations of this problem is a simple

* A third set of phases which also gives a good fit with observation must be rejected because the phases $\delta^{(2)}$, $\delta^{(4)}$ decrease instead of increasing for small increasing values of the energy.

** A treatment based on the method of resonating group structure (13.3) has been announced by WHEELER [37a], but not published.

application of the Hartree-Fock variational method (9.21) to the system of eight nucleons constituting the two α -particles: the vector $\vec{R}$ joining their centres of gravity is, however, — just as in the analogous molecular problem, — regarded as a parameter. The interaction energy between the α -particles is obtained as the difference between the total energy of the system and that of two separate, non-interacting α -particles (calculated by the same method). Let $\vec{x}^{(i)}$ ($i = 1, 2, 3, 4$) be the radius vectors from the centre of gravity of an α -particle to its constituent nucleons. The Slater-determinant of our ${}^8\text{Be}$ -system, considered as an aggregate of two α -clusters, will then involve only two distinct spatial eigenfunctions of individual nucleons, viz. $\varphi(\vec{x}^{(i)})$ and $\varphi(\vec{R} - \vec{x}^{(i)})$ ($i = 1, 2, 3, 4$), which must be multiplied by the appropriate spin and isotopic eigenfunctions; since these spatial wave-functions are not orthogonal, the Slater-determinant gets an additional normalizing factor

$$\{1 - [\int \varphi(\vec{x}) \varphi(\vec{R} - \vec{x}) dv]^2\}^{-2}.$$

The result of the first approximation of this calculation will very sensitively depend on the "exchange" properties and distance dependence of the elementary law of interaction between two nucleons. The difference between the *potential* energies of the ${}^8\text{Be}$ system and the two separate α -particles represents an attractive potential for all values of the distance R between the α -particles, with approximately the same range as the nucleon interaction potential; the repulsive contributions from the differences of kinetic and Coulomb energies will clearly predominate both at small distances, owing chiefly to the increase of the kinetic energy of the ${}^8\text{Be}$ system, and at large distances, where only the long-range Coulomb energy subsists. But the behaviour of the interaction energy at intermediate distances will, as emphasized by INGLIS [41], critically depend on the properties of the nucleon interaction potential in the domain of distances exceeding its range. If this potential has a sufficiently strong "tail", or residual intensity, the attractive part of the α -particle interaction may become large enough at such intermediate distances to overcome the Coulomb repulsion; if, moreover, the "tail" of the nucleon potential is of the ordinary type, the resulting net attraction between α -particles will have the additivity property of intermolecular van der Waals forces. In general, however, the outcome will be a first order repulsion at all distances, and in order to account for the binding of α -particles it will be necessary to investigate higher approximations.

As a potential giving rise to additive first order attraction between α -particles, Inglis suggests a superposition of two Gauss potentials of different ranges:

$$V = J \left\{ -\frac{1}{4} P_\sigma + \frac{1}{4} P_\sigma P_\tau \right\} e^{-x^2 r^2} - J' e^{-x'^2 r^2} \quad (2)$$

with

$$\begin{aligned} J &= 71.4 \text{ MeV}, & J' &= 3.1 \text{ MeV} \\ x^{-1} &= 1.34 \cdot 10^{-13} \text{ cm}, & x'^{-1} &= 2.24 x^{-1}. \end{aligned}$$

His only argument for the occurrence of the longer range "tail" is taken from the analysis of proton-proton scattering (7.13); in fact, the effective ${}^1S$ potential derived from (2) closely approximates the meson potential proposed by Breit to account for the scattering data, i.e. that corresponding to the meson mass $M_m = 326 m$. Quite apart from the small probability of such a high value of the meson mass, the special form of spin and charge dependence of the operator (2) is widely arbitrary; it is compatible with our first saturation requirement (11.13, 11.2), but it is not clear whether it satisfies the second (11.3). Inglis only shows that it gives about the right binding energy of the deuteron and the α -particle. It should especially be stressed that there is no reason for assuming the "tail" to be just an ordinary Wigner force. Inglis' argument for first order attraction between α -particles is thus far from convincing.

The straightforward assumption of a nucleon interaction represented by a simple Gauss potential has been discussed by MARGENAU [41]. For the parameters fixing the spin and charge dependence of the nucleon interaction, he takes, in conformity with the saturation requirement (11.31-2) and the relation (11.21-6) directly connected with the deuteron properties, $a_0 = 0$ and $a_\tau - a_\sigma = 0,1$; he further leaves a_σ undetermined and finds that the result of the variational procedure, in the first stage of approximation, is practically independent of this parameter. As trial functions for $\varphi(\vec{x})$ he adopts, as usual, an harmonic oscillator eigenfunction involving a variational parameter. The values of the potential constants minimizing the energy of a single α -particle are

$$J = 35,6 \text{ MeV}, \quad \kappa^{-1} = 2,25 \cdot 10^{-13} \text{ cm}; \quad (3)$$

as expected (9.21), they are not in very good agreement with those derived from the two-nucleon data (6.432). The final result, as already stated, is that in the first approximation, the α -particles will show no tendency at all to attract each other: there subsists a repulsion at all distances.

This behaviour finds an explanation of rather general scope (WERGELAND [41]) in the closed structure of the α -particles (analogous to rare gas configurations of atomic systems): mutual attraction can only arise through virtual * "polarisation" or "activation" of such structures, — a feature first appearing in the second approximation **. According to the well-known formalism of perturbation theory, this second approximation may be expressed as $-\text{av} \mathcal{V}^2 / \mathcal{E}_m$, the quantity

* The word "virtual" indicates (1.321) that we are not here concerned with actual polarization or activation processes, but only with formal features of the perturbation method involving the matrix-elements of transitions to intermediate states of excitation of the particles, which also essentially determine the probability of actual occurrence of such processes.

** The importance of virtual α -particle activation in the nuclear case, i.e. when the interacting α -particles (in contrast to the case of the He_2 molecule) come close together, has already been stressed in an early paper by ELSASSER [34].

$\text{av } \mathcal{V}^2$ being the expectation value of the square of the potential energy of the system, while $\mathcal{E}_m$ denotes a suitable average over its excitation energies. The tedious calculation of $\text{av } \mathcal{V}^2$ has not been carried out by Margenau at full length: in the first place, instead of considering the appropriate linear combination of exchange operators $1, P_\sigma, P_\tau, P_\sigma P_\tau$ in the nuclear potential, he treats the four types of potentials separately; moreover, he neglects the exchange terms in $\text{av } \mathcal{V}^2$, which means that he only retains the α -particle interactions with the longest range. Within the scope of this estimate, he shows that the second order attractive force between the α -particles has a decidedly *shorter* range than the first order repulsion. This is the essential difference from the case of intermolecular forces: in that case, the analogous term has the longer range and is thus predominant in a certain region of intermediate intermolecular distances, in which it just represents the van der Waals attraction, with the additive property inherent in interactions of the ordinary type. We see that in the α -particle case the situation is completely reversed: exchange forces of any order of approximation will here be expected to play just as essential a part as the ordinary van der Waals attractions, with the result that there can be no additivity of the interactions between more than two α -particles.

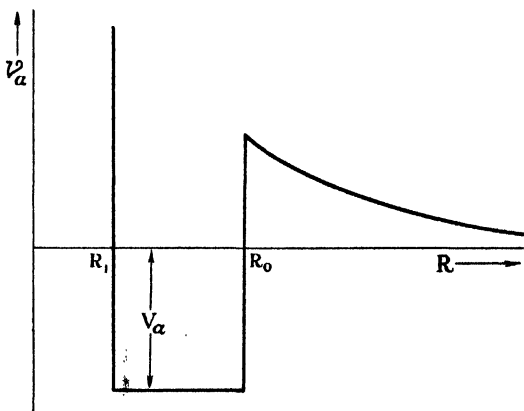


Fig. 13.12. Idealized potential for a pair of α -particles.

We shall return in a moment to this aspect of the problem. But first we must conclude the discussion of the system of two α -particles, for which the question of additivity does not arise. According to the above results, we shall in any case expect the interaction potential to reduce practically to the Coulomb repulsion at large distances, to become attractive at intermediate distances and to go over very steeply, at still smaller distances, into a strong repulsion. In order to investigate in how far this general behaviour of the interaction is able to account for the observed properties of the system, Margenau adopts the schematic representation given by fig. 13.12; he states, however, that the results he obtains by means of this rough approximation are not essentially influenced by wide variations in the shape of the potential. For $R > R_0$, a pure Coulomb field is assumed, and the "range" R_0 is fixed at $R_0 = 4.5 \cdot 10^{-13}$ cm. The depth V_α is fixed

in terms of R_1 by the condition that there should exist only one stationary state of about zero energy (13.11). Then the phase $\delta^{(0)}$ can be calculated for various values of R_1 in the interval $(0, R_0)$: at low energies there is general agreement as to order of magnitude with the result of Wheeler's analysis, but the model fails to yield the increase of $\delta^{(0)}$ in the resonance region. It can be seen that the range R_0 is too small for the type of potential under consideration to give rise to any resonance level of as low energy as ≈ 3 MeV. We here meet with one of the many shortcomings of the α -particle model in respect of quantitative description.

13.13. Binding energies of α -nuclei. It is a striking fact, brought out by the work of WEFELMEIER [37a, b] and of HAFSTAD and TELLER [38], that the binding energies of the α -nuclei would allow of a direct interpretation on the assumption of additive van der Waals interactions between α -particles. The structure of such nuclei, on the α -particle model,

13.13. Binding energies of α -nuclei					
Nucleus	Number of α -particles	Configuration	Number of bonds	α -binding energy 10^{-4} MU	α -binding per bond 10^{-4} MU
^8Be	2	straight line	1	-0.87 ± 0.70	—
^{12}C	3	triangle	3	77.00 ± 0.87	25.67 ± 0.29
^{16}O	4	tetrahedron	6	154.40 ± 1.24	25.73 ± 0.21
^{20}Ne	5	trigonal bipyramid	9	204.05 ± 2.16	22.67 ± 0.24
^{24}Mg	6	tetragonal bipyramid (octahedron)	12	310.5 ± 5.0	25.9 ± 0.4
^{28}Si	7	pentagonal bipyramid	16	365.3 or 387.3 ± 5.5	22.8 or 24.2 ± 0.5
^{32}S	8	sphenoidal bipyramid	19	499.9 ± 3.2	26.3 ± 0.2
^{56}Fe	13	centered icosahedron	42	931 ± 21	22 ± 2

can be compared with that of "geometrical models" consisting of close packings of rigid spheres: each such structure is characterized by a certain number of "bonds", or pairs of adjacent particles, and on the additivity hypothesis the binding energy of the model would simply be proportional to the number of bonds. Empirically, this binding energy can be computed as the difference between the mass of the α -nucleus and the total mass of the constituent α -particles. As the accompanying table shows, almost all the lighter α -nuclei (the only exceptions being ^8Be and ^{20}Ne) actually exhibit a surprisingly close proportionality between the empirical " α -binding energies" and the numbers of bonds; also the heavier ^{56}Fe nucleus, which, apart from 4 extra neutrons, would correspond to a symmetrical arrangement of 12 particles around a central one, obeys the same law.

Margenau's results about the interaction between two α -particles rule out any simple explanation of this remarkable regularity. As pointed out by GRÖNBLOM and MARSHAK [39], it must in reality arise from a quite

complicated compensation between energy terms of different orders of approximation, essentially involving non-additive features of α -particle interactions. In the case of the three- α -particle system constituting ^{12}C , for instance, there occur among others in the second approximation, virtual transitions in which a proton from one α -particle and a neutron from another go over into an excited state of the third: and the contribution from this type of transitions turns out to be of the same order of magnitude as that corresponding to second order interactions between pairs of α -particles. The latter will in fact just about compensate the first order repulsion, so that the main effect will be of the non-additive, many-body type; however, since the number of many-particle "bonds" increases more rapidly with the number of α -particles than the number of adjacent pair "bonds", there must also clearly occur partial compensations of the second order contributions by third order terms, and so on. The α -particle model would thus not seem to offer any more promising starting point than the statistical model for a quantitative calculation of nuclear energies.

13.14. Lability of α -clusters. Another aspect of the two- α -particle problem, strikingly illustrating the importance of the lability of the α -clusters in the determination of the energy of the system, is brought to light by the method of attack proposed by WERGELAND [41], which avoids the calculation of the second approximation. Observing that the true structure of the ^8Be nucleus must be, so to speak, something in between an aggregate of two α -particles and an assembly of eight independent nucleons, Wergeland chooses as trial eigenfunction for the application of the variational principle a linear combination of the eigenfunction Ψ_α of the α -particle model, discussed above (13.12), and that, Ψ_s , corresponding to the individual model; in the latter, the individual spatial wave-functions are again harmonic oscillator eigenfunctions, but we have now, besides that of the ground state of the oscillator, which alone occurs in the α -particle model, also to introduce that of the first excited state. We thus put

$$\Psi = \Psi_\alpha + \lambda \Psi_s \quad (4)$$

and consider λ as the variational parameter. The oscillator parameters*, on the other hand, are not varied independently, but related in such a way that for infinite separation of the α -particles ($R \rightarrow \infty$), the energy of the α -particle system is about equal to that of the 8-nucleon system: this somewhat arbitrary relation takes account of the fact that in the latter system four of the nucleons, being promoted to an excited oscillator state, have on the average larger momenta than the others. Assuming, for simplicity, a pure Majorana potential of the Gauss type, with the range and (approximately) the strength given by (3), Wergeland finds, under the conditions stated, an extremal value $\lambda = 1.34$, together with

$$A \equiv \int \Psi_\alpha \Psi_s = 0.304. \quad (5)$$

* Wergeland assumes anisotropic oscillator eigenfunctions, with two parameters.

This result can be interpreted, following a suggestion of v. WEIZSÄCKER [38], by saying that the α -particles are partly "dissociated" in the (liquid) nuclear matter; the "degree of dissociation" on this picture would be given, according to (4), (5), by

$$\delta = 1 - \frac{(\int \Psi_\alpha \Psi)^2}{\int \Psi^2} = \frac{1 - \Delta^2}{1 + 2\lambda\Delta + \lambda^2} \approx 0.25. \quad (6)$$

Such a figure once more confirms our previous conclusion as to the unsuitability of the α -particle model for energy computations.

13.15. Rotations and vibrations of α -nuclei. Notwithstanding the negative conclusions of the preceding discussion, there is a class of more qualitative predictions for which the model under discussion furnishes a sufficiently firm basis, viz. those which follow from the behaviour of the system under spatial rotations and symmetries. The justification for applying such considerations to an α -particle system depends in fact only on the relative stability of the α -clusters during time intervals larger than the periods of vibration and rotation in the stationary states of the system. Let us consider, for instance, the n -th mode of surface vibration of the nucleus, with wave-length $\lambda \approx 2\pi R/n$ (R denoting the nuclear radius); we assume, of course, that n is small enough for $\lambda/2$ to be larger than the dimensions of an α -cluster ($n \lesssim A^{1/3}$). A classical formula of Lord Rayleigh gives, for the corresponding circular wave-number ω ,

$$\omega^2 = n(n-1)(n+2) O \cdot \frac{4\pi}{3} \cdot \frac{1}{MA}, \quad (7)$$

whence for the period

$$\tau \approx n^{-1/2} A^{1/3} \cdot 0.4 \cdot 10^{-21} \text{ sec}, \quad (8)$$

if we use the value of the surface tension O deduced from (2.21-5). We may then, as proposed by WHEELER [37a], define the lability of an α -cluster in this state of vibration by the time t required by such a cluster to exchange a nucleon with a part of its surroundings moving in opposite phase: this will be, in order of magnitude, the time required for a nucleon to "diffuse" through a distance $\lambda/2$. Classically, we should have a diffusion formula of the type

$$(\lambda/2)^2 = \frac{1}{3} A_d \bar{v} t,$$

$\bar{v}$ being the average nucleon velocity and A_d a suitably defined mean free path; taking $\bar{v}$ of the order of the limiting velocity v_m of the Fermi distribution and $A_d \approx r_0$, we should get in this way

$$t \approx n^{-2} A^{1/3} \cdot 0.6 \cdot 10^{-21} \text{ sec}, \quad (9)$$

or

$$t \approx (A^{1/3}/n)^{1/2} \cdot 1.5 \tau: \quad (9a)$$

for the modes of vibration we are considering, even this classical estimate would give $t > \tau$. But the potential barriers due to the surrounding nucleons will be very effective in further reducing the rate of diffusion, at least in states of low excitation energy: for the probability of concentration on the diffusing nucleon of the "activation energy" $\approx |\epsilon| \approx 10$ MeV (2.21) necessary to surmount the barrier will contain a factor $e^{-|\epsilon|/T}$, the temperature T not exceeding ≈ 2 MeV for the excitations obtainable in ordinary transmutation experiments. We may therefore conclude that the condition for treating the nucleus as a system of permanent α -particles is fulfilled when one has to do with vibrational or rotational stationary states of sufficiently low excitation. As the excitation energy increases, however, the α -clusters will tend gradually to dissolve and the nuclear properties will become more and more comparable to those of a liquid droplet without cluster structure.

An estimate of the vibrational quanta to be expected on the α -particle model may be obtained, according to WHEELER [37a], by equating the potential energy of the quasi-elastic forces responsible for the different modes of vibration of the nucleus envisaged, taken for an amplitude roughly corresponding to the dissociation of an α -particle, to the absolute value of the binding energy of this particle to the residual nucleus, as it results from comparison of mass-defects, plus the zero-point energy of the nuclear vibrations: for the lightest α -nuclei, such an estimate yields quanta of the order of a few MeV, just as those of the lowest ($n = 2$) mode of surface vibration of the nuclear liquid drop according to (7). As regards the order of magnitude of the rotational quanta $\hbar^2/I_s$, we get for the moment of inertia I_s of a spherical homogeneous distribution of the nuclear mass

$$I_s = \frac{2}{5} M A R^2 = \frac{2}{5} M r_0^2 A^{\frac{5}{3}}, \quad (10)$$

whence

$$\hbar^2/I_s = \frac{5}{2} \cdot \frac{\hbar^2}{M r_0^2} \cdot A^{-\frac{1}{3}} \approx 54 A^{-\frac{1}{3}} \text{ MeV}, \quad (11)$$

i.e. several hundred keV for the lightest α -nuclei, but only a few keV for heavy nuclei.

The additional invariance with respect to certain rotations or symmetries introduced into the nuclear system by the assumption of an α -particle structure will, especially in the case of light α -nuclei, cause a considerable reduction in the number of low-lying levels. For instance, only the wave-functions corresponding to even values of the angular momentum satisfy the requirement of Bose statistics for the two- α -particle system ${}^8\text{Be}$. The states excluded in this way on symmetry grounds are nevertheless present in the spectrum of the α -particle system, but they appear at higher levels of excitation than in the spectrum of the corresponding system of nucleons without α -particle structure; for their eigen-

functions can only be given the required antisymmetry with respect to all nucleons by excitation of α -particles or subdivision into other than α -clusters. At higher excitation energies, the spectrum, as already mentioned, merges into that of a vibrating fluid continuum: this tendency toward a purely classical behaviour for high energies is just an aspect of the general correspondence argument of quantum theory.

Such an exclusion of low levels is, however, not likely to occur in the case of heavy nuclei, since these nuclei will generally not have any particularly symmetrical configuration. In view of the small value of the rotational quanta for such nuclei, this would at first sight seem to involve a serious difficulty, for the existence of states of such small excitation energies would be in flagrant contradiction to various observations (TELLER and WHEELER [38]): on the one hand, the spectra of natural α - or β -radioactive bodies show no trace of a fine structure of this order of magnitude; on the other, when β -rays are emitted by internal conversion in a transition from a metastable level to the ground state, they are never accompanied by γ -rays of slightly less energy, as should be the case if there were low-lying rotational levels which could be reached from the metastable state by radiative transitions. Somewhat more indirect, but still of great weight is the evidence derived from the β -active nuclei of very long life-times ^{40}K and ^{87}Rb (KONOPINSKI [43]) and from numerous cases of nuclear isomerism (FLÜGGE [41]); in all such phenomena, the small probability of a transition to the ground state is due to the large change of angular momentum involved, — an explanation which could not be upheld if there were competing transitions to rotational states of very low excitation. In fact, all the evidence concordantly points to an order of magnitude of a few hundred keV for the lowest excited states of even the heaviest nuclei.

The effect of symmetrical arrangement of the α -particles in pushing upward a number of levels of excitation of the system is, however, as emphasized by TELLER and WHEELER [38], supplemented by another interesting mechanism, very effective in producing the same result. Suppose that there is a rotation by which the nucleus is brought to a configuration not exactly but nearly coinciding with the initial one: a relatively small *translation* of the constituent particles will then suffice to restore the initial configuration. We have then to do with a "resonance" between two equilibrium configurations separated by a potential barrier, and if this barrier is sufficiently low for the frequency of oscillation from the one to the other configuration to be larger than the frequency of the corresponding rotation, a considerable disturbance of the rotation level results, the effect of which will be the same as that of a higher symmetry of the nucleus.

A simple example will serve to illustrate this peculiar resonance phenomenon. Consider a system of two particles attached by rigid rods of equal lengths to a pivot, around which they may rotate in the plane of the rods

and oscillate with respect to each other under the action of a potential V . Suppose there are two equilibrium positions of the system for the values $\vartheta_0 (< \pi)$ and $2\pi - \vartheta_0$ of the angle ϑ between the rods: the potential $V(\vartheta)$ has then the form represented on fig. 13.15. The possible energy eigenvalues of this system are

$$E = E_n + \frac{\hbar^2}{2I} K^2 \pm B_n, \quad (12)$$

(I moment of inertia, K rotational quantum number, E_n energy of harmonic vibration energy around the equilibrium positions, B_n energy change due to resonance between these two positions) and the corresponding eigenfunctions

$$\Psi \sim [\Psi_V(\vartheta_0) \mp \Psi_V(2\pi - \vartheta_0)] e^{iK\varphi} \quad (13)$$

(φ angular displacement of centre of gravity of particles). If the particles obey Bose statistics, Ψ must remain the same when the system is rotated through an angle π and brought from one equilibrium position to the other. Accordingly, the only energy values allowed in this case are

$$E = E_n + \frac{\hbar^2}{2I} K^2 - (-1)^K B_n, \quad (14)$$

the odd rotational levels are raised, the even ones depressed. As soon as the (circular) frequency $2B_n c/\hbar$ of the exchange between the two equilibrium configurations becomes larger than the rotational frequency $\hbar c K/I$, the order of the successive levels $K-1$, K is inverted. On further increase of B_n , we shall finally find that a number of even levels will all lie below the first odd level, — a situation approaching that of the rigid system with $\vartheta_0 = \pi$, for which only the even levels subsist.

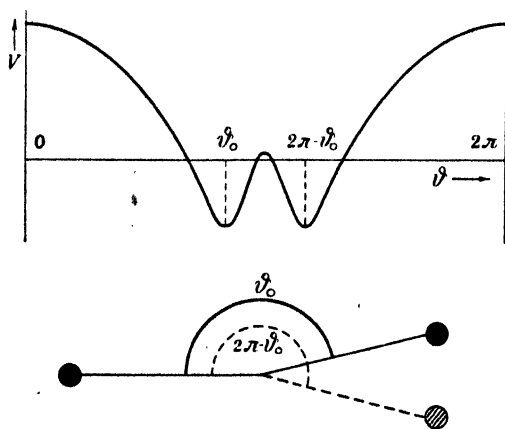


Fig. 13.15. Resonance between rotation and translation through potential barrier.

Coming back to the case we were discussing, let δ be the displacement which restores the initial configuration after it has been rotated through an angle φ ; this means that the latter rotation just corresponds to a linear displacement $\approx \delta$ and therefore to an increase in kinetic energy $\approx \frac{\hbar^2}{2MA(\delta/\varphi)^2}$. For the purpose of a rough estimation, we may identify

this with the energy change due to the resonance; let us further take $\varphi \approx \pi$ and $\delta \approx r_\alpha$, r_α being the average half-distance between two α -particles

$$r_\alpha \approx 4^{\frac{1}{2}} r_0 \approx 1,6 r_0. \quad (15)$$

The excitation energy of the state considered will thus be about *

$$\frac{\hbar^2 \pi^2}{2 M A r_\alpha^2} \approx \frac{40}{A} \text{ MeV}, \quad (16)$$

i.e. $\approx 0,2$ MeV for $A = 200$, which is indeed of the order of magnitude indicated by the empirical evidence. The quantum number J_{exc} of the first excited rotational level would accordingly be given by

$$\frac{\hbar^2 J_{\text{exc}}^2}{2 I_s} \approx \frac{\hbar^2 \pi^2}{2 M A r_\alpha^2},$$

or, by (10), (15),

$$J_{\text{exc}} \approx 1,26 A^{\frac{1}{2}} \quad (J_{\text{exc}} \approx 8 \text{ for } A \approx 200). \quad (17)$$

Another characteristic J -value is that above which the order of the rotation levels will not be disturbed, and the nucleus thus will rotate as a rigid body; we may call this value J_{coll} the limit of "collective motion". It is defined by

$$\frac{\hbar^2}{2 I_s} [J_{\text{coll}} (J_{\text{coll}} + 1) - (J_{\text{coll}} - 1) J_{\text{coll}}] \approx \frac{\hbar^2 \pi^2}{2 M A r_\alpha^2},$$

i.e.

$$J_{\text{coll}} \approx \frac{1}{2} J_{\text{exc}}^2 \approx 1,56 A^{\frac{1}{2}}, \quad (18)$$

— a condition just expressing (in agreement with the preceding argument) the equality of the frequencies of the rotation and exchange oscillation leading from the one to the other configuration. For $A = 200$, (18) yields $J_{\text{coll}} \approx 27$, corresponding to a rotation energy $\approx 2,5$ MeV.

13.2. Nuclear states on α -particle model

13.21. Stationary states of lightest α -nuclei. For the purpose of ascertaining the possible sets of quantum numbers and the parity, as well as the approximate positions, of the lowest vibrational and rotational levels of α -nuclei, it is permissible, according to the above discussion, to treat these nuclei as simple configurations of α -particles, to which the well-known methods of molecular spectroscopy may be applied. This has been done fairly completely for the three lightest α -nuclei, ^8Be , ^{12}C and ^{16}O , and to some extent also for the next, ^{20}Ne . We shall give here a brief

* The same order of magnitude for the excitation energies can also be derived on the individual Fermi gas model; cf. TELLER and WHEELER [38].

review of the main results, referring the reader to the original papers for a fuller account.

Beryllium (WHEELER [37a]). The nucleus ^8Be is compared with a simple dumbbell configuration. There is one non-degenerate mode of vibration; the rotation energy has the form $(\hbar^2/2I_c)J(J+1)$, I_c being the moment of inertia around a symmetry axis perpendicular to the line joining the centres of the α -clusters. Only the levels with even J are allowed; they are of even parity.

Carbon (WHEELER [37a]). The equilateral triangle configuration representing the ^{12}C nucleus has two distinct fundamental modes of vibration: a non-degenerate mode, corresponding to an isotropic dilatation and contraction of the triangle (quantum number n_1), and a doubly degenerate mode, describable as a tipping* of the triangle (quantum number n_2). The rotation is that of a symmetrical top, with energy

$$\frac{\hbar^2}{2I_c}J(J+1) + \frac{\hbar^2}{2}\left(\frac{1}{I_a} - \frac{1}{I_c}\right)\Delta^2; \quad (1)$$

the moments of inertia I_a , I_c refer to the symmetry axis perpendicular to the plane of the triangle (figure axis), and to a symmetry axis in this plane, respectively; Δ denotes the quantum number of the projection of the angular momentum on the figure axis, an integer limited by the condition $|\Delta| \leq J$.

The parity of the allowed levels is given by the following table; it is characterized by a parameter p , which has the value $+1$ or -1 according as the level is even or odd. The notation ± 1 means a pair of coinciding levels, one of them even, the other odd; and $n(\pm 1)$ denotes n such pairs.

13.21. Parity of levels of ^{12}C					
$ \Delta $	J	$n_1 = 0, 1, 2, \dots; n_2 =$			
		0	1	$2n$	$2n+1$
0	even	$+1$	.	$+1$	$+1$
	odd	.	.	.	-1
3λ	any	$(-1)^\Delta$	.	$(-1)^\Delta$	$+1$ -1
$3\lambda \pm 1$	any	.	± 1	$n(\pm 1)$	$n(\pm 1)$

Oxygen (WHEELER [37a], DENNISON [40]). The regular tetrahedron, which represents the ^{16}O nucleus, has 3 distinct fundamental modes of vibration: a non-degenerate isotropic dilatation and contraction (n_1), a doubly degenerate motion, in which the α -particles are paired into two dumbbells twisting with respect to each other (n_2), and a trebly degenerate

* In this mode of vibration, one side of the triangle shrinks and at the same time moves away from the third α -particle.

vibration, in which one dumbbell shortens while the other lengthens (n_3). The rotation is that of a spherical top, with energy $(\hbar^2/2I_s)J(J+1)$, I_s being defined by (13.15–10). Besides the sum of rotation and vibration energy, and the usual interactions between rotation and vibration (which, for ^{16}O , would be as large as 0.5 ... 1 MeV), two other perturbations must be taken into account. One correction arises from the fact that the third mode of vibration has an angular momentum; Dennison shows, e.g., that for the state $n_1 = n_2 = 0$, $n_3 = 1$, $J = 1$, it amounts to $\frac{9}{8} \frac{\hbar^2}{I_s}$. The other correction is due to the effect, discussed in 13.15, of exchange between two inverse configurations. In the case of ^{16}O , the inversion may be produced by a vibration of the third kind, and the additional energy term has the form $\mp G_{n_3}$ with the upper or lower sign according as the wave-function of the state remains invariant or changes sign when the inversion is performed; Dennison estimates that for $n_3 = 0$, G_0 is of the order of $1.5 \cdot 10^{-3}$ times the quantum of vibration of the third mode, and for $n_3 = 1$, $G_1 \approx 25 G_0$.

Neon (TELLER and WHEELER [38]). When no vibration is excited, the bipyramid configuration of ^{20}Ne behaves much like ^{12}C : the rotation energy has the form (1) and the allowed levels are given, with their parity, by the $n_2 = 0$ column of table 13.21. There is an additional energy term due to the exchange between two resonating configurations (fig. 13.21).

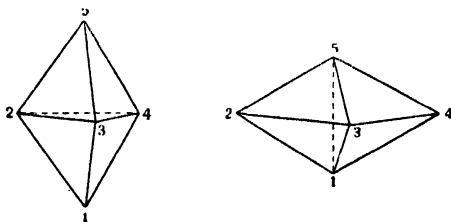


Fig. 13.21. The two resonating configurations of the ^{20}Ne nucleus.

It has been evaluated by Teller and Wheeler for the states without vibrational excitation; they find

$$-\frac{3}{2J} \varepsilon G \sum_{f=0}^{J-A} (-1)^f \binom{J+A}{f} \binom{J-A}{f} \quad (2)$$

$(\varepsilon = 2 \text{ for } A \neq 0, \quad \varepsilon = 1 \text{ for } A = 0),$

and give an estimate, probably too high, of $G \approx 0.16$ MeV.

13.211. Comparison with experiment. Although a number of excited states of the nuclei just discussed have been disclosed by the study of nuclear reactions, and their excitation energies determined, the comparison of these results with the theoretical predictions is rendered difficult by the deficient information on the parities or angular momenta of these states. Only in the case of ^{16}O could the properties of the excited states formed in the reaction $^{19}\text{F}(p, \alpha)^{16}\text{O}$ be analyzed in a sufficiently com-

plete way (OPPENHEIMER and SCHWINGER [39]). Starting from these data, DENNISON [40] was led to definite assignments of the vibration frequencies and the moment of inertia of the nucleus; the latter is found to be rather smaller than would result from formula (13.15–11).

The moments of inertia of the other nuclei can also be estimated, provided a plausible assumption is made in each case about the ratio of the axes of the spheroid of inertia. In fact, by comparing the moments of inertia of a spherical and a spheroidal mass distribution of equal volumes, it is easily proved that

$$\frac{\hbar^2}{2I_a} = \xi^2 \frac{\hbar^2}{2I_s}, \quad \frac{\hbar^2}{2I_c} = \frac{2\xi^2}{1+\xi^2} \frac{\hbar^2}{2I_s}, \quad (3)$$

ξ denoting the ratio of polar to equatorial diameter of the spheroidal mass distribution. The following table gives the results pertaining to the four

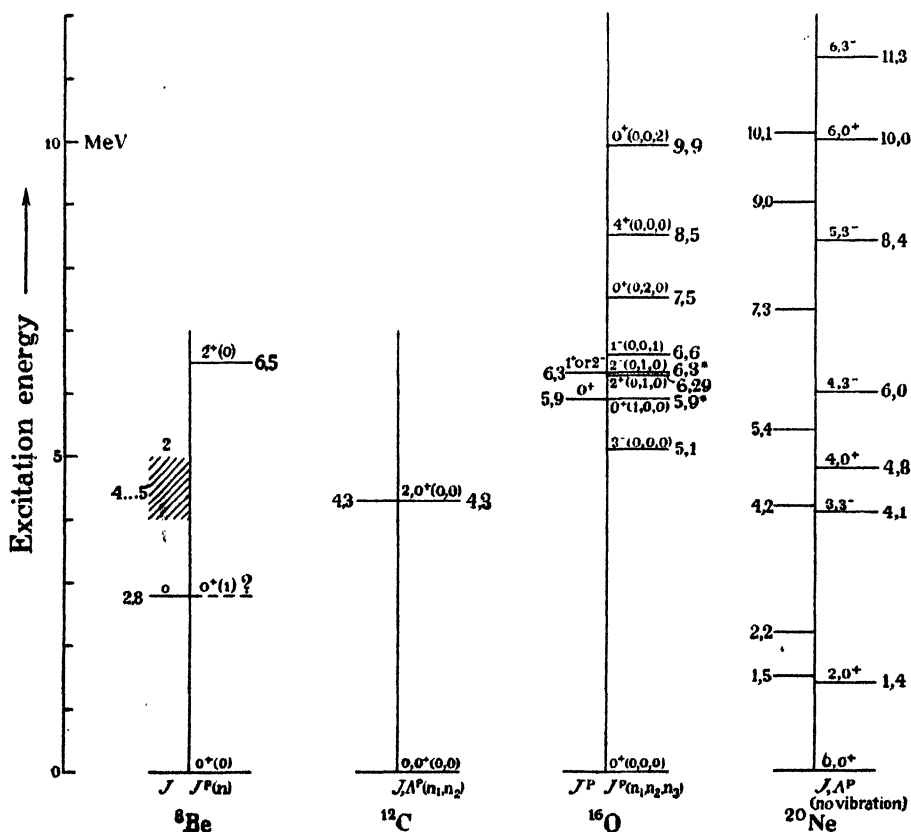


Fig. 13.211. Excited states of α -nuclei.

The empirical data are represented on the left, the theoretical spectrum on the right (all excitation energies being given in MeV). Each state is characterized by its quantum numbers, as indicated below the corresponding spectrum (p denoting the parity factor, $+1$ for even states, -1 for odd states). The ^{16}O levels marked with an asterisk are those on which Dennison's analysis is based.

lightest α -nuclei; the ξ -values are those proposed by WHEELER [37a] and TELLER and WHEELER [38]. The values of $\hbar^2/2I_s$ calculated by (13.15–11) have been multiplied by a correction factor 1.6 in conformity with Dennison's analysis of the ^{16}O spectrum; there is, of course, much uncertainty both in this correction and in the choice of the ξ 's.

13.211. Rotation parameters of light α -nuclei				
Nucleus	ξ	$\frac{\hbar^2}{2I_s}$	$\frac{\hbar^2}{2I_c}$	$\frac{\hbar^2}{2I_a} - \frac{\hbar^2}{2I_c}$
^8Be	1.5	1.34	1.07	—
^{12}C	0.67	0.69	0.74	— 0.21
^{16}O	1	0.43	0.43	0
^{20}Ne	1.46	0.30	0.24	0.14
Energies in MeV				

The degree of agreement between the computed spectra and the empirical evidence may be judged from fig. 13.211; as regards the three lightest nuclei, this diagram embodies the results of Dennison's discussion (modified as regards ^8Be in conformity with 13.11). The theoretical ^{20}Ne -spectrum is restricted to the rotation levels, without vibrational excitation; the resonance correction (2) has been omitted, since it is presumably very small. The experimental data concerning ^{20}Ne and heavier α -nuclei are collected and critically discussed in a paper by GUTHRIE and SACHS [42]; it would be premature in those cases to attempt any theoretical identification of the observed states of excitation.

13.22. Stationary states of nuclei of mass number $4a \pm 1$. The symmetry and invariance considerations which led to the above results for the light α -nuclei have been extended by HAFSTAD and TELLER [38] to the neighbouring nuclei* of mass number $4a \pm 1$, considered as aggregates of α -clusters and an additional loose nucleon or an α -cluster lacking one nucleon (cluster with a "hole"). As long as electrostatic forces are neglected, it does not matter whether the loose nucleon or hole is a neutron or a proton**. The method is based on the observation that, owing to the short range of the nuclear forces, a loose nucleon will interact strongly with only one α -particle at a time: it will therefore be possible to express its individual wave-function as a linear combination of wave-functions representing the loose nucleon in the neighbourhood of each separate α -particle. The latter wave-functions have each a node through the centre of gravity of the respective α -particle; the state of lowest energy of the system of mass number $4a + 1$ will correspond to the particular linear

* The treatment of mass number 19 is sketched by SACHS [39].

** An estimate of Coulomb energies on the present model (leading to less satisfactory results than on the individual models (3.3)) has been carried out by BROWN and INGLIS [39].

combination (of the required symmetry) which has the least number of nodes. In the case of mass number $4a - 1$, the wave-function of the hole will be built up quite similarly, but in order to get the smallest energy, we have to choose that with the largest number of nodes. A closer discussion of symmetry properties of such functions for the different mass number values leads then without difficulty to an enumeration of the lowest excited levels in each case; the rotation angular momentum must here, in conformity with spectroscopic use, be denoted by K instead of J , the latter notation being reserved for the total angular momentum, including spin, $J = K \pm \frac{1}{2}$.

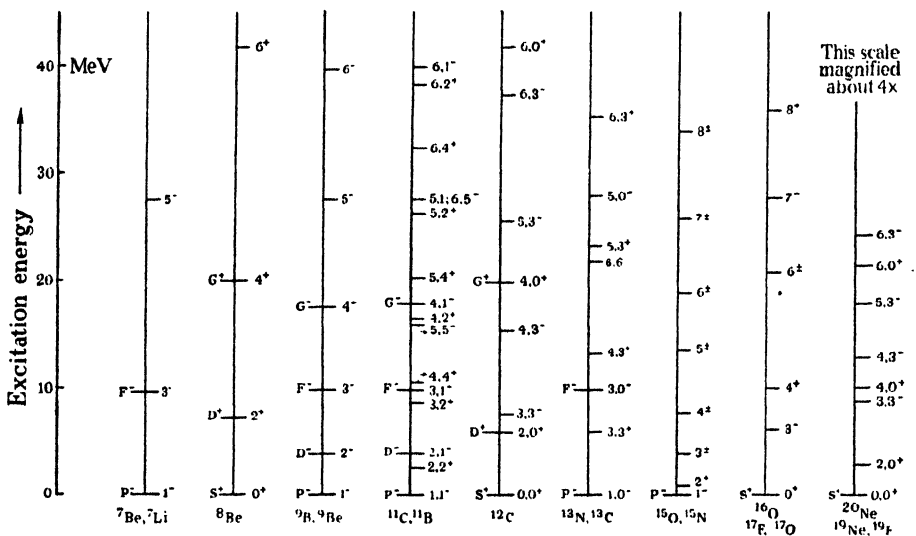


Fig. 13.22. Some excited states of light nuclei of mass numbers $4a$, $4a \pm 1$, according to the α -particle model and the quasi-atomic model.

On the left are marked the levels following from a Hartree-Fock treatment of the quasi-atomic model for the simplest configurations, on the right those given by the α -particle model, assuming only rotational excitation (the rotational quantum numbers are indicated in the order J , Δ or K , Δ). The $\pm$ -signs refer to the parity $p = \pm 1$. The energy scales are simply illustrative of the relative level separations, but have no strictly quantitative significance; they have been fitted to each other by arbitrarily identifying the calculated energies of the corresponding levels D^+ , 2^+ of ${}^8\text{Be}$.

Fig. 13.22, reproducing with a slight extension a diagram in Hafstad and Teller's paper, conveniently summarizes these results and parallels them with the analogous predictions following from the quasi-atomic model (10.2). It will be seen that both models are in complete agreement as regards angular momentum and parity of the ground state and also for excited states exhibit a striking similarity in these respects. In judging this resemblance, one should not forget that the spectra represented are very incomplete: on the quasi-atomic model, only the levels corresponding to the simplest configurations have been investigated; on the α -particle model,

the vibrational excitation has been entirely left out, and no account has been taken of levels corresponding to other wave-functions of the loose nucleon (or hole) than that with the smallest (or largest) number of nodes*.

Hafstad and Teller also show how a rough estimate of the mass-defects of the nuclei of the class under discussion can be derived from their treatment; for this aspect of the problem, however, it will suffice to refer the reader to their paper.

13.3. The resonating group structure

The prominent part played by α -clusters in nuclear structure is due to their peculiar stability. But other clusterings, in fact all conceivable groupings of the constituent nucleons into clusters, must clearly also occur in the course of time, only with widely varying probabilities. The consideration of such alternative groupings is especially important for the discussion of nuclear reactions and scattering processes, but their virtual existence should also be taken into account in any attempt to improve calculations of energies of stationary states. One is here reminded of the situation met with in the study of molecular binding energies: in many cases, a "resonance" between different types of binding (e.g. homopolar and heteropolar) in a molecule is found to have considerable influence on its energetic and spectroscopic properties. Pursuing the fruitful analogy between nuclear and molecular structures, we should likewise, as emphasized by WHEELER [37a], regard any state of a nucleus as a superposition of all possible kinds of nucleon clusters, i.e. (as Wheeler puts it) attribute to the nucleus a "*resonating group structure*". The corresponding eigenfunction should accordingly be set up as a linear combination of products of wave-functions representing the various "configurations" or groupings into clusters; the coefficients, depending on the coordinates of the centres of gravity of the clusters, should be determined as proper solutions of a certain set of equations involving these coordinates as variables. This method, which thus provides a much broader and more rational basis than the α -particle model for an attack on the properties of complex nuclei, has been carefully discussed, and its general algorithm very clearly set out, in a fundamental paper by WHEELER [37b]. To this paper the reader is referred for a detailed study; we shall here only give a brief outline of the main points of interest.

Let us, for the sake of illustration, take an especially simple case. Retaining only the most important** configurations occurring in a given

* Cf. on this point the case of ${}^7\text{Li}$, discussed in 17.52.

** The configurations which are liable to contribute most effectively to the building up of a particular nuclear state can readily be determined by a group-theory argument due to WHEELER [37a]. As long as we neglect the spin dependence of the nuclear forces, any (degenerate) state of a nucleus of mass number A belongs to a definite irreducible

nuclear state, let us suppose that they all consist of only two clusters. The total eigenfunction is then of the form

$$\Psi \sim \sum_i F_i [\vec{X}_i(\text{I}), m_{S_i}(\text{I}); \vec{X}_i(\text{II}), m_{S_i}(\text{II})] \Phi_i(\text{I}) \Phi_i(\text{II}), \quad (1)$$

the summation extending over all configurations, distinguished by an index i ; $\Phi_i(\text{I})$, $\Phi_i(\text{II})$ are the wave-functions of the two clusters I, II in configuration i , $\vec{X}_i(\text{I})$, $\vec{X}_i(\text{II})$ the radii vectores of their centres of gravity and $m_{S_i}(\text{I})$, $m_{S_i}(\text{II})$ their spin quantum numbers (expressing the spin components in an arbitrarily chosen direction). The brace over the typical term in the sum (1) means that it must be replaced by the antisymmetric combination of all similar expressions derived from it by permuting the nucleon variables. The coefficients F_i , which depend on the $\vec{X}$'s only through the differences $\vec{X}_i(\text{I}) - \vec{X}_i(\text{II})$, may conveniently be written as symbolical vectors $\mathbf{F}_i$, the components of which are numbered by the spin variables $m_{S_i}(\text{I})$, $m_{S_i}(\text{II})$. It can then be shown that for *any* type of nuclear interaction (even including many-body forces) these $\mathbf{F}_i$ must satisfy a set of integro-differential equations of the type

$$\left[\frac{\hbar^2}{2M_i} \Delta_{\vec{X}} + \mathcal{E} - \mathcal{E}_i \right] \mathbf{F}_i(\vec{X}) = \sum_j \{ \mathbf{V}_{ij}(\vec{X}) \mathbf{F}_j(\vec{X}) + \int [\mathbf{J}_{ij}(\vec{X}, \vec{\xi}) - \mathcal{E} \mathbf{I}_{ij}(\vec{X}, \vec{\xi})] \mathbf{F}_j(\vec{\xi}) d\nu_{\xi} \}; \quad (2)$$

here M_i denotes the reduced mass of the two clusters in configuration i , $\mathcal{E}_i$ the total energy of these separated clusters and $\mathcal{E}$ the energy of the nuclear state considered. The operators $\mathbf{V}_{ij}$, $\mathbf{J}_{ij}$, $\mathbf{I}_{ij}$ can be calculated in terms of the law of nuclear interaction and the cluster wave-functions Φ ; for $i = j$, they represent the interactions between the clusters in the same configuration; for $i \neq j$, the resonance effects between different configurations (in particular, the transmutations) *. The integral in (2) arises from

representation D_A of the symmetric group comprising all permutations of A elements (10.31). If now two clusters of N_1 , N_2 nucleons in states characterized by the respective representations D_{N_1} , D_{N_2} are combined into a nucleus, the states of the latter arising from this combination will directly be found by resolving the direct product $D_{N_1} \times D_{N_2}$ into its irreducible components. Since further the energies of any nuclear state can be approximately evaluated in terms of the corresponding representation (10.31), we can in each case roughly compare the energy of the state of the compound nucleus in which we are interested with the total energies of the groups of separated clusters which may occur in its formation: we have then to retain only those groups of clusters for which these two energy values are not too different from each other.

* It may be important to take account of the polarization of the clusters due to their mutual interaction. In principle, such polarization effects can be included in the general formulae (1) and (2) in either of two equivalent and mutually exclusive ways, viz. assuming the cluster wave-functions also to depend in an appropriate manner on the $\vec{X}$'s, or introducing suitably chosen configurations in which excited states of the clusters occur.

exchange terms of potential energy as well as from the lack of orthogonality of the cluster wave-functions; it can be interpreted (11.121) as expressing a velocity dependence of the cluster interactions*, — a feature not to be confused, of course, with a possible velocity dependence of the elementary interactions between nucleons.

The next step consists in transforming the system (2) to polar coordinates, thereby reducing it to a system of the same form, but containing only unknown functions of the radial coordinate. It then becomes possible to transform it further to a system of linear integral equations, the eigenvalues $\mathcal{E}$ of which are determined by equating to zero its Fredholm determinant. This demands, of course, that all $\mathcal{E}_i$ be larger than $\mathcal{E}$, for only then do we have a completely stable system. If this condition is not fulfilled (which implies the occurrence of scattering or transmutation processes), the Fredholm determinant may be made $= 0$ for any value of $\mathcal{E}$, because it contains the undetermined phases introduced by taking account of the required asymptotic behaviour of the solutions. Still, as Wheeler shows, the equation Fredholm determinant $= 0$ can be used in this case to determine almost completely the cross-sections of the scattering or transmutation processes (it has, however, to be supplemented by some other procedure in order to resolve the remaining ambiguities for one arbitrary energy value).

Up to now, the method of resonating group structure has only been applied to the simplest nuclear systems, especially the 3-nucleon systems and ${}^5\text{He}$; these applications, briefly discussed in the following Chapter, will serve to illustrate the general procedure**. However, in these instances, the last step, viz. the solution of the radial integro-differential equations, has been performed by special methods, involving certain approximations. The results are rather too inconclusive to enable one to form a judgment on the merits of the resonating group method for purposes of practical calculations. It is clear that its complication rapidly becomes overwhelming as the number of configurations is increased in order to improve the accuracy. Nevertheless, considering its extreme generality and the soundness of its theoretical foundations, this method fully deserves attention as the one most liable to lead to essential advance in the problem of the constitution of complex nuclei, the difficulty of which has been so often displayed in the preceding sections.

* This general property of cluster interactions has been known for a long time in the special case of the force exerted by a nucleus on one of its constituent nucleons ("van Vleck's potential"). Cf. B & B, § 37.

** Applications of the method to the 4-nucleon (α -particle) system by PARKER [37] and the system of two α -particles by WHEELER [37a] have been announced, but not published. For its application to ${}^7\text{Li}$, see KITTEL [42].

CHAPTER XIV

PROPERTIES OF LIGHT NUCLEI

14.0. The methods of treatment of complex nuclei, discussed in the preceding Chapters, have also been applied, with varying degrees of success, to the investigation of the lightest nuclei consisting of more than two nucleons. Especially the systems of 3 or 4 nucleons have been studied in a rather detailed way, either by the Hartree-Fock variational method (9.21) or by the more refined resonating group method (13.3); we shall in this Chapter confine ourselves to these systems and give a brief account of the present state of their theoretical and experimental investigation. Since the interaction between two neutrons cannot be studied directly from phenomena involving only two such particles, the 3-nucleon systems are the simplest ones liable to yield indirect information about this interaction; unfortunately, the issue is obscured by the possible occurrence of many-body forces (1.34). The following discussion will be based on the assumption of central interactions between pairs of nucleons only; within this limitation, it will appear that important indications as regards the charge dependence of the nuclear interaction can in fact be derived from the analysis of the scattering of nucleons by deuterons.

14.1. The three-nucleon systems

14.11. Stationary states on resonating group method. According to the conception of resonating group structure, a system of 3 nucleons (not all of them protons or all neutrons) has to be represented as a combination of a deuteron and a loose nucleon (neutron or proton). Clearly, this representation is especially suited for the treatment of scattering problems; but it can just as well be used for describing the stationary states of binding of the nuclei ${}^3\text{H}$ and ${}^3\text{He}$. Its most serious drawback is its inability to take account easily of the polarization of the deuteron in the field of the third nucleon. The total wave-function is written as a symmetrical combination

$$\Psi(1,2,3) = \psi(\widetilde{12},3) + \psi(\widetilde{23},1) + \psi(\widetilde{31},2) \quad (1)$$

of wave-functions such as $\psi(\widetilde{12},3)$, antisymmetrized with respect to the pair (1,2), constituting the deuteron. The function $\psi(\widetilde{12},3)$ is further expressed as a product of three factors:

$$\psi(\widetilde{12},3) = \varphi(\overline{12},3) \sigma(\overline{12},3) \tau(\widetilde{12},3), \quad (2)$$

containing the spatial, spin and isotopic coordinates, respectively. If the

deuteron group is assumed to be in the ground state, the isotopic factor,

$$\tau(\tilde{1}\bar{2},3) = {}^1(\tau)_0(1,2) u_{\pm}(3), \quad (3)$$

is antisymmetric in 1 and 2, while the spin factor, involving ${}^3(\sigma)(1,2)$, is symmetrical in these particles; consequently the space factor is also symmetrical.

The stationary states can be classified, according to the spin multiplicity, into a doublet and a quartet system. The respective spin eigenfunctions are readily found to be:

Doublet system:

$$\begin{aligned} {}^2\sigma_1(\bar{1}\bar{2},3) &= \frac{1}{\sqrt{3}} [{}^3(\sigma)_0(1,2) v_+(3) - \sqrt{2} \cdot {}^3(\sigma)_1(1,2) v_-(3)] \\ {}^2\sigma_{-1}(\bar{1}\bar{2},3) &= \frac{1}{\sqrt{3}} [{}^3(\sigma)_0(1,2) v_-(3) - \sqrt{2} \cdot {}^3(\sigma)_{-1}(1,2) v_+(3)] \end{aligned} \quad (4)$$

Quartet system:

$$\begin{aligned} {}^4\sigma_3(\bar{1}\bar{2},3) &= {}^3(\sigma)_1(1,2) v_+(3) \\ {}^4\sigma_1(\bar{1}\bar{2},3) &= \frac{1}{\sqrt{3}} [\sqrt{2} \cdot {}^3(\sigma)_0(1,2) v_+(3) + {}^3(\sigma)_1(1,2) v_-(3)] \\ {}^4\sigma_{-1}(\bar{1}\bar{2},3) &= \frac{1}{\sqrt{3}} [\sqrt{2} \cdot {}^3(\sigma)_0(1,2) v_-(3) + {}^3(\sigma)_{-1}(1,2) v_+(3)] \\ {}^4\sigma_{-3}(\bar{1}\bar{2},3) &= {}^3(\sigma)_{-1}(1,2) v_-(3). \end{aligned} \quad (5)$$

The spatial dependence of the wave-function is further split into two factors:

$$\varphi(\bar{1}\bar{2},3) = \chi(|\vec{x}^{(1)} - \vec{x}^{(2)}|) F[\vec{x}^{(3)} - \frac{1}{2}(\vec{x}^{(1)} + \vec{x}^{(2)})]; \quad (6)$$

in this formula, χ is the radial eigenfunction of the deuteron, F the probability amplitude pertaining to the position of the third nucleon with respect to the centre of gravity of the deuteron. This last function F is the only one which is not specified beforehand. It has to satisfy a certain integro-differential equation, which is obtained by inserting into the wave-equation for Ψ its expression (1) with the choice described above of the various factors of the ψ 's. This equation for F has been first given by WHEELER [37a] for the case in which there is only a proton-neutron interaction, and again derived for the most general form of central nuclear potential by BUCKINGHAM and MASSEY [41] and by HÖCKER [42]. We shall not reproduce the details of the derivation, which may best be looked up in Buckingham and Massey's paper*. The resulting form of the

* There is a strange error in this otherwise excellent paper. The authors give two separate treatments of the doublet states, (a) by using isotopic coordinates, (b) by treating

fundamental equation of the resonating group treatment of the 3-nucleon problem is

$$\left\{ \frac{3\hbar^2}{4M} \Delta + \varepsilon' - a U(r) - \eta C(r) \right\} F(\vec{x}) = \\ (\frac{4}{3})^3 \int \{ \beta Q(\vec{x}, \vec{x}') + \gamma [P(\vec{x}, \vec{x}') + (\varepsilon' - \frac{5}{3} \varepsilon_0) N(\vec{x}, \vec{x}')] \} F(\vec{x}') d\vec{v}'. \quad (7)$$

Here, $\varepsilon' = \varepsilon - \varepsilon_0$ denotes the difference between the total energy ε of the system and the binding energy ε_0 of the deuteron; in a state of relative motion of the third nucleon with respect to the deuteron, ε' is the kinetic energy of this motion, equal to $2/3$ the kinetic energy of the nucleon in the laboratory system (in which the deuteron is at rest). The other symbols are defined as follows: putting

$$\vec{u} = \frac{2}{3} (\vec{x} + 2\vec{x}'), \quad \vec{v} = \frac{2}{3} (\vec{x}' + 2\vec{x}), \quad (8)$$

and adopting the form (8.2-1 or 4) for the nuclear potential, — so that $J(r)$ represents the absolute value of the effective potential in the ground state of the deuteron, — one has

$$U(r) \equiv \int |\chi(r')|^2 J(|\vec{x} + \frac{1}{2}\vec{x}'|) d\vec{v}' \\ C(r) \equiv \int |\chi(r')|^2 \frac{e^2}{|\vec{x} + \frac{1}{2}\vec{x}'|} d\vec{v}' \\ Q(\vec{x}, \vec{x}') \equiv \chi(u) J(|\vec{u} - \vec{v}|) \chi(v) \\ P(\vec{x}, \vec{x}') \equiv -\chi(u) \left[\frac{1}{3} (J(u) + J(v)) + \eta \frac{e^2}{|\vec{u} - \vec{v}|} \right] \chi(v) + \frac{4\hbar^2}{3M} \text{grad}_u^2 \chi(u) \cdot \text{grad}_u^2 \chi(v) \\ N(\vec{x}, \vec{x}') \equiv \chi(u) \chi(v). \quad (9)$$

The constants a, β, γ are linear combinations of the interaction parameters, different for doublet and quartet states:

Doublet:

$$a = 2W - B - H + \frac{1}{3}M \\ \beta = \frac{1}{3}W - B - H + 2M \\ \gamma = -\frac{1}{3}$$

Quartet:

$$a = 2W + 2B - H - M \\ \beta = -W - B + 2H + 2M \\ \gamma = 1, \quad (10)$$

or, by (8.2-5),

$$a = 2(a_0 - 2a_\sigma) \quad a = 2(a_0 + a_\sigma) \\ \beta = \frac{1}{3}[a_0 - 5a_\sigma + 3(a_\tau - 5a_{\sigma\tau})] \quad \beta = -[a_0 + a_\sigma + 3(a_\tau + a_{\sigma\tau})] \\ \gamma = -\frac{1}{3} \quad \gamma = 1. \quad (11)$$

protons and neutrons as distinct particles; they seem to expect — and actually find — different results in the two cases, whereas both treatments are of course equivalent (4.14). In case (a), however, they use a wrong spin function (which actually represents a mixture of doublet and quartet states). Everything that concerns case (a) in their paper should therefore be omitted; their case (b) contains the correct results.

The constant η is, of course, 0 or 1 according as the third nucleon is a neutron or a proton.

14.111. Doublet states on individual model. The wave-function of a doublet state appears, on the individual model, in a somewhat different form. On this model, it is in fact natural to group the two like nucleons together. If we write the total wave-function in the form, analogous to (1),

$$\Psi(1,2,3) = \psi_{\text{ind}}(\widetilde{12},3) + \psi_{\text{ind}}(\widetilde{23},1) + \psi_{\text{ind}}(\widetilde{31},2), \quad (12)$$

the isotopic factor of the function $\psi_{\text{ind}}(\widetilde{12},3)$ will be

$$\tau(\widetilde{12},3) = {}^3(\tau)_{\pm 1}(1,2) u_{\mp}(3), \quad (13)$$

and we are primarily concerned with the antisymmetrical spin states

$${}^2\sigma_{\pm \frac{1}{2}}(\widetilde{12},3) = {}^1(\sigma)_0(1,2) v_{\pm}(3). \quad (14)$$

However, the symmetrical states ${}^2\sigma_{\pm \frac{1}{2}}(\widetilde{12},3)$ will also occur to some extent; the function $\psi_{\text{ind}}(\widetilde{12},3)$ will be a linear combination of the type

$$\psi_{\text{ind}}(\widetilde{12},3) = [\varphi_{\text{ind}}(\widetilde{12},3) \cdot {}^2\sigma(\widetilde{12},3) + \varphi_{\text{ind}}(\widetilde{12},3) \cdot {}^2\sigma(\widetilde{12},3)] \cdot \tau(\widetilde{12},3). \quad (15)$$

Formally, of course, our resonating group solution (1), (2), (3), (4) can be put into the individual model form (12), (13), (14), (15); one has just to take

$$\begin{aligned} \varphi_{\text{ind}}(\widetilde{12},3) &= \frac{3}{\sqrt{2}} [\varphi(\overline{31},2) + \varphi(\overline{23},1)] \\ \varphi_{\text{ind}}(\widetilde{12},3) &= \frac{1}{\sqrt{2}} [\varphi(\overline{31},2) - \varphi(\overline{23},1)]. \end{aligned} \quad (16)$$

But the procedure generally followed when treating the states of binding of ${}^3\text{H}$ or ${}^3\text{He}$ on the individual model consists in adopting for the $\varphi_{\text{ind}}(\widetilde{12},3)$ and $\varphi_{\text{ind}}(\widetilde{12},3)$ some simple analytical form involving a small number of parameters, which are then varied so as to minimize the expression for the total energy of the system.

After carrying out the summations over spin and isotopic coordinates, this last quantity (in the case of ${}^3\text{H}$) may be written

$$\begin{aligned} \epsilon = \frac{1}{N_s + N_a} \left[-\frac{\hbar^2}{2M} (K_s + K_a) + a_s J_s + a_a J_a + a'_s J'_s + a'_a J'_a \right. \\ \left. + a_s^{\text{ex}} J_s^{\text{ex}} + a_a^{\text{ex}} J_a^{\text{ex}} \mp a_{\times} J_{\times} \mp a_{\times}^{\text{ex}} J_{\times}^{\text{ex}} \right], \end{aligned} \quad (17)$$

with the following notation: the N 's are the normalization integrals

$$N_s = \int [\varphi(\widetilde{12},3)]^2 dv^{(1)} dv^{(2)} dv^{(3)}, \quad N_a = \int [\varphi(\widetilde{12},3)]^2 dv^{(1)} dv^{(2)} dv^{(3)}; \quad (17a)$$

the K 's pertain to the kinetic energy, the J 's to the potential energy, the

indices s and a signifying that the functions $\varphi(\bar{1}2,3)$ or $\varphi(\bar{1}\bar{2},3)$, respectively, enter into their definition:

$$\begin{aligned} K &= \int \varphi(12,3) (\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}) \varphi(12,3) dv^{(1)} dv^{(2)} dv^{(3)} \\ J &= \int \varphi(12,3) J(12) \varphi(12,3) dv^{(1)} dv^{(2)} dv^{(3)} \\ J' &= \int \varphi(12,3) J(23) \varphi(12,3) dv^{(1)} dv^{(2)} dv^{(3)} \\ J^{\text{ex}} &= \int \varphi(12,3) J(23) \varphi(31,2) dv^{(1)} dv^{(2)} dv^{(3)}; \end{aligned} \quad (17b)$$

finally,

$$\begin{aligned} J_{\times} &= \int \varphi(\bar{1}2,3) J(23) \varphi(\bar{1}\bar{2},3) dv^{(1)} dv^{(2)} dv^{(3)} \\ J'_{\times} &= \int \varphi(\bar{1}\bar{2},3) J(23) \varphi(\bar{3}\bar{1},2) dv^{(1)} dv^{(2)} dv^{(3)}. \end{aligned} \quad (17c)$$

The double signs in (17) refer to the substates with magnetic quantum numbers $\pm \frac{1}{2}$. The numerical coefficients a can be expressed in the following equivalent ways in terms of the interaction parameters (8.2-5, 9, 12):

$$\begin{aligned} a_s &= W - B - H + M = a_0 - 3a_\tau + a_\tau - 3a_{\tau\tau} = -q \\ a_a &= W + B - H - M = a_0 + a_\tau + a_\tau + a_{\tau\tau} = -^3p \\ a'_s &= 2W + B = 2(a_0 - a_\tau) = -\frac{1}{4}[3 + q + ^3p + ^1p] \\ a'_a &= 2W - B = 2(a_0 - 2a_\tau - a_\tau + 2a_{\tau\tau}) = -\frac{1}{4}[1 + 3q + ^3p + ^3^1p] \\ a_s^{\text{ex}} &= H + 2M = -2(a_\tau + 3a_{\tau\tau}) = -\frac{1}{4}[3 + q - ^3p - ^1p] \\ a_a^{\text{ex}} &= -H + 2M = 2(a_\tau - 5a_{\tau\tau}) = -\frac{1}{4}[1 + 3q - ^3p - ^3^1p] \\ \alpha_{\times} &= \sqrt{3}B = 2\sqrt{3}(a_s - a_{\tau\tau}) = -\frac{\sqrt{3}}{4}[1 - q + ^3p - ^1p] \\ \alpha_{\times}^{\text{ex}} &= \sqrt{3}H = -2\sqrt{3}(a_\tau - a_{\tau\tau}) = -\frac{\sqrt{3}}{4}[1 - q - ^3p + ^1p]. \end{aligned} \quad (17d)$$

Once the ^3H -problem has been solved, the energy of ^3He is easily obtained by adding the expectation value of the Coulomb interaction between the protos:

$$\epsilon(^3\text{He}) = \epsilon(^3\text{H}) + \frac{C_s + C_a}{N_s + N_a}, \quad C = \int \varphi(12,3) \frac{e^2}{r(12)} \varphi(12,3) dv^{(1)} dv^{(2)} dv^{(3)}. \quad (18)$$

Calculations performed according to this scheme (BROWN [39]) have shown that the amount of symmetrical spin state in (15) is relatively small; this affords some justification for the simplified procedure adopted in most treatments of the problem, viz. the complete neglect of $\varphi(\bar{1}2,3)$, so that all integrals with indices a and $\times$ disappear from equation (17). If, moreover, one chooses for $\varphi(\bar{1}\bar{2},3)$ a form completely symmetrical in all pairs of nucleons, the remaining J_s , J'_s and J_s^{ex} become equal, and on account of (17d) the problem further reduces to minimizing the expression

$$s = \frac{1}{N} \left[-\frac{3\hbar^2}{2M} K' - 3\frac{1+q}{2} J \right], \quad K' = \int \varphi(\bar{1}\bar{2}3) \Delta^{(1)} \varphi(\bar{1}\bar{2}3) dv^{(1)} dv^{(2)} dv^{(3)}. \quad (19)$$

It will be better to postpone the discussion of the results obtained for ^3H and ^3He by the above methods until we are in a position to compare them with those pertaining to the ^4He and ^5He nuclei. We shall now pass to the consideration of various reactions involving three nucleons.

14.12. Scattering of neutrons by deuterons. The total cross-section for elastic scattering of neutrons by deuterons has often been determined together with that of the proton (and other nuclei), and by the same methods. The data are collected in table 14.12. The measurements for thermal neutrons, which have been performed with heavy water as

14.12. Scattering cross-section of deuteron for neutrons		
Reference	Neutron energy MeV	Deuteron scattering cross-section 10^{-24} cm^2
GOLDHABER and BRIGGS [37]	thermal (300° K)	$4,0 \pm 0,8$
POWERS <i>et al.</i> [38]	id. (300° K)	$5,3 \pm 0,2$
	id. (120° K)	$6,5 \pm 0,3$
BEYER and WHITAKER [40]	id. (300° K)	$6,95 \pm 0,2$
CARROLL [41]	id. (300° K)	5,7
MARSHALL [46]	id. (300° K)	3,1
KIKUCHI and AOKI [39a]	$2,45 \pm 0,05$	$1,98 \pm 0,10$
AOKI [39]	2,21	$2,25 \pm 0,10$
	2,48	$2,19 \pm 0,10$
ZINN, SEELY and COHEN [39]	2,76	$2,10 \pm 0,10$
	$2,85 \pm 0,04$	$2,17 \pm 0,08$
AMALDI <i>et al.</i> [43]	4,1	$1,79 \pm 0,08$
	12,5	$0,78 \pm 0,12^*$
	13,5	$0,89 \pm 0,07^*$
	14	$0,86 \pm 0,03^*$
NUCKOLLS, BAILEY <i>et al.</i> [46]	0,35	$3,53 \pm 0,32$
	0,46	$3,32 \pm 0,16$
	0,72	$3,46 \pm 0,13$
	0,97	$2,97 \pm 0,14$
	$1,0 \pm 0,1$	$3,11 \pm 0,20$
	1,6	$2,90 \pm 0,12$
	2,0	$2,60 \pm 0,08$
	2,6	$2,34 \pm 0,06$
	3,0	$2,24 \pm 0,09$
	3,5	$2,04 \pm 0,07$
	4,0	$1,70 \pm 0,07$
	4,5	$1,69 \pm 0,05$
	5,0	$1,46 \pm 0,05$
	5,5	$1,52 \pm 0,08$
	6,0	$1,30 \pm 0,07$
* It has been verified that the reaction $D + n \rightarrow 2n + p$ (14.15) has a negligible cross-section.		

scatterer, have not been corrected for the effect of binding. The reduction factor necessary to get the free deuteron cross-section is hard to estimate; it probably lies between 0.5 and 0.7 (the lower limit (6.213–29b) being 0.44).

The *angular distribution* of the recoil deuterons* has been studied in the case of scattering of $D(d, n)$ -neutrons. KRUGER *et al.* [38], using the cloud chamber method (KRUGER *et al.* [37]; 6.42) to investigate the angular range $\Theta' = 0^\circ \dots 80^\circ$ for an energy of 2.6 MeV of the incident neutrons, found a peak in the angular dependence of the differential cross-section at about 60° ; but this result, based on the analysis of only 328 recoil deuterons does not seem to agree with BARSCHALL and KANNER's [40] ionization chamber investigation with neutrons of energy 2.44 ... 2.5 MeV, which do not show any departure from isotropy in the barycentric system in the range $\Theta' = 30^\circ \dots 55^\circ$, i.e. $\vartheta = 70^\circ \dots 120^\circ$. Recently, this investigation has been extended by COON, DAVIS and BARSCHALL [46a, b] to the whole range $\Theta' = 0^\circ \dots 55^\circ$, i.e. $\vartheta = 70^\circ \dots 180^\circ$. They found that the differential cross-section, as a function of ϑ , while remaining constant for $\vartheta = 70^\circ \dots 110^\circ$, rises from $\vartheta = 110^\circ$ to a peak value for $\vartheta = 180^\circ$, about twice as large as the value for $\vartheta = 110^\circ$. Finally, DARBY and SWAN [48], analyzing 902 recoil tracks by Dee and Gilbert's method (6.42), find, for neutrons of 2.53 MeV energy, an angular distribution qualitatively similar to that just described, but with a relatively much higher peak at $\vartheta = 180^\circ$.

From the theoretical point of view, the whole problem has been thoroughly discussed by BUCKINGHAM and MASSEY [41]**. Their work is based on the general equation (7) of the resonating group method; for the distance dependence of the nuclear potential, they adopt the exponential type, with the constants of table 6.432. The deuteron eigenfunction is then given by 5.211. The wave-amplitude $F(\vec{x})$ is expanded in harmonics:

$$F(\vec{x}) = \sum_l \frac{1}{r} f_l(r) P_l(\cos \vartheta); \quad (20)$$

* The relation (6.212–20) between the angle of recoil Θ' in the laboratory system and the scattering angle ϑ in the barycentric system is independent of the masses of the collision partners. The scattering angle Θ in the system in which the scatterer of mass AM is at rest is related to ϑ by the general formula

$$\operatorname{tg} \Theta = \sin \vartheta / \left(\cos \vartheta + \frac{1}{A} \right)$$

whence

$$d\Omega_{\text{bar}} = \left[\frac{2}{A} \cos \Theta + \frac{A^2 + \cos 2\Theta}{A\sqrt{A^2 - \sin^2 \Theta}} \right] d\Omega_{\text{lab}}.$$

** See also MASSEY and BUCKINGHAM [47a].

similarly, the kernels P , Q , N are expressed by series of harmonics of the type

$$Q(\vec{x}, \vec{x}') = \sum_l \frac{2l+1}{4\pi r r'} q_l(r, r') P_l(\cos \vartheta), \quad (21)$$

ϑ being the angle between the directions of $\vec{x}$ and $\vec{x}'$. The amplitudes $f_l(r)$ then satisfy integro-differential equations of the same general form as (7):

$$\frac{d^2 f_l}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} \right] f_l = \quad (22)$$

$$\frac{4M}{3\hbar^2} \left[\alpha U(r) + \left(\frac{4}{3} \right)^3 \int (\beta q_l(r, r') + \gamma [p_l(r, r') + (\epsilon' - \frac{5}{3}\epsilon_0) n_l(r, r')]) f_l(r') dr' \right],$$

$$k^2 \equiv \frac{4M}{3\hbar^2} \epsilon';$$

for positive ϵ' , they have the asymptotic form

$$f_l \simeq \sin(kr - \frac{1}{2}l\pi + \delta_l). \quad (23)$$

The phases δ_l , for both doublet and quartet waves, have to be determined by numerical integration of (22), the kernels q_l , p_l , n_l being themselves evaluated by numerical methods. This formidable task has been carried out by the authors for four values of the kinetic energy of the impinging

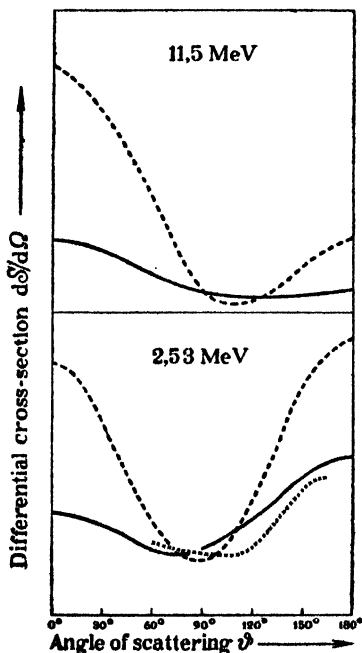


Fig. 14.12-1. Differential scattering cross-section of deuterons for neutrons. The differential cross-section (in arbitrary units) is given in terms of the scattering angle ϑ for neutrons of 11.5 MeV (upper curves) and 2.53 MeV (lower curves) kinetic energy; the full curves refer to the symmetrical theory, the dotted ones to the neutral theory. The crossed curve represents the observations of COON and BARSCHALL [46b].

neutrons, covering the range 0.25 ... 11.5 MeV (in the laboratory system), and for three different choices of the spin and charge dependence of the nuclear potential. For the energies involved, only S - and P -waves ($l=0, 1$)

need be considered. The three forms of nuclear interaction investigated include the neutral and symmetrical theories (8.32–16, 17), to which we shall confine ourselves here *. The information thus provided is indeed very valuable, since the angular distributions predicted by these two theories turn out to be widely different, even for relatively low neutron energies (fig. 14.12–1). Over the wide energy range surveyed by the calculations, the P -phases turn out to be much smaller on symmetrical than on neutral theory, with the result that the angular distribution is more uniform in the former than in the latter case.

The empirical material can be compared — disregarding the slight difference between the respective neutron energies — with the lower curves of fig. 14.12–1. Unfortunately, the discrepancy mentioned above between Coon and Barschall's results on the one hand and Darby and Swan's on the other is so large as to preclude any discrimination between the two theoretical possibilities. In fact, while the first named experiments, as shown on the figure, tend to support the predictions of the symmetrical

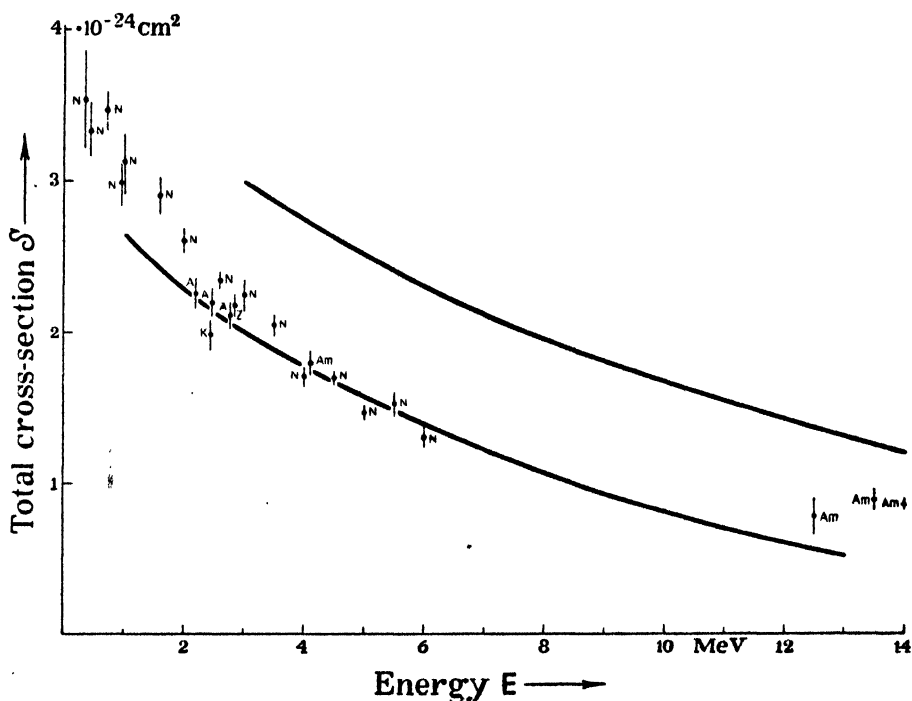


Fig. 14.12–2. Total elastic scattering cross-section of deuteron for neutrons, in terms of the kinetic energy of the neutrons (in the laboratory system). Lower curve according to symmetrical theory, upper curve according to neutral theory. Experimental points:

A Aoki K Kikuchi and Aoki Z Zinn, Seely and Cohen.
Am Amaldi *et al.* N Nuckolls, Bailey *et al.*

* The third type, a combination of Majorana and Heisenberg forces, yields results very similar to those of the symmetrical type.

theory, those of Darby and Swan seem to fit nicely the curve corresponding to the neutral theory.

A more definite conclusion can, however, be derived from the consideration of the total scattering cross-section. The theoretical values of this quantity as function of neutron energy, on both the neutral and the symmetrical theory, are illustrated by fig. 14.12-2, in which the relevant experimental results listed in the above table have also been plotted. The general trend of these results undoubtedly points to the range of values expected on symmetrical theory. The analysis of the saturation properties of the nuclear interactions which, as we have seen (11.33), also leads us to this theory, cannot claim the simplicity and directness of the evidence just discussed. It thus appears that *the analysis of the scattering of neutrons by deuterons furnishes at present the weightiest argument in favour of the symmetrical form of nuclear interaction*. In view of the importance of this conclusion, it is highly desirable that the experiments be repeated and extended with increased accuracy.

Further calculations of the S-wave scattering cross-section for different values of the range of the nuclear potential show that at higher neutron energies this quantity does not depend very much on the value assumed for the range, but that for very slow neutrons (for which it represents practically the whole cross-section) it becomes quite sensitive to this value: it increases with increasing range. Unfortunately, the slow neutron cross-section is, as shown by the table above, not sufficiently well known experimentally to allow of a determination of the range; at any rate, the range value derived from the proton-proton scattering data would seem not to be inconsistent with the neutron-deuteron evidence.

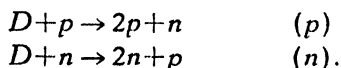
An independent calculation of the S-wave scattering cross-section for the neutron energy range 0 ... 3 MeV has been performed by HÖCKER [42], using essentially the same method, but somewhat different assumptions on the nuclear potential. He uses the symmetrical form of interaction, adopting a Gauss potential with quite inadequate constants, viz. $J = 26.4$ MeV, $\kappa^{-1} = d$, $q = 0.5$; he solves the integro-differential equation (22) for f_0 by a variational method. The values he gets for the cross-section are rather too low.

14.13. Scattering of protons by deuterons. The scattering of protons by deuterons is in serious need both of experimental investigation and of theoretical treatment. The only available measurements — discarding older ones of TUVE *et al.* [36] — are those of TASCHEK *et al.* [42, 47], with protons in the energy intervals 0.2 ... 0.3 MeV and 0.825 ... 3.53 MeV (using the same apparatus as in the study of proton-proton scattering, 7.12), and of POWELL *et al.* [47b], who investigated by the photographic method (6.42) the scattering of 4.2 MeV protons in deuterium and of 6.7 MeV deuterons in hydrogen. A detailed phase-shift analysis of the main data has been given by CRITCHFIELD [48]. The theory of the process

is dealt with rather summarily in two older papers by PRIMAKOFF [37] and OCHIAI [37]; however, more up-to-date calculations by MASSEY and BUCKINGHAM [48], along the same lines as their investigation of the neutron-deuteron system, are in progress. In the preliminary note quoted, the authors compare their theoretical results with the ratio of observed to Rutherford scattering at various angles, interpolated for 1.85 MeV. protons: at this energy, the symmetrical theory is found to lead to a remarkably good agreement, whereas the predictions from the neutral theory deviate markedly from the observed values.

14.14. Radiative capture of thermal neutrons by deuterons. The cross-section for the radiative capture of thermal neutrons by deuterons has been estimated by BORST and HARKINS [40] at $2 \dots 3 \cdot 10^{-28}$ cm² from measurements of the β -activity of the tritium (³H) formed in this reaction. The effect is connected, just as the analogous capture of neutrons by protons (6.52), with a change in magnetic moment of the system. Its theory, worked out by HÖCKER [42] (on the assumptions mentioned at the end of 14.11) and more exactly by BURHOP and MASSEY [48] (using the resonating group method outlined above), yields the right order of magnitude for the cross-section.

14.15. Disintegration of the deuteron by proton or neutron impact. Sufficiently fast nucleons can disintegrate a deuteron according to either of the reactions



Since a minimum energy of 2.18 MeV must be available in the barycentric system to make any one of these reactions possible, the threshold energy of the impinging nucleon is $2.18 \cdot \frac{3}{2} = 3.27$ MeV. The (*p*) reaction has been produced by BARKAS and WHITE [39b] with protons of 5.1 MeV. The deuterium gas was contained in a chamber with insulated lead lining for collecting the protons. The neutrons were slowed down by paraffin and measured with a silver detector. The cross-section was estimated at $\approx 1.4 \cdot 10^{-26}$ cm². The (*n*) reaction has hitherto not been studied directly*; in the course of their proton-neutron scattering experiments, AMALDI *et al.* [43] derived an estimate of the (*n*) cross-section for 14 MeV neutrons, viz. $4 \dots 9 \cdot 10^{-26}$ cm².

For the theoretical discussion of the (*n*) reaction, the reader is again referred to HÖCKER's paper [42]. The cross-section of the (*p*) reaction can, as also shown by Höcker, be derived from that of the (*n*) reaction by multiplying the latter by the penetrability of the deuteron barrier for

* The cloud chamber tracks of recoil particles from deuterium exposed to Rn-Be neutrons, observed by BAGGE [42], are of doubtful interpretation (see HÖCKER [42], AMALDI *et al.* [43]).

the proton (2.1). The theoretical results come out an order of magnitude larger than the crude empirical estimates just quoted.

14.2. Binding energies of the lightest nuclei

14.20. Numerous attempts have been made, starting from various assumptions on the form of the central potential of interaction between a pair of nucleons, to compute the binding energies of the nuclei ${}^3\text{H}$, ${}^3\text{He}$ and ${}^4\text{He}$. The discussion of 14.111, leading to formula (19), shows that the energy of the ground state of ${}^3\text{H}$ depends primarily on the interaction constants κ , J and q , i.e. on the range and the strengths of the effective potentials for even states. One can therefore try, in the first place, to adjust these parameters so as to account simultaneously for the observed values of the binding energy of ${}^3\text{H}$ and of the energies of the ${}^3\text{S}$ and ${}^1\text{S}$ levels of the deuteron (which we shall here call, for brevity, the "deuteron data"). It then remains to be seen whether agreement can also be obtained, in conformity with the charge-independence property, with the data from proton-proton scattering (the " p - p data") and with the observed mass-defect of ${}^4\text{He}$. Of course, other groupings of the items of empirical evidence just enumerated are possible, e.g. adjusting the parameters to the deuteron and p - p data and testing the resulting potential by the calculation of the binding energies of ${}^3\text{H}$ and ${}^4\text{He}$. In view of the margin left by the inaccuracy of the methods of computation of the latter quantities, these various possibilities may be helpful in elucidating different aspects of the consistency problem thus raised. In this section, we propose to review the discussions to which this problem has given rise and to draw our own conclusions. We begin with some preliminary remarks, carrying further the theory developed in 14.11 and 14.111.

14.21. Data and methods. Let us first of all collect the data of interest. The binding energies of the ${}^2\text{S}$ ground state of ${}^3\text{H}$ and the ${}^1\text{S}$ ground state of ${}^4\text{He}$, respectively, are

$$\begin{aligned}\varepsilon({}^3\text{H}) &= \varepsilon_3 = -8,3 \text{ MeV} \\ \varepsilon({}^4\text{He}) &= \varepsilon_4 = -28,2 \text{ MeV}.\end{aligned}\tag{1}$$

The triton ${}^3\text{H}$ is β -active, going over into ${}^3\text{He}$ with the emission of a rather slow electron: the maximum kinetic energy K_β of the electron spectrum is less than 15 keV (WATTS and WILLIAMS [46]). From the energy balance (1.13-1) of the process, with (1.22-5), we thus get

$$\varepsilon({}^3\text{He}) - \varepsilon({}^3\text{H}) = 1,47 m - K_\beta \quad (K_\beta < 0,03 m).\tag{2}$$

The calculation of this energy difference, by (14.111-18), affords a useful test of the accuracy of the method of approximation applied to the determination of ε_3 . We may state at this point that in all the papers which

will be discussed below, this test has been carried out with a satisfactory outcome; this will dispense us from mentioning the treatment of ${}^3\text{He}$ in the sequel. Finally, we recall for later reference, that the ground state of ${}^5\text{He}$ is a virtual level, involving the spontaneous decay of this nucleus into a neutron and an α -particle; its energy is 0.8 MeV above that of the latter, i.e.

$$\epsilon({}^5\text{He}) = -27.6 \text{ MeV}. \quad (3)$$

The methods of computation at our disposal are the resonating group method (14.11), the variational method (14.111), the variation-iteration method (5.24) and the ordinary perturbation method. As regards Svartholm's variation-iteration scheme, it has hitherto only been applied to the problem in hand, with very encouraging results, by Svartholm himself. The resonating group method, which might be expected to be the most reliable one, has been put to the test by Miss WAY [39b] in the case of ${}^3\text{H}$ and ${}^5\text{He}$: the conclusion of this study is rather disappointing. In the case of ${}^3\text{H}$, Miss Way carried through the whole calculation according to the general prescriptions of the method (13.3), but found that this yielded hardly better results than a simplified procedure, consisting in choosing for the wave-amplitude $F(\vec{x})$ a suitable analytic expression with variational parameters, thus reducing the problem to the usual variation principle*. She applied this method also to the ${}^2P(1s)^4 2p$ and ${}^2S[(1s)^4 2s + (1s)^3 (2p)^2]$ configurations of ${}^5\text{He}$. Both nuclei had been treated by the ordinary variational method, using very nearly the same specification of the nuclear potential (Gauss type), viz. ${}^3\text{H}$ by MARGENAU and WARREN [37] and ${}^5\text{He}$ by TYRRELL [39]. This refined variational treatment, involving trial functions of complicated form, proved superior to the resonating group method. In particular, only the former treatment is able to bring out the existence of a ${}^2S$ level of ${}^5\text{He}$ with nearly the same energy as the ${}^2P$ state (which is probably lower). The failure of the resonating group method in the case of ${}^3\text{H}$ might be attributed to the neglect of the polarization of the deuteron cluster by the loose neutron; but in the case of ${}^5\text{He}$, such polarization effects should be unimportant.

The methods most extensively used can be described as more or less elaborate applications of the variation principle; sometimes the perturbation method has also been employed. A critical study of the technical aspects

* In the course of his treatment (by the resonating group method) of the radiative capture of neutrons by deuterons (14.14), Höcker likewise tries to set up a simple expression for the amplitude $F(\vec{x})$ corresponding to the ground state of the triton (final state of the capture process). He does not, however, determine the "best" parameter values by the variation principle, but contents himself with fixing them in such a way that the expectation value of the energy is equal to the observed value. Burhop and Massey, on the other hand, replace the kernel of the integro-differential equation for $F(\vec{x})$ by an approximate expression, making possible an analytic solution of the equation; this procedure yields (fortuitously) an excellent result.

of the calculations and convergence questions involved will be found in a paper by WARREN and MARGENAU [37]. We shall here only point out that a rapid, but not always very reliable, orientation may be gained with the help of a peculiar mode of estimate of the energies ε_3 , ε_4 of ${}^3\text{H}$ and ${}^4\text{He}$, known as the reduction to "equivalent two-body problems", the idea of which goes back to WIGNER [33a]. The simple form in which it will be presented here is due to RARITA and PRESENT [37]. Let us first write down the variation principle, pertaining to ${}^4\text{He}$, in a simplified form analogous to equation (14.111–19) for ${}^3\text{H}$. We adopt a wave-function entirely symmetrical in the space coordinates:

$$\Psi = \varphi(1234) \frac{1}{\sqrt{4!}} \sum_P (\pm) P [{}^3(r)_1(\bar{1}2) \cdot {}^3(r)_{-1}(\bar{3}4) \cdot {}^1(o)_0(\bar{1}2) \cdot {}^1(o)_0(\bar{3}4)]; \quad (4)$$

in this formula, the summation extends over all permutations P of the numbers 1234, affected with the + or — sign according as they are even or odd. One then easily finds:

$$\varepsilon = \frac{1}{N} \left[-4 \frac{\hbar^2}{2M} K' - 6 \frac{1+q}{2} J \right], \quad (5)$$

the symbols N , K' , J being defined in complete analogy to those referring to ${}^3\text{H}$: in the formulae (14.111–17a, 17b, 19), one has just to replace $\varphi(123)$ by $\varphi(1234)$ and to integrate also over $dv^{(4)}$. To this ε must be added the average Coulomb energy C/N , with C analogous to the integral in (14.111–18). Now, it turns out that if the trial functions $\varphi(123)$ or $\varphi(1234)$ are taken as products of "Gaussian" functions of the type $\exp(-\frac{1}{2}\mu r^{(ik)2})$:

$$\varphi = \exp(-\frac{1}{2}\mu \sum_{i>k} r^{(ik)2}), \quad (6)$$

both equations (14.111–19) and (5) reduce to the same simple form, which is just the expression of the variation principle for a two-body problem. Elementary integrations yield in fact the following results:

$$\begin{aligned} {}^2\text{H} : \quad & \varepsilon = \frac{3}{2}\mu - J W(\mu) \\ {}^3\text{H} : \quad & \varepsilon = \frac{9}{2}\mu - 3 \frac{1+q}{2} J W(\frac{3}{2}\mu) \\ {}^4\text{He} : \quad & \varepsilon = 9\mu - 6 \frac{1+q}{2} J W(2\mu), \end{aligned} \quad (7)$$

with

$$W(\mu) = \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-x^2} w\left(\frac{x}{\sqrt{\mu}}\right) x^2 dx. \quad (8)$$

In other words, the variation problems for ${}^3\text{H}$ and ${}^4\text{He}$ are identical with

that of a two-nucleon system; one has just to establish between the respective potential constants the following correspondence:

$$\begin{aligned} {}^2\text{H} : & \quad \propto \quad J \\ {}^3\text{H} : & \quad \propto \sqrt{2}, \quad 3 \frac{1+q}{2} J \\ {}^4\text{He} : & \quad \propto \sqrt{3}, \quad 6 \frac{1+q}{2} J. \end{aligned} \quad (9)$$

The method of the "equivalent two-body problem" consists in taking as an estimate of the energy ε_3 or ε_4 the exact eigenvalue of the two-body wave-equation involving the corresponding interaction parameters as given by (9). This procedure has the advantage of simplicity, but it clearly lacks any theoretical justification. For some forms of distance dependence of the nuclear potential (Gauss, exponential), it may yield rather close approximations; but in other instances, e.g. the meson potential, in which the Gaussian function is but a poor representation of the deuteron eigenfunction, it is far less reliable.

14.22. Discussion of results. Computations of varying accuracy have been carried out for different types of nuclear interaction, viz. the exponential potential (RARITA and PRESENT [37]), the Gauss potential (MARGENAU and collaborators [37, 38a], SVARTHOLM [45]) and the meson potential (HYLLERAAS and RISBERG [41], BROWN [39], SVARTHOLM [45], FRÖHLICH *et al.* [46, 47], HUMBLET). Insofar as the simplified variation problems (14.111–19) and (5) are considered, or the calculation by some other method is limited to the first approximation, only the effective potentials for even states are concerned and the precise choice of the spin and charge dependence of the potential is immaterial (except for the charge independence property). For more accurate calculations, as well as for the investigation of excited states*, it is, strictly speaking, necessary to adopt a definite form: in the papers just quoted, either the symmetrical theory or a mixture of Majorana and Heisenberg forces has been used**; however, in practice it hardly makes any difference which choice is made. In fact, when judging the degree of accuracy of such calculations, one should not forget that they imply the complete neglect of possible non-central forces and, above all, of the contributions from direct 3-body and 4-body interactions. Especially the latter circumstance has not been properly recognized until recently; otherwise, the laborious investigations of which an account will now be given would probably have been led into more promising channels.

The consistency problem was raised for the first time in connexion

* MARGENAU and TYRRELL [38a] discuss some excited states of the four-nucleon (${}^4\text{He}$) system.

** Svathholm uses a mixture of Wigner and Bartlett forces and applies a correction to go over to the Majorana-Heisenberg case.

with Rarita and Present's treatment of the *exponential type of potential*. They found a set of potential constants (those given in table 6.432) consistent with both ϵ_3 and deuteron data; we may add (7.13) that they turn out to be consistent also with the p - p data. But when the ${}^4\text{He}$ energy $|\epsilon_4|$ is computed with these constants, it comes out considerably *too large*.

Margenau and his collaborators then took up the problem using a *Gauss potential*. By very elaborate variational calculations, they determined a set of parameters:

$$\kappa^{-1} = 2.25 \cdot 10^{-13} \text{ cm}, \quad J = 35.6 \text{ MeV}, \quad q = 0.6, \quad (10)$$

with which they were able to account for ϵ_3 and the deuteron data. The binding energy $|\epsilon_4|$, however, came out slightly too small; the observed value of ϵ_4 could be obtained by increasing the parameter q to 0.65, but with such a high (absolute) value of the ${}^1S$ effective potential, the ${}^1S$ level of the deuteron becomes an actual state of binding. Moreover, when the p - p data became available, they proved clearly incompatible with the constants (10): a smaller range was required (7.13). Resuming the calculation, by his variation-iteration method, with this smaller range, Svartholm could further show that the adjustment of the potential to the deuteron data and ϵ_3 was possible, but again yielded too large a value of $|\epsilon_4|$. Summing up, the two following conclusions emerge:

- (1) a set of parameters compatible with the deuteron and p - p data and with ϵ_3 yields too large a binding of ${}^4\text{He}$;
- (2) a set of parameters yielding the right binding energies of ${}^2\text{H}$, ${}^3\text{H}$ and ${}^4\text{He}$ involves a larger range than allowed by the p - p data and gives rise to a bound ${}^1S$ state of the deuteron.

Both these conclusions have been confirmed in the case of the *meson potential*. As shown by Hylleraas and Risberg, the second one readily follows from the application to this case of the "equivalent two-body" mode of estimate; the numerical results obtained by these authors are probably exaggerated by the inadequacy of the method, but their relations, as formulated in the above statement (2), are undoubtedly correct. The first conclusion results from the work of Brown and Svartholm: the former finds that the meson potential corresponding to the mass-value $M_m = 211 m$, — the strengths ${}^3J^{(0)}$, ${}^1J^{(0)}$ being given in terms of M_m by (6.432–8), — will suit both ϵ_3 and deuteron data; Svartholm then shows that an excessive binding of ${}^4\text{He}$ results. Moreover, Svartholm establishes the same conclusion for the mass-value $M_m = 326 m$ suggested as a possible one by the p - p data (7.13).

The first paper by FRÖHLICH *et al.* [46] contains a determination of the meson mass from the deuteron data and ϵ_3 , very similar to that of Brown, though based on the simplified variation principle (14.111–19); this leads to $M_m = 225 m$. An improved calculation, carried out in the second paper [47], yields $M_m = 210 m$ in complete agreement with Brown*. On the

* Fröhlich and his collaborators adopt the symmetrical theory, while Brown assumes a mixture of Majorana and Heisenberg forces.

other hand, it is very strange that — in contrast with all previous results, and especially with Svartholm's — the value of $|\epsilon_4|$ computed by Fröhlich *et al.* is smaller than the observed one (less than half the experimental value). There is presumably some error in this computation, which, owing to the special form adopted for the trial wave-functions, is very intricate. At any rate, a renewed calculation of $|\epsilon_4|$ by HUMBLET, using the usual "Gaussian" trial functions, yields results much nearer to Svartholm's. By varying the mass of the meson, Humblet shows that the value of the ${}^4\text{He}$ binding energy is very sensitive to this parameter, and increases (in absolute value) with increasing meson mass; it is true that he finds an $|\epsilon_4|$ smaller than the observed value for M_m as high as $\approx 240 m$, which slightly disagrees with Svartholm's calculation, but on the whole his results do not deviate from the general trend expressed by the above statements (1) and (2).

Now, what is the significance of these statements? Considering (2), it is clear that we must renounce any consistent account of the binding energies of the lightest nuclei by means of central two-body interactions alone. Conclusion (1) then embodies the most that we can expect in the way of consistent interpretation on this basis and tells us at the same time in which direction a first discrepancy arises. This discrepancy has often been commented on with undue pessimism, when one reflects on the narrowness of the fundamental assumption just recalled about the nature of the nuclear forces at play. It is indeed reasonable to suppose that the excess of binding of ${}^4\text{He}$ should be removed by forces which make their first appearance in a system of four nucleons, but are inoperative in systems composed of fewer particles; i.e. by direct four-body interactions, which in the special case of ${}^4\text{He}$ should have the effect of a repulsion. This inference, drawn both by Svartholm and by Fröhlich *et al.**, seems very cogent indeed. Of course, it still needs quantitative verification; but in its present state, meson field theory is not yet ripe for such problems.

14.23. Argument for like-nucleon forces. Early calculations of binding energies of ${}^3\text{H}$ and ${}^4\text{He}$ (B & B, §§ 20—21) were carried out with the help of a nuclear potential not presupposing the charge independence property, and the strengths of neutron-proton and like-nucleon potentials were determined independently so as to fit deuteron data as well as ϵ_3 and ϵ_4 . In this way, the existence of forces between like nucleons received a further confirmation and their order of magnitude was found to agree with that derived from the p - p data.

14.24. Stationary states of ${}^6\text{Li}$ and ${}^6\text{He}$. According to rough deductions from the quasi-atomic model (11.41), the ground state of the stable odd nucleus ${}^6\text{Li}$ should be a ${}^3\text{S}$ state, and the first excited level a ${}^3\text{D}$ state

* Owing to their aberrant result on ϵ_4 , these authors are naturally led to postulate a 4-nucleon attraction. On the other hand, they rightly observe that since ϵ_4 appears as the difference of two large quantities, viz. the kinetic and potential energies, nearly balancing each other, a relatively small contribution from 4-body forces to the potential energy would be sufficient to produce the big change required in their ϵ_4 -value.

(10.22). Detailed calculations by MARGENAU and CARROLL [38*b*], combining variational and perturbation methods, confirmed this, but disclosed at the same time the unreliability, from a quantitative point of view, of first order results concerning the spacing of levels of different orbital quantum numbers: in fact, the second order perturbations decrease considerably the ${}^3D$ - ${}^3S$ interval, without, however, reversing the order of the levels. A similar effect has been found by Tyrrell in his calculation (14.21) of the ${}^2S$ - ${}^2P$ interval in ${}^5\text{He}$.

The energy difference between the ground states of the isobaric nuclei ${}^6\text{He}$ and ${}^6\text{Li}$, connected by a β -transition ${}^6\text{He} \xrightarrow{\beta} {}^6\text{Li}$, can be deduced from the limiting energy of the β -rays; after allowing for the Coulomb energy, there remains a difference in proper nuclear energy amounting to 3.8 ± 0.6 MeV. This arises, of course, from the fact that the ${}^6\text{He}$ configuration, due to the exclusion principle, is a ${}^1S$ one. Assuming charge independence of the nuclear forces, we may say that the difference in question represents the ${}^1S$ - ${}^3S$ separation for 6-nucleon systems. Its theoretical calculation has been carried out by MARGENAU [39]; in this case, in which we are dealing with states of the same orbital momentum, the second order effects have actually little influence. In principle, we might therefore hope to derive from comparison with experiment definite information about the spin dependent part of the nuclear potential. However, — quite apart from the uncertainty regarding possible contributions from other than central interactions, — the empirical value itself is too inaccurate to allow of any such deductions.

SUMMARY OF PART III

In this Part we have been concerned with two main lines of argument. On the one hand, we have discussed how far it is possible to account for the properties of complex nuclei, according to various *structural models*, on the assumption of a central interaction between pairs of nucleons. In particular, a tendency has been revealed towards the formation, within such a nucleus, of transient clusters of nucleons, especially α -clusters (12.23); and nuclear models have been described in which account is taken of this *cluster structure* (13.3).

On the other hand, we have considered the additional evidence from complex nuclei concerning the exact form of the *isotopic factor of the law of nuclear interaction*. In this respect, the saturation properties of nucleon binding (11.33) and the scattering of neutrons by deuterons (14.12) have both been found to yield evidence in favour of the *symmetrical* form of nuclear potential (8.31). This conclusion completes the determination of the elementary law of force, insofar as it may be — at least in first approximation — regarded as a central two-body interaction.

There are, however, quite definite indications as to the necessary corrections to be applied to this description. The computation of the binding energies of the lightest nuclei has revealed the far from negligible role played by many-body forces (14.22). Moreover, as we know (6.12), a certain amount of non-central coupling is necessary to explain the electromagnetic properties of the deuteron. While little can be said at present about many-body forces, considerable work has already been devoted to a closer study of non-central interactions. Our last task will now be to survey the preliminary results, as yet inconclusive, pertaining to this important question.

PART IV

NON-CENTRAL AND NON-STATIC COUPLINGS

CHAPTER XV

GENERAL TYPES OF NON-CENTRAL AND NON-STATIC NUCLEAR FORCES

15.1. General empirical evidence

The analysis of the electromagnetic properties of the ground state of the deuteron (6.12, 6.121, 6.122) has made it clear that the proton-neutron interaction involves a non-central term, as the result of which the ${}^3S$ ground state contains a small admixture of ${}^3D_1$ state. The occurrence of the non-central coupling is a direct consequence of the existence of an electric quadrupole moment in this state. That the admixture of ${}^3D$ state in the case of the deuteron is very small can be concluded from a comparison of the actual magnetic moment μ_d with the value $\mu_d^0 = \mu_n + \mu_p$ pertaining to a pure ${}^3S$ state. The values of the magnetic moments of the other stable odd nuclei confirm the inferences drawn from the deuteron data and illustrate the increasing importance of the non-central couplings with increasing complexity of the nucleus (A2.26).

Other evidence from nuclei more complex than the deuteron, such as that derived from the fine structure of the levels or, more indirectly, from the magnetic moments, is, as we shall see, much less definite. In fact, the general conclusions which the preceding study of heavier nuclei from the standpoint of central interaction permits do not open very encouraging prospects of a more detailed analysis of this interaction: on the one hand, the gross features of nuclear properties are to a surprising extent insensitive to even large variations of the law of nuclear force; on the other, the investigation of finer details, which might in principle yield useful information on the precise form of this law, would require an accuracy of theoretical treatment which present methods would generally not permit. However, very general and elementary arguments of covariance will at least carry us a first step forward, by enabling us to enumerate the simplest forms of interaction potentials possible *a priori*.

15.2. Survey of nuclear interactions

15.21. Static and first order interactions. In trying to determine the various possible types of nuclear forces, we shall first of all neglect many-body forces (1.34) and restrict the discussion to interactions between pairs of nucleons. The question is, of course, essentially non-relativistic: a relativistic formulation of interaction laws is not possible without the help of the field concept. The introduction of an interaction potential is only justified insofar as the velocities of the constituent nucleons remain on the average sufficiently small (2.3), and we may therefore consistently limit

ourselves to expressions involving no higher power of the momenta than the first. We need not consider any explicit dependence on the variables q_i , since the latter variables can always be reduced to the momenta. We have thus to construct interaction operators of the general form $\frac{1}{2}\Sigma\mathcal{V}^{(ik)}$, in which $\mathcal{V}^{(12)}\equiv\mathcal{V}(Q^{(1)}, Q^{(2)})$ is a linear function of the momenta of the two nucleons (1) and (2), symmetric with respect to the sets of coordinates $Q^{(1)}, Q^{(2)}$, and further satisfying all invariance requirements listed under (a) ... (f) in 4.31. The following discussion has been given in two papers by WIGNER [37a] and EISENBUD and WIGNER [41].

Let us first fix our attention on the rotation and spatial symmetry covariance. As regards the spatial variables, we have at our disposal (suppressing the indices (12))

$$\begin{aligned} 2 \text{ scalars:} & \quad r \equiv |\vec{x}|, & \quad \vec{x} \cdot \vec{p}; \\ 2 \text{ polar vectors:} & \quad \vec{x}, & \quad \vec{p}; \\ 1 \text{ axial vector:} & \quad \vec{x} \wedge \vec{p}; \\ 2 \text{ tensors:} & \quad T_{lm}^{(x)} = x_l x_m, & \quad T_{lm}^{(xp)} = x_l p_m. \end{aligned} \tag{1}$$

With the axial vectors $\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}$ we can further form

$$\begin{aligned} 2 \text{ scalars:} & \quad 1, \quad \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}; \\ 3 \text{ axial vectors:} & \quad \vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}, \quad \vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}, \quad \vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)}; \\ 1 \text{ tensor:} & \quad T_{lm}^{(\sigma)} = \sum_n \sigma_{ln}^{(1)} \sigma_{nm}^{(2)}. \end{aligned} \tag{2}$$

In writing the tensor components, we denote by x_i, p_i ($i = 2, 2, 3$) the components of $\vec{x}$ and $\vec{p}$ and likewise we put

$$\sigma_{12} \equiv \sigma_z \quad (\text{cycl}).$$

We have to combine the various quantities (1) and (2) in such a way as to fulfil in the first place the requirements (b), (c), (d) of dynamical and spatial invariance, which do not involve the isotopic variables. To obtain invariants from the tensors $T^{(x)}, T^{(xp)}, T^{(\sigma)}$ in the most convenient form, it is best to start from the traceless tensors $T_{lm} - \frac{1}{3} \delta_{lm} T$ (T denoting the trace $\sum_l T_{ll}$). If $T^{(\alpha)}, T^{(\beta)}$ are any two of our tensors, $T^{(\sigma)}$ being symmetrized if necessary, we define an invariant by

$$\begin{aligned} I^{(\alpha\beta)} & \equiv \sum_{l,m} (T_{lm}^{(\alpha)} - \frac{1}{3} \delta_{lm} T^{(\alpha)}) (T_{lm}^{(\beta)} - \frac{1}{3} \delta_{lm} T^{(\beta)}) \\ & = \sum_{l,m} T_{lm}^{(\alpha)} T_{lm}^{(\beta)} - \frac{1}{3} T^{(\alpha)} T^{(\beta)}; \end{aligned} \tag{3}$$

the traces of the tensors we have to use are

$$T^{(x)} = r^2, \quad T^{(xp)} = \vec{x} \vec{p}, \quad T^{(\sigma)} = -2 \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}. \quad (4)$$

An easy discussion leads to the following independent types of invariants:

$$1, \quad \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \quad (5a)$$

$$r^2 \mathcal{D}^{(12)} \equiv (\vec{x} \vec{\sigma}^{(1)}) (\vec{x} \vec{\sigma}^{(2)}) - \frac{1}{3} r^2 \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \quad (5b)$$

$$(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \cdot (\vec{x} \wedge \vec{p}) \quad (5c)$$

$$(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) \cdot (\vec{x} \wedge \vec{p}) \quad (5d)$$

$$(\vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)}) \cdot (\vec{x} \wedge \vec{p}); \quad (5e)$$

$$\begin{aligned} & \vec{x} \vec{p}, \quad (\vec{x} \vec{p}) (\vec{\sigma}^{(1)} \vec{\sigma}^{(2)}), \quad \vec{x} \vec{p} \mathcal{D}^{(12)}, \\ & \frac{1}{2} [(\vec{x} \vec{\sigma}^{(1)}) (\vec{p} \vec{\sigma}^{(2)}) + (\vec{x} \vec{\sigma}^{(2)}) (\vec{p} \vec{\sigma}^{(1)})] - \frac{1}{3} (\vec{x} \vec{p}) (\vec{\sigma}^{(1)} \vec{\sigma}^{(2)}). \end{aligned} \quad (6)$$

Each of these may be multiplied by an arbitrary function of the distance r .

Turning to the charge invariance requirement (f), we see that it can be satisfied by six distinct operators built up with the help of the isotopic variables $\tau^{(1)}$, $\tau^{(2)}$, viz.

$$1, \quad \tau^{(1)} \tau^{(2)}, \quad \tau_3^{(1)} \tau_3^{(2)}, \quad \tau_3^{(1)} + \tau_3^{(2)} \quad (7a)$$

$$\tau_3^{(1)} - \tau_3^{(2)} \quad (7b)$$

$$(\tau^{(1)} \wedge \tau^{(2)})_3 = 2i (\Pi^{(1)\dagger} \Pi^{(2)} - \Pi^{(1)} \Pi^{(2)\dagger}) = 2 \Pi^{(12)}. \quad (7c)$$

In (7c) use has been made of the notation introduced in (4.41-2). The most general interaction operator for a pair of nucleons will be a linear combination of these six operators, the coefficients of which will be suitable functions of the other variables. The allowed combinations of isotopic factors (7) with spatial invariants (5) and (6) are essentially limited by the requirement (e) of invariance for a change of sign of the time. On that account, the isotopic factors (7a) and (7b) can only be combined with the invariants of types (5a) to (5d); but since the resulting expression must also be invariant for an interchange of *all* coordinates of the two nucleons, (7a) is compatible only with (5a), (5b), (5c), while (7b) belongs to (5d) only. Since (7c) changes sign when the Π 's and $\Pi^\dagger$'s are interchanged, it can only be associated with either (5e) or (6), which likewise change sign when the signs of the $\vec{p}$'s and $\vec{\sigma}$'s are changed; but the symmetry with respect to the sets of coordinates of both nucleons excludes all invariants of types (6). The requirement (a) of being Hermitian, which is the last we still have to consider, is obviously fulfilled by all remaining combinations.

As a result of the preceding discussion, we are thus left with a relatively small number of possible forms of nuclear potentials, each of which can be represented by a product of three factors: one of them, expressing the distance dependence, remains arbitrary; the second one embodies the dependence on spin, angle and momentum variables, the third is the isotopic factor. The following table enumerates the various types, with the names and symbols by which they will henceforth be denoted; the notation $\vec{x}_0$ has been used for the unit vector $\vec{x}/r$.

15.21. Types of static and first order nuclear interactions		
Dependence on spin, angle and momentum variables		Dependence on charge (isotopic factor)
I. central	$1, \quad \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$	$\left. \begin{array}{l} \text{any linear} \\ \text{combination} \\ \text{of the factors} \end{array} \right\} \begin{array}{l} 1. \text{ neutral } 1 \\ 2. \text{ symm. } \tau^{(1)} \tau^{(2)} \\ 3. \text{ charged } \tau^{(12)} \end{array}$
II. axial dipole	$\mathcal{D}^{(12)} \equiv (\vec{\sigma}^{(1)} \vec{x}_0) (\vec{\sigma}^{(2)} \vec{x}_0) - \frac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$	
III. spin-orbit	$\mathcal{M}^{(12)} \equiv (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) (\vec{x} \wedge \vec{p})$	
IV. spin-orbit of the 2nd kind	$\mathcal{M}_2^{(12)} \equiv (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) (\vec{x} \wedge \vec{p})$	
V. spin-orbit of the 3rd kind	$\mathcal{M}_3^{(12)} \equiv (\vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)}) (\vec{x} \wedge \vec{p})$	
		$\Pi^{(12)} \equiv \frac{1}{3} (\tau^{(1)} \wedge \tau^{(2)})_3$

With respect to the velocity dependence, we distinguish static and non-static couplings; the static ones may be either central (type I) or non-central (type II), while the non-static ones are all angle dependent. The interaction $\mathcal{D}^{(12)}$ is the familiar expression occurring in the interaction energy of two magnetic dipoles (6.13–20); hence its name *axial dipole coupling*. The non-static interaction $\mathcal{M}^{(12)}$ also represents a type well-known especially in atomic theory under the name *spin-orbit coupling*. The names of the isotopic factors which can be associated with any of the types I, II, III, of interaction have been chosen with reference to meson field theory (8.31). The factor $\tau_3^{(1)} + \tau_3^{(2)}$ has been left out, since its occurrence is excluded by the equality of proton-proton and neutron-neutron nuclear interactions (3.3). The types IV and V of non-static couplings are only listed for the sake of completeness; they are indeed very unlikely to play any role owing to their peculiar isotopic factors, which have non-vanishing matrix-elements only for proton-neutron interaction and are thus incompatible with the charge independence property of the nuclear potential.

15.22. Second order interactions. The preceding survey must be completed by an examination of the interactions of second order in the nucleon velocities which are compatible with the requirement of Lorentz invariance of the Hamiltonian up to that order. This discussion will be of importance in showing, first, that a potential energy, function of the variables of the nucleon pair, may be introduced in the relativistic Hamiltonian of the two-nucleon system as long as effects of the first order only

are considered (4.31); secondly, that second order interactions are generally so small compared with those of lower order that it is permissible in most cases to neglect them. Attacked for the first time by INGLIS [36], the problem has later been treated more systematically by BREIT [37a, 38a], to whose papers the reader is referred for a detailed study. We shall here discuss in full a simple case by way of illustration and only briefly mention more general results obtained in essentially the same way.

Let us first consider the behaviour of the nucleon moving in a field of force described in unrelativistic approximation by a potential energy $\mathcal{V}_n$; on the quasi-atomic model, this might be taken as the average interaction of the nucleon with the other constituent particles of a nucleus. At the same time we shall also introduce an electrostatic potential energy $\mathcal{V}_e$, in order to bring out the characteristic differences in the treatments of the nuclear energy $\mathcal{V}_n$ and the electromagnetic energy $\mathcal{V}_e$. The latter, in fact, behaves as the time-component of a four-vector; whereas, according to an elegant procedure suggested by FURRY [36], we may regard the nuclear energy term as a scalar, and accordingly add it to the mass term of the Hamiltonian. In this way, we get (4.221-9) the invariant expression

$$\mathcal{H} = \varrho_1 \vec{\sigma} \cdot \vec{p} + T_- \mathcal{V}_e + \varrho_3 (M + \mathcal{V}_n) - \frac{1}{2} (g_s - 2T_-) \mu_0 \vec{\sigma} \cdot \vec{H}_e, \quad (8)$$

in which g_s denotes the Landé factor $2\mu_n$ or $2\mu_p$ of the nucleon (A2.22-11) and $\vec{H}_e$ the magnetic field acting on its anomalous magnetic moment. In fact, the energy contribution due to the anomalous moment, being invariant*, can be calculated in the system of reference in which the nucleon is momentarily at rest; in this system, the nucleon is acted upon by a magnetic field $\vec{H}_e = \vec{E} \wedge \vec{v}/c$, $\vec{E}$ being the electric field $-\frac{1}{c} \text{grad } \mathcal{V}_e$ and $\vec{v} \approx \vec{cp}/M$ the velocity of the nucleon in the original system of reference.

The wave-equation derived from the Hamiltonian (8) can be reduced in the usual way to an equation for the "large" components only (4.21, 4.33). The latter corresponds to the following reduced Hamiltonian, exact to the second order in the nucleon velocities:

$$\begin{aligned} \mathcal{H}_{\text{red}} = & \frac{p^2}{2M} + T_- \mathcal{V}_e + \mathcal{V}_n - \frac{1}{2M} [(\varepsilon - T_- \mathcal{V}_e)^2 - \mathcal{V}_n^2] \\ & + \frac{i\hbar}{(2M)^2} \text{grad } (\mathcal{V}_n - T_- \mathcal{V}_e) \cdot \vec{p} - \frac{\hbar}{(2M)^2} \vec{\sigma} [\text{grad } (\mathcal{V}_n - T_- \mathcal{V}_e) \wedge \vec{p}] \\ & + \frac{\hbar}{(2M)^2} (g_s - 2T_-) \vec{\sigma} [\text{grad } \mathcal{V}_e \wedge \vec{p}]; \end{aligned} \quad (9)$$

In this formula, ε denotes the unrelativistic energy of the system. The last two terms can be combined to give a second order interaction

$$\mathcal{V}^{(2)} = -\frac{\hbar}{(2M)^2} \vec{\sigma} \{ \text{grad } [\mathcal{V}_n - (g_s - T_-) \mathcal{V}_e] \wedge \vec{p} \}, \quad (10)$$

which, in the case of central potentials $\mathcal{V}_n$ and $\mathcal{V}_e$, takes the form of a spin-orbit coupling

$$\mathcal{V}^{(2)} = -\frac{\hbar}{(2M)^2} \frac{1}{r} \frac{d}{dr} [\mathcal{V}_n - (g_s - T_-) \mathcal{V}_e] \vec{\sigma} (\vec{r} \wedge \vec{p}). \quad (11)$$

INGLIS [36] gives a more physical analysis of this *relativistic spin-orbit coupling*, by observing that it consists of the superposition of two distinct contributions. In the first place, the magnetic dipole due to the spin of the nucleon, moving in the electric field $\vec{E}$, can be regarded, in the reference system R in which the particle is momentarily at rest, as acted upon by the magnetic field $\vec{H}_e$ and accordingly performing, in this reference system, a Larmor precession of angular velocity (divided by c)

$$\vec{\omega}_L = -\frac{e}{2M} g_s \vec{H}_e = \frac{g_s}{2M^2} (\text{grad } \mathcal{V}_e \wedge \vec{p}). \quad (12)$$

When we transform to the original system of reference, we have to introduce, owing to the acceleration of the nucleon, the well-known Thomas precession*, affecting the whole reference system R . Its angular velocity is

$$\vec{\omega}_T = \frac{1}{2} \frac{d}{cdt} \left(\frac{\vec{v}}{c} \right) \wedge \frac{\vec{v}}{c} = -\frac{1}{2M} \text{grad } (\mathcal{V}_n + T_- \mathcal{V}_e) \wedge \frac{\vec{p}}{M}. \quad (13)$$

The total (kinetic) energy change due to these precessions amounts to $\frac{1}{2} \hbar \sigma (\vec{\omega}_L + \vec{\omega}_T)$, which, according to (12) and (13), is just (10). While the nuclear potential $\mathcal{V}_n$ naturally gives rise only to a Thomas precession, the electric potential contributes also a Larmor term; as a result, the effect of $\mathcal{V}_n$ is enhanced by that of $\mathcal{V}_e$ in the case of a proton, counteracted in the case of a neutron: this follows indeed, according to (10), from the fact that $\mathcal{V}_n$ and $\mathcal{V}_e$ are of opposite signs (the former attractive, the latter repulsive), while $g_s - T_-$ is positive for a proton, negative for a neutron.

As already stated, the Hamiltonian (8) or (9) might be used to describe, on the quasi-atomic model, the stationary states of an odd-mass nucleus in which all the constituent nucleons but one are in their lowest configuration with saturated spins. Such stationary states will be doublets, and in the absence of any other non-central coupling, the splitting of these doublets into their components $J = L \pm \frac{1}{2}$ would be entirely effected by the spin-orbit coupling $\mathcal{V}^{(2)}$. This effect is most readily visualized by a comparison with the well-known case of an atomic system with one optical electron. The atomic problem is formally included in our formulae, if we put $\mathcal{V}_n = 0$, $g_s = 2$, $T_- = 1$; according to (10), we have here

$$\mathcal{V}_{\text{atom}}^{(2)} = \frac{\hbar}{(2m)^2} \sigma (\text{grad } \mathcal{V}_{e, \text{atom}} \wedge \vec{p}). \quad (14)$$

Since $\mathcal{V}_{e, \text{atom}}$ is attractive just as $\mathcal{V}_n$, the sign of $\mathcal{V}_{\text{atom}}^{(2)}$ will be opposite to that of $\mathcal{V}^{(2)}$ in the nuclear case, provided the nuclear potential is predominant. This sign inversion arises from the large Larmor effect in the atomic case: according to (12), (13), one has here $\vec{\omega}_L = -2\vec{\omega}_T$, so that the sign of $\mathcal{V}_{\text{atom}}^{(2)}$ is fixed by $-\vec{\omega}_T$, as against $+\vec{\omega}_T$ in the nuclear case. We conclude that *the order of the doublet terms in an odd-mass nucleus, as a result of the relativistic spin-orbit coupling, will in general be the reverse of that of the one-electron atomic spectra*. In the latter case, the order of the levels would be regular: the lower state would have the lower

* See, e.g., DANCOFF and INGLIS [36].

J-value. In the nuclear spectrum, the doublets should therefore be inverted: the lower state should have the higher *J*-value. This rule, enunciated by INGLIS [36], will hold generally if the odd nucleon is a proton; if it is a neutron, it will only hold provided the Thomas effect of nuclear origin is larger than the Larmor effect (of opposite sign) due to the neutron's anomalous magnetic moment.

In atomic theory, it is an easy matter to extend the discussion of multiplet splitting to many-electron spectra (assuming normal coupling conditions): the electrons can be regarded as moving in a central field, and their mutual interaction neglected in the initial approximation. One finds regular or inverted multiplets according as the last orbital shell is less or more than half filled up. On the quasi-atomic nuclear model, we might repeat, *mutatis mutandis*, the same argument. The electrostatic energy $\mathcal{V}_e$ would be much smaller in absolute value than the fictitious central field $\mathcal{V}_n$ representing the bulk of nuclear interaction: Inglis' rule could therefore be generalized for any nuclear multiplets in the simple form: *nuclear multiplets will be inverted or regular according as the last orbital shell is less or more than half filled*. As to the application of this theoretical rule to actual nuclei, however, one should not lose sight of the two very questionable assumptions on which it rests: the idealization of the quasi-atomic model and the assumption that, besides the relativistic spin-orbit coupling, there would be no other non-central coupling of comparable or even larger order of magnitude. In fact, the doubtful validity of either assumption in most cases renders the above rules quite unreliable; they have been mentioned here chiefly to warn the reader against their uncritical use*.

In order to obtain a more general expression for the relativistic interaction between pairs of nucleons, it suffices to discuss the case of two nucleons. The extension of Furry's method to this case leads to simple results when the static nuclear potential does not depend on the spins of the nucleons (Wigner and Heisenberg forces). The corrections $\mathcal{H}^{(2)}$ to be added to the Hamiltonian

$$\mathcal{H}_{\text{free}} = \vec{a}^{(1)} \vec{p}^{(1)} + \vec{a}^{(2)} \vec{p}^{(2)} + (\varrho_3^{(1)} + \varrho_3^{(2)}) M \quad (15)$$

of a pair of free nucleons (we have put $\vec{a} = \varrho_1 \vec{\sigma}$) in order to take account of the interaction in an approximately invariant way (exact to the second order in the velocities) have been derived by BREIT [37a, 38a] for the case that the non-relativistic potential is treated either as the time-component of a 4-vector (we then call it $\mathcal{V}_e$) or as a scalar $\mathcal{V}_n$: he finds

$$\mathcal{H}_e^{(2)} = \mathcal{V}_e - \frac{1}{2} \left[\vec{a}^{(1)} \vec{a}^{(2)} - (\vec{a}^{(1)} \vec{x})(\vec{a}^{(2)} \vec{x}) \frac{1}{r} \frac{d}{dr} \right] \mathcal{V}_e \quad (16)$$

and

$$\mathcal{H}_n^{(2)} = -\varrho_3^{(1)} \varrho_3^{(2)} \mathcal{V}_n - \frac{1}{2} \left[\vec{a}^{(1)} \vec{a}^{(2)} + (\vec{a}^{(1)} \vec{x})(\vec{a}^{(2)} \vec{x}) \frac{1}{r} \frac{d}{dr} \right] \mathcal{V}_n. \quad (17)$$

respectively. The interaction $\mathcal{H}_e^{(2)}$, which is, of course, entirely analogous to

* See, for instance, the comparative estimates of the relativistic spin-orbit coupling of ${}^7\text{Li}$ on the quasi-atomic and α -particle models (17.53).

the electromagnetic one, gives rise, when the reduction to large components is carried out, to 3 second order terms of coupling between the nucleons*:

(a) a Thomas spin-orbit coupling

$$\frac{\hbar}{(2M)^2} \frac{1}{r} \frac{d\mathcal{V}_e}{dr} [(\vec{x} \wedge \vec{p}^{(1)}) \vec{\sigma}^{(1)} - (\vec{x} \wedge \vec{p}^{(2)}) \vec{\sigma}^{(2)}],$$

(b) a "Larmor" spin-orbit coupling

$$-2 \frac{\hbar}{(2M)^2} \frac{1}{r} \frac{d\mathcal{V}_e}{dr} [(\vec{x} \wedge \vec{p}^{(2)}) \vec{\sigma}^{(1)} - (\vec{x} \wedge \vec{p}^{(1)}) \vec{\sigma}^{(2)}],$$

(c) an interaction analogous to that between two magnetic dipoles**

$$\frac{\hbar^2}{(2M)^2} \left\{ \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \cdot \frac{2}{3} \Delta \mathcal{V}_e + \left[\frac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} - (\vec{x}_0 \vec{\sigma}^{(1)}) (\vec{x}_0 \vec{\sigma}^{(2)}) \right] \vec{r} \frac{d}{dr} \left(\frac{1}{r} \frac{d\mathcal{V}_e}{dr} \right) \right\}.$$

The interaction $\mathcal{H}_n^{(2)}$ gives rise only to a Thomas spin-orbit coupling of opposite sign to that resulting from $\mathcal{H}_e^{(2)}$.

If the nuclear potential involves a spin exchange operator P_σ (Bartlett and Majorana forces), the expression to be added to $\mathcal{H}_{\text{free}}$ becomes more complicated; Breit gives it only for the case that $\mathcal{V}_e P_\sigma$ is treated as the time-component of a 4-vector, in the form

$$\begin{aligned} \mathcal{H}_e^{(2)} = & \mathcal{V}_e P_\sigma - \frac{1}{2} \left[\vec{a}^{(1)} \vec{a}^{(2)} - (\vec{x} \vec{a}^{(1)}) (\vec{x} \vec{a}^{(2)}) \right] \frac{1}{r} \frac{d}{dr} \mathcal{V}_e P_\sigma \\ & + \frac{i}{2M} \left[(\vec{a}^{(1)} \vec{x}) \frac{1}{r} \frac{d\mathcal{V}_e}{dr} (\vec{x} \wedge \vec{p}^{(2)}) \vec{\sigma}^{(2)} - (\vec{x} \wedge \vec{p}^{(2)}) \vec{\sigma}^{(1)} (\vec{a}^{(2)} \vec{x}) \frac{1}{r} \frac{d\mathcal{V}_e}{dr} \right. \\ & \left. + \mathcal{V}_e \vec{a}^{(1)} (\vec{p}^{(12)} \wedge \vec{\sigma}^{(2)}) - (\vec{p}^{(12)} \wedge \vec{\sigma}^{(1)}) \vec{a}^{(2)} \mathcal{V}_e \right] P_\sigma. \end{aligned} \quad (18)$$

But Breit observes further that Furry's method does not yield the most general expressions compatible with the requirement of approximate invariance. He constructs expressions of the same general form but involving several arbitrary constants; the interactions derived by Furry's method appear as special cases of those general expressions: on account of their origin, they might perhaps be more likely than others. For spin independent nuclear forces, the total spin-orbit coupling can be of the form

$$- \frac{\hbar}{(2M)^2} \frac{1}{r} \frac{d\mathcal{V}}{dr} \left\{ [\vec{x} \wedge (\vec{a}' \vec{p}^{(1)} + (1-\vec{a}') \vec{p}^{(2)})] \vec{\sigma}^{(1)} - [\vec{x} \wedge (\vec{a}'' \vec{p}^{(2)} + (1-\vec{a}'') \vec{p}^{(1)})] \vec{\sigma}^{(2)} \right\}$$

* Besides these terms, there also appear other relativistic corrections involving, for a system of nucleons, the operator $\sum_{i,k} \vec{p}^{(ik)} \text{grad}^{(i)} \mathcal{V}$. Since the latter commutes with the total orbital momentum $\sum_i \vec{x}^{(i)} \wedge \vec{p}^{(i)}$, such terms only give rise to small displacements of the energy eigenvalues, but cause no perturbation of the orbital and spin quantum numbers of the stationary states.

** For $\mathcal{V}_e = 1/r$, it just reduces to the magnetic interaction (6.13-20).

with two arbitrary constants a' , a'' ; with $\vec{p}_b = \vec{p}^{(1)} + \vec{p}^{(2)}$, this may be written

$$-\frac{\hbar}{(2M)^2} \frac{1}{r} \frac{dV}{dr} \left\{ (\vec{x} \wedge \vec{p}^{(12)}) [(2a' - 1)\vec{\sigma}^{(1)} + (2a'' - 1)\vec{\sigma}^{(2)}] + \frac{1}{2} (\vec{x} \wedge \vec{p}_b) (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) \right\}; \quad (19)$$

the above cases of V_e, V_n correspond to $a' = a'' = -1$ or $+1$, respectively. For a potential of the form $V P_\sigma$ we can have a spin-orbit coupling consisting of the expression (19) multiplied on the right by P_σ and the additional term

$$-\frac{i}{(2M)^2} V (\vec{p}^{(1)} \wedge \vec{p}^{(2)}) (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) P_\sigma. \quad (20)$$

With reference to table 15.21, we see that the relativistic couplings may, besides an axial dipole coupling, involve spin-orbit couplings of types III and IV, as well as other couplings of types (20) and

$$\mathcal{M}_4^{(12)} (\vec{x} \wedge \vec{p}_b) (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}); \quad (21)$$

the isotopic factors of these new types of interactions may be any combination of the isotopic factors 1, 2 and 3. The Thomas and "Larmor" terms are respectively proportional to

$$\mathcal{M}^{(12)} + \frac{1}{2} \mathcal{M}_4^{(12)} \quad \text{and} \quad \mathcal{M}^{(12)} - \frac{1}{2} \mathcal{M}_4^{(12)}. \quad (22)$$

The preceding considerations disclose the general property that *if we neglect second order effects altogether, we have just to use the unrelativistic potential energy without any correction*. As to the order of magnitude of the relativistic corrections, several estimates have been carried out in connexion with the interpretation of the fine structure of certain nuclear levels; we shall come back to this question later (17.43, 17.53). At the moment, we shall just mention the results relating to the two-nucleon system. The displacements of the ground state and of the virtual ^1S level of the deuteron due to second order effects have been calculated by SHARE and BREIT [37b]) on the basis of the expressions (16), (17), (18). For the resulting contribution to the ^1S - ^3S separation, they find (assuming a well potential of width d) $0.36m$, $0.24m$ and $0.81m$, respectively. The second order effects to be expected for proton-neutron and proton-proton scattering have been estimated by HOISINGTON [40] and found to be quite small even for energies of 16 MeV.

15.3. Non-central and non-static couplings on meson theory

15.30. Invariance considerations will not give us any information on the distance dependence of the nuclear potential and they leave us the choice between several possible types of dependence on the other coordinates, or combinations of these types. If we want a more precise specification, from a theoretical point of view, it is natural that we should turn our attention to field theories of nuclear interaction. In this section, we shall discuss the forms of nuclear potential which can be derived from meson field theory.

15.31. Static interactions due to meson fields. On the hypothesis of weak coupling between nucleons and meson fields, we shall expect the main contribution to the interaction energy of slowly moving nucleons to arise from the static part of the meson field (8.31). Each one of the four possible types of meson fields (1.32) gives rise to a definite form of static interaction between a pair of nucleons. By way of illustration, we shall outline the derivation in the case of a pseudoscalar field and simply quote the results for the other types *. It will of course suffice to treat the case of a neutral theory.

The equations of the neutral pseudoscalar meson field can be derived from the Hamiltonian

$$\mathcal{H}_\psi = \frac{1}{2} \int [\Phi^2 + (\text{grad } \Psi - \vec{P})^2 + \kappa^2 \Psi^2] dv - \int (N\Psi + Q\Phi) dv, \quad (1)$$

in which Ψ is the field amplitude and Φ its canonical conjugate, obeying the commutation rule

$$[\Phi(\vec{x}, t), \Psi(\vec{x}', t)] = \frac{\hbar}{i} \delta(\vec{x} - \vec{x}'). \quad (2)$$

The quantities N , $\vec{P}$, Q are the source densities, consisting of a pseudo-scalar

$$N = \sqrt{4\pi} f_1 n, \quad n \equiv \sum_i q_2^{(i)} \delta(\vec{x} - \vec{x}^{(i)}) \quad (3)$$

and a pseudovector

$$\begin{aligned} \vec{P} &= \sqrt{4\pi} \frac{f_2}{\kappa} \vec{p}, & \vec{p} &\equiv \sum_i \vec{\sigma}^{(i)} \delta(\vec{x} - \vec{x}^{(i)}) \\ Q &= \sqrt{4\pi} \frac{f_2}{\kappa} q, & q &\equiv \sum_i q_1^{(i)} \delta(\vec{x} - \vec{x}^{(i)}); \end{aligned} \quad (4)$$

the formulae (3), (4) give the representation of the source density operators in the configuration space of the nucleons; the coupling constants f_1 , f_2 have the same dimension as an electric charge and play a role entirely analogous to that of the elementary charge e in electromagnetic theory: they are specific parameters attached to a nucleon, which give a measure of the intensity of the interaction between this nucleon and the meson field. An essential difference from the electromagnetic case is that one has here to introduce two independent densities, one of which has the same covariance properties as the meson field itself, while the other transforms like the derivatives of the field. (In electromagnetism, the former is the charge and current density, transforming like the potential; the latter would be an intrinsic density of electric and magnetic polarization.)

* On this subject, see, e.g., MøLLER and ROSENFELD's paper [40], in which further references are given.

With the help of the commutation rules (2), one gets from the Hamiltonian (1) the field equations

$$\begin{aligned}\dot{\Psi} &= \Phi - Q \\ \dot{\Phi} &= \text{div} (\text{grad } \Psi - \vec{P}) - \kappa^2 \Psi + N.\end{aligned}\quad (5)$$

Denoting by an index $\circ$ the static approximation, and observing that

$$\begin{aligned}\dot{Q} &= \dot{N} = 0 \\ \dot{\vec{P}} &= \vec{P},\end{aligned}\quad (6)$$

one finds that the static field obeys the equations

$$\begin{aligned}\dot{\Phi} &= 0 \\ \Delta \dot{\Psi} - \kappa^2 \dot{\Psi} &= \text{div } \vec{P},\end{aligned}\quad (7)$$

with the solution

$$\dot{\Psi}(\vec{x}) = -\frac{1}{4\pi} \int \text{div } \vec{P}(\vec{x}') \cdot \varphi(r) dv'; \quad (8)$$

in this last formula, $\varphi(r)$ represents the meson potential $e^{-\kappa r}/r$. The static energy is obtained by inserting (6) and (7) into (1), which gives, after partial integrations,

$$\dot{H}_q = -\frac{1}{2} \int \vec{P} \text{grad } \dot{\Psi} dv; \quad (9)$$

a term $\frac{1}{2} \int \vec{P}^2 dv$ of contact interaction, which would give rise to an unacceptable singularity (5.132), has here been left out. This may be justified by the remark that it is always possible, by adding to the Hamiltonian an invariant — in this case $\frac{1}{2} \int (Q^2 - \vec{P}^2) dv$ — without influence on the field equations, to reduce any static term of contact interaction to a similar one of the *second* order in the nucleon velocities: now, as we shall see presently in more detail, the meson field theory cannot claim to be reliable up to that order, and all second order non-static effects have to be discarded. In this (admittedly far from elegant) way we can get rid of all unwanted contact interactions.

When the expression (8) for $\dot{\Psi}$ is inserted in the static energy (9), the latter takes a form involving only the nucleon variables. This includes in the first place self-energy terms, which are of course infinite when a point model for the nucleons is adopted, as is implied by the δ -functions occurring in the source densities (3), (4). We are only interested in the interaction terms; taking account of these expressions for the source densities, the static interaction energy may be written

$$\dot{V}_q = -\frac{1}{2} \sum_{i,k} \left(\frac{f_2}{\kappa} \right)^2 (\vec{\sigma}^{(i)} \text{grad}^{(i)}) (\vec{\sigma}^{(k)} \text{grad}^{(k)}) \varphi(r^{(i,k)}). \quad (10)$$

On the symmetrical theory, we would have obtained a similar expression, with additional isotopic factors $\tau^{(i)}\tau^{(k)}$. The form (10) is characteristic of the pseudoscalar theory. Although, e.g., a scalar meson field has a Hamiltonian of exactly the same form as (1), the meaning of the source densities

is quite different: in contrast to (6), only $\vec{P}$ vanishes in this case; this explains how one gets an entirely different outcome for the static interaction energy. The accompanying table summarizes the results obtained for

15.31. Static interactions according to meson field theory					
Types of meson fields		Source intensity constants		Static interaction between a pair of nucleons on neutral theory *	
Spin 0	scalar	scalar f'_1	vector f'_2	$-f'_1{}^2 \varphi(r)$	
	pseudoscalar	ps.-sc. f_1	ps.-vect. f_2	$f_2^2 \mathcal{S}^{(12)}$	
Spin 1	vector	vector g_1	tensor. g_2	$[g_1^2 + g_2^2 \overset{\rightarrow}{\sigma}^{(1)} \cdot \overset{\rightarrow}{\sigma}^{(2)}] \varphi(r) - g_2^2 \mathcal{S}^{(12)}$	
	pseudovector	ps.-vect. g'_1	ps.-tens. g'_2	$-g_1'^2 \overset{\rightarrow}{\sigma}^{(1)} \cdot \overset{\rightarrow}{\sigma}^{(2)} \varphi(r) - (g_1'^2 - g_2'^2) \mathcal{S}^{(12)}$	
* On symmetrical theory, multiply by isotopic factor $\tau^{(1)} \tau^{(2)}$.					

the four types of mesons. It will be seen that, besides central forces, only the type of interaction defined by the operator

$$\mathcal{S}^{(12)} = -\frac{1}{\kappa^2} (\vec{\sigma}^{(1)} \cdot \text{grad}^{(1)}) (\vec{\sigma}^{(2)} \cdot \text{grad}^{(2)}) \varphi(r) \quad (11)$$

occurs.

On account of these results, a *purely* scalar meson theory can readily be dismissed. On a neutral scalar theory, a separation of ^3S and ^1S states of the deuteron could only appear as a non-static perturbation; on a symmetrical theory, the isotopic factor would indeed give rise to different effective potentials for ^3S and ^1S states, but the ^3S potential would be repulsive. In the three other kinds of meson theories, the interaction of type $\mathcal{S}^{(12)}$ occurs effectively; let us therefore look more closely at this operator (ROSENFELD [45b]). It can readily be transformed into

$$\begin{aligned} \mathcal{S}^{(12)} &= \frac{1}{\kappa} \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \varphi(r) + \mathcal{D}^{(12)} F(r) - \frac{1}{\kappa} \mathcal{A}^{(12)}, \\ F(r) &\equiv \left[1 + \frac{3}{\kappa r} + \frac{3}{(\kappa r)^2} \right] \varphi(r) \\ \mathcal{A}^{(12)} &\equiv \frac{3}{4\pi\kappa^2} \lim_{\vec{\xi} \rightarrow 0} \int \delta(\vec{x} - \vec{\xi}) (\vec{\sigma}^{(1)} \cdot \vec{\xi}) (\vec{\sigma}^{(2)} \cdot \vec{\xi}) \xi^{-2} d\Omega_{\xi}. \end{aligned} \quad (12)$$

It thus reduces to a combination of central and axial dipole interactions, to which must be added a peculiar contact interaction $\mathcal{A}^{(12)}$. By considering the matrix representation of the latter with respect to the stationary states

of the two-nucleon system, one can immediately reduce it to the usual type of contact interaction *

$$A^{(12)} = \frac{1}{x^2} \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \delta(x). \quad (12a)$$

The essential point about $S^{(12)}$ is the distance dependence $F(r)$ of the axial dipole coupling. Here is the critical issue around which the whole discussion about the adequacy of meson field theory for the description of nuclear forces centres. In fact, on account of its pole of the third order at the origin, the potential $F(r)$ cannot be regarded as an exact expression of the law of force at very small distances: for the radial eigenfunction of a ${}^3P_1$ state of the deuteron, e.g., which is not mixed with any other state (4.331), satisfies a wave-equation to which the axial dipole term $\mathcal{D}^{(12)}F(r)$ contributes an effective potential $\frac{2}{3}F(r)$; such an equation has no regular solution (5.11) if the singularity of $F(r)$ at the origin is not reduced to a lower order. Clearly, then, the idealization of point nucleons, which underlies the present formalism of meson theory, fails to provide even an unambiguous derivation of the exact law of force at small distances; and the trouble is that we have no better scheme to guide us in this search. We are therefore driven to the necessity of introducing some arbitrary modification in the theory in order to get rid of the r^{-3} singularity of the non-central coupling: a good deal of the uncertainty still prevailing as regards the true nature of the nuclear forces has its root in this unfortunate circumstance. For there are, of course, several ways of removing the singularity in question and the available evidence does not yet permit us quite definitely to eliminate any of them.

15.32. Cut-off and mixture. The procedures followed to free the law of interaction from the r^{-3} singularity belong to two essentially distinct classes, briefly denoted by the epithets "*cut-off*" and "*mixture*". The most radical method consists in *cutting off* the singularity: the validity of the form $F(r)$ for the distance dependence of the non-central interaction is limited to distances larger than some critical radius r_c , while for smaller distances, some other, permissible expression, is substituted for it. Following BETHE [40], who first introduced them, two special forms of cut-off have been discussed most: the *zero cut-off*, according to which the potential is put equal to zero for $r < r_c$, and the *straight cut-off*, in which the constant value $F(r_c)$ is assumed for all $r < r_c$. However, the arbitrary character of the cut-off procedure does not lie so much in the infinite variety of possible assumptions for the substitute potential at small distances **

* Owing to the radial dependence of the eigenfunctions, the only non-vanishing matrix-elements are those pertaining to transitions between S states, for which the equivalence of the operators (12) and (12a) is obvious.

** In all cut-off theories that have been discussed, the cut-off has also been applied to the central part of the interaction, in spite of the fact that there is no cogent reason for doing so.

(this does not affect very much the prediction of the effects of experimental interest), as in the introduction of an additional parameter, the critical radius, the choice of which involves, as we shall see, the gravest difficulties.

The alternative possibility of avoiding the occurrence of the r^{-3} singularity consists in introducing a suitably chosen *mixture* of meson fields of different types, in such a way as to achieve a compensation of the singular contributions due to each of them. More precisely, one assumes that the nucleons are the sources of two or more meson fields, which give independent contributions to the interaction energy. In the most general conception of the "mixture" (or "mixed") theory, suggested by SCHWINGER [42], these independent mesons may even have different masses. The requirement that the total static interaction should not have any third order pole imposes a condition on the intensity constants of the various source densities. For instance, we may consider a mixture of a pseudoscalar and a vector field with respective ranges κ_0 , κ_1 , provided the source intensities fulfil the relation

$$\frac{f_2^2}{\kappa_0^2} = \frac{g_2^2}{\kappa_1^2}. \quad (13)$$

A glance at table 15.31 will show that many other combinations are possible *a priori*; the determination of the suitable mixture must be completed by different considerations, involving a comparison with the experimental data. The expansion

$$F(r) = \frac{1}{r} \left[-\frac{3}{(\kappa r)^2} - \frac{1}{2} + \frac{3}{2} \kappa r + \dots \right] \quad (14)$$

further shows that after compensation of the r^{-3} singularity, the remaining one will be a simple pole.

According to Schwinger's general scheme, just outlined, the static interaction of a mixed theory would include both central and axial dipole couplings, just as the cut-off theories. The special form of mixed theory originally proposed by MØLLER and ROSENFELD [40], on the other hand, starts from the simpler assumption of a single value of the meson mass for all kinds of fields entering into the mixture. The elimination of the r^{-3} singularity entails in this case the complete disappearance of non-central interactions. In view of the evidence summed up in 15.1, this would at first sight appear as an over-simplification. But we must not forget that there are further non-static contributions to the interaction energy, which may be of the required form and magnitude. We shall now turn our attention to these non-static effects, which prove to be of the greatest importance in judging the consistency and suitability of the variants of meson theory just introduced.

15.33. Non-static interactions due to spin precession and charge exchange. As already stated (8.31), we have to distinguish two kinds of

non-static effects, viz. those depending on the translation velocities of the nucleons, and those arising from the precessions of their spins and isotopic variables. Let us discuss the latter first; we may limit ourselves to the treatment of the spin precession, as that of the charge exchange gives much the same results. Let us consider two nucleons at a fixed distance r apart. In first approximation, the equation of motion of the spin $\vec{\sigma}^{(1)}$ (say) will be

$$\dot{\vec{\sigma}}^{(1)} = \frac{i}{\hbar} [\hat{H}_\psi, \vec{\sigma}^{(1)}], \quad (15)$$

$\hat{H}_\psi$ being the static energy of the system. In general, this quantity will involve a spin dependent part of the form

$$G_\sigma \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \varphi(r) + G_D \mathcal{D}^{(12)} F(r), \quad (16)$$

the isotopic factor (or rather its average value in the state considered) being included in the coefficients G . On account of (16), equation (15) takes the form

$$\dot{\vec{\sigma}}^{(1)} = \vec{\Omega}^{(1)} \wedge \vec{\sigma}^{(1)}, \quad (17)$$

with

$$\vec{\Omega}^{(1)} = \frac{2}{\hbar} \{ G_\sigma \varphi(r) \vec{\sigma}^{(2)} + G_D F(r) [(\vec{\sigma}^{(2)} \vec{x}_0) \vec{x}_0 - \frac{1}{3} \vec{\sigma}^{(2)}] \}. \quad (18)$$

This may be described as a precession of angular velocity (divided by c) $\vec{\Omega}^{(1)}$. Formula (18) shows that for small values of the distance r , the order of magnitude of this angular velocity is $G/\hbar \kappa^2 r^3$ in a cut-off theory, and $G/\hbar r$ in a mixed theory (G denoting the order of magnitude of the relevant constants G_σ , G_D). This difference has an important bearing on the range of validity of the static approximation in both kinds of theories.

Indeed, it is clear that if the periods of the spin precessions are not large compared with the time of propagation of the main part of the non-static meson fields through a distance of the order of the range of the static nuclear forces, there is no justification in regarding these static forces as a sufficient approximation to the true interaction. This condition defines a critical value R_c of the distance r , such that the validity of the static approximation will be limited to distances larger than R_c . Since the time of propagation of the non-static fields through the distance κ^{-1} is of the order of magnitude $(\kappa c)^{-1}$, the critical distance R_c may be defined by

$$\begin{aligned} \kappa R_c &= (G/\hbar)^{\frac{1}{2}} && \text{on cut-off theory,} \\ \kappa R_c &= G/\hbar && \text{on mixed theory.} \end{aligned} \quad (19)$$

Since $G/\hbar$, as we shall see, is of the order of 0.1, the limitation imposed

* If several ranges are assumed, κ^{-1} denotes the longest.

by the existence of R_c is much more severe on cut-off than on mixed theory. It even turns out that R_c in the former case is of the same order of magnitude as the cut-off radius itself: this means, indeed, that a cut-off prescription based on the sole consideration of the static potential is essentially inconsistent. *This is in the present state of meson field theory the most serious objection against the cut-off procedure.* It must be conceded that such an objection cannot be regarded as entirely convincing; the advocates of the cut-off idea may in fact point out that the quantum field theory anyway fails to give a reliable account of the forces in the immediate neighbourhood of a particle, and that the cut-off is intended as a rough substitute for the *complete* interaction at small distances which would result from a consistent field theory. Even so, the inconsistency arising from the limitation (19) for cut-off theories enhances their arbitrary and hypothetical character.

On the other hand, while no similar inconsistency occurs in mixed theories, it must be emphasized that such theories are not free from arbitrariness either; only, the element of arbitrariness is introduced, so to speak, in a more subtle way. In fact, the ambiguities of the present formalism of quantum field theory can only be avoided by imposing considerable restrictions on its interpretation. The canonical equations derived from the Hamiltonian are not treated as an exact system of equations, but solved by a process of successive approximations in which, starting from a suitably defined unperturbed system, the calculation of the solution corresponding to a given initial state of this system should not be carried further than the first step leading to a non-vanishing result for the effect under consideration; and such results should of course only be considered as reliable if even this first step does not involve any ambiguity. In quantum electrodynamics, a certain justification for this procedure can be derived from a "correspondence argument", i.e. from the comparison of the quantized problem with the analogous classical calculation; at the same time, the correspondence treatment indicates that the unperturbed Hamiltonian must be taken to include the electrostatic interaction between the charged particles of the system, which can be separated from the non-static part of the electromagnetic field by a canonical transformation. From a purely formal point of view, the general procedure just recalled, including the separation of the static energy by means of a canonical transformation, can be carried over, *mutatis mutandis*, into meson theory (MØLLER and ROSENFELD [40]); but in contrast to the case of electromagnetism it here lacks all physical basis, since there is no empirical evidence on the classical counterpart of the quantized meson fields.

In the last resort, the "correspondence" point of view, thus extended to meson field theory, comes very near to a cut-off prescription, since it also purports to avert the paradoxes of quantum field theory pertaining to the effects at close distance from the sources. In meson theory, such effects arise not only in connexion with the single nucleons (in the form

of self-energy, anomalous magnetic moment and the like), but even in the interaction between them. The mixture method rigorously eliminates the latter difficulty (at any rate from the static interaction) and as regards the former it follows a course entirely analogous to that adopted in quantum electrodynamics. It has, however, to pay for this display of elegance by a greater complexity and lack of cogency, due to the introduction of several kinds of fields with unknown source intensity parameters. The cut-off theory, rough and arbitrary as it is, involves only one type of field, with a smaller number of undetermined parameters.

The limitation of the domain of validity of the law of static interaction symbolized by the critical distance R_c , given by (19), has actually been derived in the spirit of the "correspondence" argument, i.e. by disregarding the quantization of the meson fields. According to the ideas put forward by HEISENBERG [39], other effects, essentially depending on this field quantization, lead to a further, still more restrictive, limitation. This arises from the increase of the probabilities of *explosive processes* in systems of nucleons and mesons, when the energy involved becomes large compared with the rest-mass of the mesons. The corresponding critical length λ_c is the meson wave-length for which the explosive character of the process sets in; one readily finds

$$\kappa \lambda_c = \sqrt{G/\hbar}. \quad (20)$$

If, as argued by Heisenberg, this critical length has a universal significance, in the sense that the usual concepts of field theory would not be applicable within regions of a linear extension smaller than λ_c , the law of distance dependence of nuclear force derived from any form of meson theory could only claim validity for distances larger than λ_c . Even so, there remains, *on mixed theories*, an interval of distances between λ_c and κ^{-1} , in which the *static potential* has a significant value and can consistently be regarded as representing the main part of the nuclear interaction.

15.34. Velocity dependent coupling on mixed meson theory. It follows from the preceding discussion that on cut-off theory there is no point in taking into account the non-static interaction terms which depend on the translation velocities of the nucleons. On mixed theory, however, such terms may be taken into consideration; they should be treated as perturbations and their contributions retained insofar as they appear as unambiguous results of the calculations. The general expression for the energy operator, exact to the first order in the nucleon velocities, can be derived for each of the four types of meson fields by applying to the original Hamiltonian the canonical transformation which separates the static part of the field; the details of this operation may be found in MØLLER and ROSENFELD's paper [40]. Again, we shall here take as an example the pseudoscalar case and quote the final results obtained in the other cases.

With the notation of 15.31, the first order interaction energy due to

the pseudoscalar field can simply be written

$$\mathcal{W}_\psi = - \int N \dot{\Psi} dv = \frac{f_1 f_2}{\kappa} \int n(\vec{x}) \vec{p}(\vec{x}') \text{grad } \varphi dv dv', \quad (21)$$

by (3), (4), (8). The velocity dependent matrices ϱ_1, ϱ_2 occurring in the expressions for the source densities can be eliminated by a transformation analogous to the well-known Gordon decomposition of the current density in the Dirac electron theory. For instance, from

$$\begin{aligned} \frac{d}{cdt} (\varrho_1^{(i)} \delta(\vec{x} - \vec{x}^{(i)})) &= \frac{i}{\hbar} [\varrho_1^{(i)} \vec{\sigma}^{(i)} \vec{p}^{(i)} + \varrho_3^{(i)} M, \varrho_1^{(i)} \delta(\vec{x} - \vec{x}^{(i)})] \\ &= \text{div}^{(i)} (\vec{\sigma}^{(i)} \delta(\vec{x} - \vec{x}^{(i)})) - \frac{2M}{\hbar} \varrho_2^{(i)}, \end{aligned}$$

we get

$$n(\vec{x}) = - \frac{\hbar}{2M} \text{div } \vec{p} - \frac{\hbar}{2M} \frac{dq}{cdt}. \quad (22)$$

In this formula, the last term is of higher order in the velocities and must consistently be neglected; the first term, though formally independent of the nucleon velocities, is nevertheless of the first order, for the divergence operator is equivalent, as to the order of magnitude, to a factor λ^{-1} , λ representing the de Broglie wave-lengths of the nucleons. Other source density operators give rise to a decomposition of the type (22) containing a further term which depends explicitly on the momenta of the nucleons. Inserting (22) in (21) and using the notation (11), we get

$$\mathcal{W}_\psi = f_1 f_2 \frac{M_m}{M} \cdot \frac{1}{2} \sum_{i,k} \mathcal{S}^{(ik)}. \quad (23)$$

For the other types of meson fields, the derivation proceeds along the same lines (ROSENFELD [45b] *). As in the static case, the final outcomes are very different on account of the different forms of source densities. They are summarized in table 15.34, in which the same notation has been used as in 15.2, with in addition

$$\mathcal{M}_3^{(12)} \equiv (\vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)}) (\vec{x} \wedge \vec{p}_b). \quad (24)$$

The table exhibits a striking disparity between the pseudovector case on the one hand and the pseudoscalar and vector cases on the other. The expression pertaining to the former case is not at all of a form which would be expected on general invariance considerations for first-order velocity dependent interactions between a pair of nucleons (15.21); in fact, it vanishes for any two-nucleon system whose centre of gravity is at rest.

* In this paper, however, the treatment of the scalar and pseudovector cases is incomplete; the correct results are given in table 15.34.

15.34. Velocity dependent interactions on meson field theory		
Type of meson field		First order interaction between a pair of nucleons on neutral theory*
Spin 0	scalar	$\frac{2 f_1' f_2'}{\hbar} \frac{M_m}{M} \frac{\vec{p} \cdot \vec{x}}{\kappa^2 r} \frac{\varphi'(r)}{\kappa^2 r}$
	pseudoscalar	$f_1 f_2 \frac{M_m}{M} \mathcal{S}^{(12)}$
Spin 1	vector	$-g_1 g_2 \frac{M_m}{M} \left\{ \left(1 + \frac{\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}}{\sigma^2} \right) [\varphi(r) - \frac{1}{\kappa^2} \delta(\vec{x})] - \mathcal{S}^{(12)} \right\} - \frac{2 g_1 g_2}{\hbar} \frac{M_m}{M} \frac{\varphi'(r)}{\kappa^2 r} \mathcal{M}^{(12)}$
	pseudovector	$\frac{g_1' g_2'}{\hbar} \frac{M_m}{M} \frac{\varphi'(r)}{\kappa^2 r} \mathcal{M}_5^{(12)}$

* On symmetrical theory, multiply by isotopic factor $\tau^{(1)} \tau^{(2)}$.

The couplings due to pseudo-scalar or vector fields have the expected form and include non-central interactions of both axial dipole and spin-orbit types.

It might at first sight seem disturbing that the distance dependence of the non-static couplings also exhibits a r^{-3} singularity. The purpose of the mixture method, viz. to eliminate such a singularity from the static interaction, would seem to be defeated by its reappearance in the next approximation (FERRETTI [43b], HU [45]). At this juncture it is essential to remember that the mixture method is inseparable from the general "correspondence" prescription enunciated above (15.33), in virtue of which the non-static interaction cannot be treated as an exact contribution to the Hamiltonian, but only as a perturbation. As such, however, the third order pole does not offer any fundamental difficulty: the situation, in fact, is exactly the same as in the case of a magnetic dipole interaction, touched upon in 6.13. However, the strong increase of the non-static potential at small r impairs considerably the convergence of perturbation calculations. The situation may be judged by considering the ratio of non-static to static potential; if we assume that all source constants (which enter in different combinations into the two potentials) are of the same order of magnitude, this ratio is of the order $(M_m/M) (\kappa r)^{-2}$. In the significant range of distances $\lambda_c \dots \kappa^{-1}$, it varies, according to (20), from ≈ 1 to ≈ 0.1 : even with the small factor M_m/M , which represents the ratio of mean nucleon velocity to velocity of light for stationary states of nuclei, the non-static contribution to non-central interactions is thus by no means negligible (unless special assumptions are made about the source constants).

15.35. Spin and life-time of mesons. The considerations of this section are of a preliminary character. In the next Chapters, we shall have to

confront the predictions of the various forms of meson theories we have been discussing with the properties of nuclear systems, with a view to specifying, if possible, the particular form of theory best adapted to the evidence. But first, we shall enquire in how far cosmic ray data could yield more direct information on the kinds of mesons actually produced in nuclear processes and consequently contributing to the interaction between nucleons. The possibility of detecting mesons of a particular type is, of course, intimately connected with the life-time of these mesons. Theoretically, if we first consider the decay of the meson into a lepton pair, the *relation between its life-time and covariance properties* is determined by the form adopted for the Hamiltonian describing the interaction between the meson and lepton fields. Now, this Hamiltonian, and especially the source densities which govern the production of the meson field by leptons, can be set up in a form identical with that of the meson-nucleon system; the only difference being in the numerical values of the source constants, which will be distinguished from those of table 15.31 by a $\checkmark$ mark (e.g. $\check{g}_2$). A straightforward calculation (SAKATA [41]) leads to the following results for the life-time t_0 of a meson at rest:

$$\begin{aligned}
 \text{scalar:} \quad & \frac{1}{t_0} = c\kappa \left[\frac{\check{f}_1^2}{\hbar} + \frac{\check{f}_2^2}{\hbar} \left(\frac{m}{M_m} \right)^2 \right] \\
 \text{pseudoscalar:} \quad & \frac{1}{t_0} = c\kappa \left[\frac{\check{f}_1}{\sqrt{\hbar}} - \frac{\check{f}_2}{\sqrt{\hbar}} \frac{m}{M_m} \right]^2 \\
 \text{vector:} \quad & \frac{1}{t_0} = c\kappa \left[\frac{2}{3} \frac{\check{g}_1^2}{\hbar} + \frac{1}{3} \frac{\check{g}_2^2}{\hbar} \right] \\
 \text{pseudovector:} \quad & \frac{1}{t_0} = c\kappa \left[\frac{2}{3} \frac{\check{g}_1'^2}{\hbar} + \frac{1}{3} \frac{\check{g}_2'^2}{\hbar} \right].
 \end{aligned} \tag{25}$$

Since any of these formulae contains two undetermined constants, it cannot be directly compared with the experimental results. An additional theoretical aspect has to be adduced; this will be, of course, the conception of the *process of β -decay* proper to meson theory (1.322). Let us consider the expression for the life-time of a β -active nucleus to which this theory leads when one assumes that the interaction between nucleons and leptons is brought about by meson fields of a single type; we shall further restrict ourselves to allowed β -transitions of very light nuclei (A1.11). The result is of the form corresponding either to Fermi interaction (scalar meson theory), to Gamow-Teller interaction (pseudoscalar and pseudovector meson theories), or to a linear combination of both (vector meson theory). The life-time is thus given by a formula of the type (A1.11-1) or the sum of two such expressions; the constants $1/\tau_0$ in the different cases are as follows:

15.35. Value of β -decay parameter $1/\tau_0$ on meson theory			
Type of field		Fermi interaction term	Gamow-Teller interaction term
Spin 0	scalar	$\frac{8}{\pi} \kappa c \left(\frac{m}{M_m}\right)^5 \left(\frac{f_1' \check{f}_1'}{\hbar}\right)^2$	—
	ps.-scalar	—	$\frac{8}{3\pi} \kappa c \left(\frac{m}{M_m}\right)^5 \left(\frac{f_2 \check{f}_2}{\hbar}\right)^2$
Spin 1	vector	$\frac{8}{\pi} \kappa c \left(\frac{m}{M_m}\right)^5 \left(\frac{g_1 \check{g}_1}{\hbar}\right)^2$	$\frac{32}{3\pi} \kappa c \left(\frac{m}{M_m}\right)^5 \left(\frac{g_2 \check{g}_2}{\hbar}\right)^2$
	ps.-vect.	—	$\frac{1}{3\pi} \kappa c \left(\frac{m}{M_m}\right)^5 \left[32 \left(\frac{g_1' \check{g}_1'}{\hbar}\right)^2 + \left(\frac{g_2' \check{g}_2'}{\hbar}\right)^2 \right]$

The numerical coefficients in this table are those given by SAKATA [41] *; it should be observed, however, that the choice of the Hamiltonian is not unique, owing to the possibility of adding to it terms of direct coupling between nucleons and leptons with arbitrary coefficients: since the available evidence on β -spectra is not sufficiently accurate to allow of a complete determination of these coefficients, we can only say that Sakata's choice is a plausible one, and that the order of magnitude of the $1/\tau_0$ -values could only be changed by very improbable assumptions about the direct couplings **. If a mixture of meson fields is adopted, there may appear further interference terms slightly modifying the energy dependence of the life-time (cf. A1.11, footnote); this has been thoroughly discussed, on very general assumptions, by ROZENTAL [41a, 45] *** for the case of a pseudoscalar-vector mixture. We shall here disregard this complication, which is not essential for our argument.

The expression for $1/\tau_0$, given by table 15.35, is more favourable than that for the meson life-time $1/t_0$ to get an insight into the order of magnitude of the lepton source density constants. In the first place, since the empirical evidence is strongly in favour of Gamow-Teller interaction, we infer the unsuitability of a purely scalar theory to account for the phenomenon of β -decay. In any case, the constants $\check{f}_1'$ and $\check{g}_1$ must be assumed to vanish, or at any rate to be of a smaller order of magnitude than the others. As regards the latter, if we adopt for $1/\tau_0$ the value (A1.21-4), for the meson source density constants (such as f_2 , g_2 , etc.) the general estimate $g^2/\hbar \approx 0.1$ and for the meson mass $M_m \approx 200 m$, we get for the relevant lepton source constants a rough estimate

$$\frac{\check{g}^2}{\hbar} \approx 10^{-14} \quad (\check{g} \equiv \check{f}_2; \check{g}_2; \check{g}_1', \check{g}_2'). \quad (26)$$

* A misprint in Sakata's paper, affecting the pseudoscalar case, has been corrected.

** On this point, see ROZENTAL [41a].

*** The special choice of parameters advocated in the last paper (ROZENTAL [45]) is questionable.

If we now turn to the formulae (25) for the meson life-time, we see at once that there is an essential difference between mesons of spin 1 and mesons of spin 0. The former would have a life-time of the order of 10^{-9} sec, much shorter than the observed one (1.332) of $\approx 2 \cdot 10^{-6}$ sec. *In order to account for β -decay, spin 1 mesons should have a much shorter life-time than those observed at sea-level:* this consequence of meson theory has especially been emphasized by NORDHEIM [39]. On the other hand, the observed life-time of the order of the microsecond can easily be reconciled with the hypothesis of meson fields of spin 0. A scalar field, according to what has just been found, could at most occur in combination with another one and the choice of the constants $\check{f}'_1, \check{f}'_2$, is thus indeterminate, the only restriction being the small (or vanishing) value of $\check{f}'_1$. Owing to the peculiar form of the expression of $1/t_0$ in the pseudoscalar case, a value of $\check{f}'_2$ of the order of magnitude (26), combined with a suitable choice of $\check{f}'_1$ ($\ll \check{f}'_2$), will yield a life-time of the required magnitude. Remembering the evidence (1.333) pointing to a spin 0 for the sea-level mesons, we come to the conclusion that *the cosmic ray evidence about spin and life-time of mesons as well as the evidence about light β -active nuclei is compatible with a purely pseudoscalar meson theory;* a conclusion first enunciated by ROZENTAL [41b].

We may now ask about the bearing of the foregoing discussion on the conception of the mixture theory. Confronting the general results of this discussion with the requirement of obtaining a static potential without third order pole, we immediately see that *any mixture of meson fields*, in order to be compatible with the cosmic ray and β -decay data, *must include mesons of both spins, those of spin 1 having a very short life-time.* If the two kinds of mesons are assumed to have different masses, we have to take account of another type of decay process, by which the heavier meson may go over into the lighter one with emission of a γ -ray. This process involves as an intermediate step the virtual creation of a nucleon pair; it belongs to the same category as that leading to the decay of a neutral meson into photons (8.311) and, like this last process, has a very large probability of occurrence. This implies that in a mixed theory involving mesons of different masses, one would have to assume that the heavier ones have spin 1. Taking, e.g., a mixed pseudoscalar and vector meson theory, with a mass ratio 1.6 (16.41), the " γ -decay" process just described would lead for the vector mesons to life-times of the order of magnitude of 10^{-18} sec (FINKELSTEIN [47]).

CHAPTER XVI

NON-CENTRAL COUPLINGS AND TWO-NUCLEON SYSTEMS

16.0. The different forms of non-central couplings derived on general grounds in the preceding Chapter must now be tested by comparing their consequences with experiment. In this Chapter, we shall carry out this test insofar as the properties of two-nucleon systems are concerned: we shall review successively the properties of the ground state of the deuteron, the neutron-proton scattering, the proton-proton scattering and the photo-disintegration of the deuteron. First of all, it appears that the only type of non-central coupling having any influence on the behaviour of two-nucleon systems is that we have called axial dipole coupling. We shall first approach the problem from a general point of view, making no assumption on the form of distance dependence of the forces, or schematically representing this dependence by potential wells. Then we shall discuss some variants of meson theories, of both the cut-off and the mixture type.

16.1. General considerations

16.11. *The ${}^3D$ -admixture to the ground state of the deuteron.* From the two possible kinds of non-central couplings evolved in the discussion of the foregoing Chapter, viz. the axial dipole coupling $\mathcal{D}^{(12)}$ and the spin-orbit coupling $\mathcal{M}^{(12)}$, only the first can contribute to the admixture of ${}^3D$ state to the ${}^3S$ ground state of the deuteron, necessary to account for the electromagnetic properties of this nucleus (6.12). Indeed, the other commutes with the orbital momentum and can thus at most give rise to displacements of the levels; it is readily verified that it even vanishes for all singlet states and for ${}^3S$ states*. On the other hand, *the axial dipole operator $\mathcal{D}^{(12)}$, which does not commute with the orbital momentum, is just of the type required to bring about the necessary ${}^3S + {}^3D$ combination:* this, of course, follows immediately from the fact that the derivations of $\mathcal{D}^{(12)}$ as a possible type of interaction and of the ${}^3S + {}^3D$ mixture as the only possible one are based on the same invariance considerations. One can also prove explicitly (6.121) that $\mathcal{D}^{(12)}$ leaves the total angular momentum and the spin multiplicity unaltered, and combines S and D

* In fact, on account of (4.32-16), we may write

$$\mathcal{M}^{(12)} = -\hbar [l(l+1) + s(s+1) - j(j+1)]$$

with $s = 0$ for singlet states and $s = 1$ for triplet states.

states. From the easily verified relation

$$\mathcal{D}^{(12)} |^1(o)_0 = 0, \quad (1)$$

one further concludes that *all singlet states of the two-nucleon system are entirely unaffected by the axial dipole coupling.*

With the help of the formulae (4.32–17) for the spin and angle dependence of the eigenfunctions, the matrix-elements of the operator $\mathcal{D}^{(12)}$ can readily be found (BETHE [40]); the diagonal elements are:

$$\begin{aligned} (j=l, \quad l | \mathcal{D}^{(12)} | j=l, \quad l) &= \frac{2}{3} \\ (j=l+1, l | \mathcal{D}^{(12)} | j=l+1, l) &= -\frac{2}{3} \frac{l}{2l+3} = -\frac{2}{3} \frac{j-1}{2j+1} \\ (j=l-1, l | \mathcal{D}^{(12)} | j=l-1, l) &= -\frac{2}{3} \frac{l+1}{2l-1} = -\frac{2}{3} \frac{j+2}{2j+1}; \end{aligned} \quad (2)$$

the non-diagonal elements, which determine the mixture of states with orbital momenta $j-1, j+1$, have the expression

$$(j, l=j\pm 1 | \mathcal{D}^{(12)} | j, l'=j\mp 1) = -2 \frac{\sqrt{j(j+1)}}{2j+1}. \quad (3)$$

Using (2) and (3) we are able to write down explicitly the set of radial equations (4.331–25) for all triplet states. The nuclear potential consists of the sum of the central potential (8.2–1) $J(r) \mathcal{O}$ and an axial dipole potential which we write in the form $F(r) |^{(12)} \mathcal{D}^{(12)}$, including an isotopic factor $|^{(12)}$. For the latter, we shall chiefly consider either the neutral type $|^{(12)} = 1$ or the symmetrical type $|^{(12)} = \tau^{(1)} \tau^{(2)}$, and incidentally also the charged type $|^{(12)} = T^{(12)}$ (table 15.21). We thus write

$$\mathcal{V}_{\text{nuc}} = J(r) \mathcal{O} + F(r) |^{(12)} \mathcal{D}^{(12)}. \quad (4)$$

In particular, for the ground state of the deuteron, represented by (6.12–3, 4) we have to find the two radial eigenfunctions

$$\chi_0(r) \equiv R_0(r) \cos \omega, \quad \chi_2(r) \equiv R_2(r) \sin \omega \quad (5)$$

as solutions of the system of differential equations

$$\begin{aligned} \frac{d^2 \chi_0}{dr^2} + \frac{M}{\hbar^2} [\epsilon_0 + J(r)] \chi_0 &= 2 \sqrt{2} \frac{M}{\hbar^2} F(r) \chi_2 \\ \frac{d^2 \chi_2}{dr^2} - \frac{6}{r^2} \chi_2 + \frac{M}{\hbar^2} [\epsilon_0 + J(r) - 2F(r)] \chi_2 &= 2 \sqrt{2} \frac{M}{\hbar^2} F(r) \chi_0. \end{aligned} \quad (6)$$

The *symmetrical* isotopic factor has here been adopted; on a neutral theory, $F(r)$ should be replaced by $-\frac{1}{3} F(r)$; if the charged form of isotopic factor is assumed, it should be replaced by $+\frac{1}{3} F(r)$. The sign of $F(r)$ is uniquely related to that of the electric quadrupole moment. The positive sign found

for the latter quantity means, in fact, that the spatial density distribution in the ground state is prolate with respect to the axis of total angular momentum (which is in this case the common direction of the spins). Now, in such a configuration, the mean value of the operator $\mathcal{D}^{(12)}$ is positive; in order that the potential energy (4) be a minimum, the factor $F(r) |^{(12)}$ must be negative. By the same argument, a negative quadrupole moment, or oblate density distribution, would correspond to a positive sign of $F(r) |^{(12)}$. We thus get the following correspondence:

Sign of quadrupole moment	+	-
Density distribution	ϕ	ϕ
Sign of $F(r)$ $\left\{ \begin{array}{l} \text{Symmetrical} \\ \text{or charged type} \\ \text{Neutral type} \end{array} \right.$	+	-
	-	+

(7)

16.12. The axial dipole coupling as a perturbation. In view of future application to mixed meson theory, we shall first derive some formulae valid on the assumption that the axial dipole coupling can be regarded as a perturbation; we shall only write down the formulae pertaining to a *symmetrical* isotopic factor. If in equations (6) we assume that the terms containing the product $F(r) \chi_2$ can be neglected, we may express χ_2 in terms of the eigenfunctions ${}^3R_\epsilon^{(2)}(r)$ of the continuum states corresponding to the central potential $-J(r)$:

$$\left\{ \frac{d^2}{dr^2} - \frac{6}{r^2} + \frac{M}{\hbar^2} [\epsilon + J(r)] \right\} {}^3R_\epsilon^{(2)}(r) = 0. \quad (8)$$

In fact, if we multiply by ${}^3R_\epsilon^{(2)*}$ the second equation (6), in which χ_0 on the right may be replaced by R_0 , and integrate over r , we get, on account of (8),

$$\int_0^\infty {}^3R_\epsilon^{(2)*} \chi_2 dr \approx -2 \sqrt{2} \frac{1}{\epsilon - \epsilon_0} F_\epsilon, \quad (9)$$

with

$$F_\epsilon \equiv \int_0^\infty {}^3R_\epsilon^{(2)*}(r) F(r) R_0(r) dr.$$

Hence,

$$\chi_2(r) \approx -2 \sqrt{2} \int_0^\infty \frac{d\epsilon}{\epsilon + |\epsilon_0|} F_\epsilon {}^3R_\epsilon^{(2)}(r). \quad (10)$$

The normalization condition yields the amount of D -admixture in the form

$$\sin^2 \omega = \int_0^\infty |\chi_2|^2 dr = 8 \int_0^\infty \frac{d\varepsilon}{(\varepsilon + |\varepsilon_0|)^2} |F_\varepsilon|^2, \quad (11)$$

while, according to (6.12-5), the expression for the quadrupole moment reduces in first approximation to

$$\begin{aligned} Q &= -\frac{1}{10} \Re \int_0^\infty R_0^* \chi_2 r^2 dr \\ &= \frac{2}{5} \int_0^\infty \frac{d\varepsilon}{\varepsilon + |\varepsilon_0|} \Re [F_\varepsilon \int_0^\infty R_0^* R_\varepsilon^{(2)} r^2 dr]. \end{aligned} \quad (12)$$

On the other hand, the correction to the value of the binding energy of the ground state owing to the axial dipole term takes the form, on account of (6.121-16),

$$\Delta \varepsilon_0 = \frac{9}{1/2} \Re \int_0^\infty R_0^* F \chi_2 dr = -18 \int_0^\infty \frac{d\varepsilon}{\varepsilon + |\varepsilon_0|} |F_\varepsilon|^2; \quad (13)$$

it is a quantity of the *second* order.

16.13. Stationary states of not too high energy. So long as the energy involved in the state or process envisaged is sufficiently small (compared with the unit $\hbar^2 \kappa^2 / M$, 5.11), a general survey of the influence of a non-central coupling can be obtained, as shown by HEPNER and PEIERLS [42], without making any assumption on the form of this coupling. The procedure utilized by these authors consists in neglecting, in all radial integrals, the contributions from the "inside" region, in which the radial wave-functions are distorted by the nuclear potential. In the "outside" region, i.e. for r larger than some distance r' , the radial components χ_0 , χ_2 can be replaced by the asymptotic expressions

$$\begin{aligned} \chi_0(r) &\simeq e^{-r/a_0} \\ \chi_2(r) &\simeq e^{-r/a_0} \left[\left(\frac{a_0}{r} \right)^2 + \frac{a_0}{r} + \frac{1}{3} \right]; \end{aligned} \quad (14)$$

in these formulae, a_0 denotes the characteristic length (6.21-14) $a_0 \equiv \hbar / \sqrt{M |\varepsilon_0|}$ directly related to the binding energy. If some plausible value ($\frac{1}{2}$, say) is assumed for the ratio r'/a_0 , the amount of D -admixture can be calculated, by the normalization condition, from the observed value of the quadrupole moment, and the eigenfunction of the ground state is then completely fixed in the outside region.

This suffices for a renewed discussion of the *photo-disintegration* of the deuteron (6.51) by quanta of not too high energy. The possible admixture of F state in the final P state of the photoelectric process may be disregarded, and the modification of the photoelectric cross-section is found to

be quite negligible. The photomagnetic effect now comprises an additional D - D transition, but its contribution to the cross-section is too small to be observed.

For the treatment of the triplet contribution to the *scattering cross-section of neutrons by protons*, we start from a set of equations analogous to (6), but with a positive energy value ε instead of ε_0 . The asymptotic solution is a superposition of S - and D -waves, each consisting of an ingoing plane wave and an outgoing spherical wave. The phases have to be determined from the conditions of continuity to be satisfied at the "boundary" $r = r'$ of the inside and outside regions. The unknown behaviour of the inside wave-functions enters into the result only through three parameters which indirectly represent the influence of the nuclear interactions. One finds that for small values of the energy ε only the S -wave contributes appreciably to the scattering cross-section. The combination of the three parameters entering into the expression for this quantity can finally be reduced, by a procedure generalizing the argument of 5.32, to the characteristic length a_0 pertaining to the corresponding bound state of relatively low binding energy. The details of the argument will be found in Hepner and Peierls' paper. The outcome of the discussion is again to bring out the insignificant influence of the axial dipole coupling in the energy region considered.

16.2. The Rarita-Schwinger theory

16.20. To proceed further, it will be necessary to make more detailed assumptions on the distance dependence of the nuclear forces. The schematic representation of both the central part $J(r)$ and the non-central part $F(r)$ by wells of suitable widths and depths has the obvious advantage of simplicity and would seem sufficient to give at any rate an idea of the order of magnitude of the various effects investigated. Calculations on this basis have been carried out by RARITA and SCHWINGER [41a, b, c]; these authors, however, have introduced an additional simplification, viz. the assumption of wells of equal widths for $J(r)$ and $F(r)$, which detracts to some extent from the generality of the theory. We shall therefore limit ourselves to a brief survey of the main points of interest; for details, the reader may turn to the original papers*.

* Some of the results of Rarita and Schwinger's discussion have recently been extended by MASSEY and HU [47b] to other forms of interactions (viz. the exponential, Gauss and meson potential), the essential assumption being to take for both $J(r)$ and $F(r)$ the same form of potential and the same range. In the same category fall some (partly incorrect) calculations by FLÜGGE [39], who took for $J(r)$ and $F(r)$ exponential potentials of equal ranges, and a recent work by MOSZKOWSKI and SACHS [48], using Gauss potentials (unfortunately with an inadequate choice of the range).

It should also be mentioned that results similar to Rarita and Schwinger's had been obtained a little earlier by KLIQMAN [40], but escaped notice owing to their publication in Russian.

16.21. The ground state of the deuteron. The eigensolutions of the system (16.11–6), when $J(r)$ and $F(r)$ are assumed to be potential wells of the same width D and depths J and $F \equiv \alpha J$, can be calculated explicitly (RARITA and SCHWINGER [41b]): outside the well, they have exactly the form (16.13–14); the inside part can be adjusted with the help of the continuity conditions, and the remaining amplitudes completely fixed by the normalization condition. The numerical values of the parameters must be chosen so as to account for both the binding energy ε_0 and the electric quadrupole moment Q . For every assumed value of the range D , this fixes the depths J and F .

In particular, for $D = d$, one finds

$$J = 13.9 \text{ MeV}, \quad \alpha = 0.775. \quad (1)$$

Comparing this result with that obtained with purely central forces, viz. (table 6.431–3) $J = 20.9 \text{ MeV}$, we see that — although the amount of D -admixture turns out to be as small as 4 % — the contribution of the non-central force to the binding energy would be very large indeed. In fact, the strength of the central potential for ^3S states is reduced almost to the same value as that for ^1S states (viz. 11.8 MeV), so that practically the whole separation of the ^3S and ^1S levels arises from the non-central coupling. This is the most striking feature of the Rarita-Schwinger theory, — a feature it shares, as we shall see, with most variants of meson theory; for brevity, we shall speak in such cases of a “large” non-central coupling.

This situation sharply contrasts with the possibility of a non-central coupling allowing of a perturbation treatment (16.12), which will be called the case of “small” non-central coupling. It must clearly be expected that the use of the perturbation method would not be allowed for the Rarita-Schwinger potential; indeed, an early calculation by CHRISTY and KUSAKA [39] using this method led, for the same width value $D = d$, to a value of $\alpha \approx 0.18$, i.e. to an extreme underestimate of the influence of the axial dipole coupling on the binding energy.

16.22. Proton-neutron scattering. For the scattering of neutrons of not too high energy by protons, the accurate computations of RARITA and SCHWINGER [41b] of course confirm the general conclusion arrived at by Hepner and Peierls (16.13). Compared with the expression derived on the assumption of central forces, the triplet contribution to the scattering cross-section is reduced by a few percent; while this goes towards decreasing the outstanding discrepancy (6.431) between the experimental values and those calculated for a well width $D = d$, the correction is too small to be of any avail in judging the reality of this discrepancy.

Interesting new results are obtained in the domain of higher energies of the two-nucleon system, when states of higher orbital momenta, and in the first place the P states, have to be taken into account. A distinction must then be introduced with respect to the dependence on the isotopic

variables, for the properties of the odd states are essentially different for different choices of this dependence (8.32). RARITA and SCHWINGER [41c] take a nuclear potential of the general form

$$\mathcal{V}_{\text{nuc}} = l^{(12)} (\alpha_0 + \alpha_\sigma \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} + \alpha_D \mathcal{D}^{(12)}) J(r), \quad (2)$$

thus assuming the same isotopic factor for the central and the non-central part of the potential: this assumption appears, of course, quite natural, especially from the point of view of field theory*. They discuss three cases of isotopic dependence: symmetrical, charged and neutral. The relations between the properties of odd-state potentials in these cases are more complicated than those derived in 8.32 for central interactions. Of course, the formulae (8.32–18, 19) retain their validity, provided the symbols occurring in them are understood to denote potential operators; they may be supplemented by analogous relations pertaining to the charged case:

$$\sigma \mathcal{V}_{\text{charged}}^{(\text{odd})} = - \sigma \mathcal{V}_{\text{charged}}^{(\text{even})} \quad (\sigma = 1, 3; \text{deuteron}). \quad (3)$$

But the notion of (attractive or repulsive) effective potential cannot be upheld in general.

In the case of P states, however, if one neglects the coupling with F states for $j=2$, it is still possible to define effective potentials and to study their signs in the different cases of isotopic dependence. These potentials will be different in the three substates ${}^3P_0$, ${}^3P_1$, ${}^3P_2$. The parameters α_0 , α_σ , α_D in (2) are fixed by the requirement that the ${}^3S + {}^3D$ and ${}^1S$ potentials have the respective forms

$$\begin{aligned} \mathcal{V}({}^3S + {}^3D) &= -(1 + 3\alpha \mathcal{D}^{(12)}) J(r) \\ \mathcal{V}({}^1S) &= -q' J(r), \end{aligned} \quad (4)$$

α being the depth ratio introduced above in the discussion of the ground state (16.21). This implies, on the one hand,

$$\alpha_0 - 3\alpha_\sigma = -q', \quad (5)$$

on the other,

	symm.	charged	neutral	
$\alpha_0 + \alpha_\sigma =$	$\frac{1}{3}$,	1,	-1	(6)
$\alpha_D =$	α ,	3α ,	-3α .	

If we now write the effective potentials in ${}^3P_j$ states on symmetrical theory in the form

$$\mathcal{V}({}^3P_j)_{\text{symm}} = \frac{1}{3} {}^3p_j J(r), \quad (7)$$

* There is a certain similarity between the operator (2) of the symmetrical Rarita-Schwinger theory and that of the corresponding pseudoscalar meson theory (16.31–1). In fact, a slight reduction of the value assumed for the range of force to $D = 2.7 \cdot 10^{-13}$ cm causes the constant α_0 to vanish, so that, by formula (6) below, the coefficient α_σ takes the value $\frac{1}{3}$. In this case, according to (5), $q' = 1$, i.e. the binding of the deuteron is entirely due to the axial dipole term.

we have in the first place, on account of (8.32–18, 19), (3) and (6),

$$\begin{aligned}\mathcal{V}(^3P_J)_{\text{charged}} &= {}^3p_J J(r) \\ \mathcal{V}(^3P_J)_{\text{neutral}} &= -{}^3p_J J(r).\end{aligned}\quad (7a)$$

Further, with the help of the formulae (16.11–2) for the diagonal elements of the operator $\mathcal{D}^{(12)}$, we find

$${}^3p_0 = -(4\alpha - 1), \quad {}^3p_1 = 1 + 2\alpha, \quad {}^3p_2 = 1 - \frac{2}{3}\alpha. \quad (8)$$

With a value of α of the order of magnitude indicated in (1), we see that ${}^3p_0$ is negative, while ${}^3p_1$ and ${}^3p_2$ are positive. *In the symmetrical or charged Rarita-Schwinger theory, therefore, the ${}^3P_0$ potential is attractive, while the ${}^3P_1$ and ${}^3P_2$ potentials are repulsive; in the neutral theory, the situation is reversed.* On account of the smaller statistical weight of the ${}^3P_0$ state, its effect will generally not upset the trend of the two others, so that the situation, so far as no higher than P states are concerned, is qualitatively much the same as that noted in 8.32 for central interactions.

This is indeed apparent in the proton-neutron scattering problem. Numerical computations of the differential cross-section have been carried out for a kinetic energy $E = 15.3$ MeV of the incident neutrons, using the range value $D = d$ and the corresponding parameter values (1), together with the ${}^1S$ strength ${}^1J^{(0)} = 11.8$ MeV (i.e. $q' \approx 0.85$). A weakness of such calculations, which equally affects those pertaining to meson theories with large axial dipole interaction, presently to be discussed, is that (owing to the prohibitive amount of numerical computation involved) they only include the contributions from P -waves but neglect those from waves of higher orbital momenta. The experience with analogous calculations for central forces (8.32) leads us to believe that such a neglect may involve serious errors. Be this as it may, Rarita and Schwinger's results exhibit the same contrast between symmetrical and neutral theory as those of central force theory, even in a still more accentuated manner: the anisotropy ratio $A = S(\pi)/S(\frac{1}{2}\pi)$ (8.32–23) has the values

$$A_{\text{symm}} = 1.16, \quad A_{\text{neut}} = 0.525 \quad (E = 15.3 \text{ MeV}); \quad (9)$$

the charged theory, with

$$A_{\text{charged}} = 0.92. \quad (9a)$$

occupies an intermediate position. The symmetrical theory is quite compatible with the results of the Bristol experiments, which, on the other hand, would lead to the rejection of the neutral theory.

This last conclusion is confirmed in an interesting way by the discussion of the total cross-sections:

$$\begin{aligned}S_{\text{symm}} &= 0.621 \cdot 10^{-24} \text{ cm}^2 \\ S_{\text{charged}} &= 0.666 \cdot 10^{-24} \text{ cm}^2 \quad (E = 15.3 \text{ MeV}) \\ S_{\text{neut}} &= 0.983 \cdot 10^{-24} \text{ cm}^2.\end{aligned}\quad (10)$$

While the two first figures are of the expected magnitude (6.413), the last one shows a considerable deviation from this value, due to the fact that in this case the contribution of the P -wave to the *total* cross-section is not small. This contradicts the criterion enunciated in 6.431. All fast neutron scattering experiments therefore concur to exclude the neutral variant of the Rarita-Schwinger theory; on the other hand, they are compatible with the symmetrical form of the theory.

16.23. Radiative processes. For relatively small energies, RARITA and SCHWINGER [41b] have verified that no significant evidence regarding the axial dipole coupling can be expected from measurements in this region of either the photo-disintegration of the deuteron or the radiative capture of neutrons by protons. At higher energies, however, the influence of states of higher orbital momenta again makes itself felt in a characteristic way (RARITA, SCHWINGER and NYE [41a], RARITA and SCHWINGER [41c]). The calculations have been performed for the γ -rays emitted in the reaction ${}^7\text{Li}(p, \gamma)$, of energy $h\nu = 17,5$ MeV. For such energies, the photomagnetic contribution, as well as that due to electric quadrupole transitions, is quite negligible on any theory, so that we have essentially to do with the photo-electric effect, due to electric dipole transitions.

The most striking effect of the non-central coupling is to modify the simple $\sin^2\theta$ -law of angular distribution of the ejected nucleons. This modification arises from the fact (16.22) that the radial components of the ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$ final states are affected in different ways by the axial dipole interaction, so that there appears in the differential cross-section, besides the $\sin^2\theta$ term, an additional term which is isotropic:

$$d\Phi_{\text{el}}(\theta) \sim \sin^2\theta + \Phi_0. \quad (11)$$

The ratio H (8.34) of the intensities of ejected nucleons in directions parallel and perpendicular to that of incidence of the radiation has therefore, in contrast to the case of central forces, a non-vanishing value

$$H = \frac{\Phi_0}{1 + \Phi_0} \quad (12)$$

even at high energies. For the three cases of isotopic dependence, one finds:

	Φ_0	H	$(h\nu = 17,5 \text{ MeV}) \quad (13)$
Symmetrical	0,015	0,015	
Charged	0,077	0,071	
Neutral	0,36	0,265	

The reason for the large difference between the symmetrical and the neutral theory is to be sought in the predominant influence of the strong ${}^3P_1$ attraction in the latter theory, which produces a relatively large

perturbation of the corresponding state. Clearly, since the occurrence of the isotropic term Φ_0 is entirely the result of the perturbations of the ^3P states, an appreciable value of this term will then be expected.

On the other hand, the effect of such an attraction in the final P state will be to decrease the matrix-element of the electric dipole if the wavelength of the ejected nucleons is comparable with the radius of the deuteron in its ground state. The *total* photoelectric cross-section will thus be smaller on neutral than on symmetrical theory. Indeed, the computation yields

$$\begin{aligned}\Phi_{\text{symm}} &= 0.768 \cdot 10^{-27} \text{ cm}^2 \\ \Phi_{\text{charged}} &= 0.723 \cdot 10^{-27} \text{ cm}^2 \quad (\hbar\nu = 17.5 \text{ MeV}) \\ \Phi_{\text{neutral}} &= 0.376 \cdot 10^{-27} \text{ cm}^2.\end{aligned}\tag{14}$$

It appears from these results that measurements, with high energy γ -rays, of the total disintegration cross-section and, still better, of the angular distribution of the ejected nucleons could lead to a decision between the neutral and the symmetrical Rarita-Schwinger theory. But if we have to deal with a symmetrical isotopic factor, it would be very difficult indeed to trace the difference between the Rarita-Schwinger interaction, with large non-central coupling, and a purely central potential.

16.3. Meson theories with cut-off

16.30. Just as it is to be regretted that the assumptions of the Rarita-Schwinger theory are too narrow to permit a general survey of the properties of axial dipole couplings, one must deplore the lack of any systematic discussion of meson field theories using some cut-off device to eliminate the singularities of interactions at small distances. The curse of all meson theories, especially those with large non-central coupling, is the amount of laborious numerical computations involved in any problem. As a result of the chaotic situation prevailing among research workers in theoretical physics, the outcome of the active discussions on this topic is a medley of fragmentary information from which it is difficult to gain a general idea, while the gaps, on account of the prohibitive numerical work involved, cannot readily be filled. This section will therefore be nothing more than a very unsatisfactory attempt at piecing together the disconnected results available on the subject. The general conclusion, so far as it goes, is very unfavourable for cut-off theories.

16.31. Symmetrical pseudoscalar theory. Among the meson theories using only one definite kind of field, the choice, from general considerations, would seem to be fairly obvious: in order to secure agreement with the observations on the spin of the cosmic ray mesons and to get a non-central term of static interaction, we must adopt the *pseudoscalar* type of meson field. According to the rule (16.11-7), the sign of the deuteron quadrupole moment, considering the form of the static interaction given by table 15.31 together with formula (15.31-12), further imposes the *symmetrical* com-

bination of charged and neutral fields. The nuclear potential is thus, on this theory,

$$\mathcal{V}_{\text{ps. scal.}} = f_2^2 \tau^{(1)} \tau^{(2)} \left[\frac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \varphi(r) + \mathcal{D}^{(12)} F(r) \right]. \quad (1)$$

The binding of the ground state of the deuteron is entirely determined by the axial dipole term, and depends therefore sensitively on the cut-off radius. Once a definite value has been assumed for the range of the force, i.e. for the meson mass, the cut-off radius and the source constant are fixed by the binding energy of the deuteron and the slow neutron scattering cross-section. *All other properties of the ground state, in particular the amount of D-admixture and the electric quadrupole moment, then follow unambiguously and allow a test of the theory: this is an attractive feature of any cut-off theory working with only one source constant.*

Unfortunately, when this procedure is carried out (FERRETTI [41b]), assuming for the meson mass the (somewhat low) value $M_m = 177 m$, the cut-off radius turns out to be larger than the range κ^{-1} of the potential. This means, of course, that the specification of the distance dependence by the meson potential becomes illusory. The manner of cutting-off is immaterial; besides the zero and straight cut-off (15.32), Ferretti tried the extreme assumption of a very large repulsive potential at small distances, without achieving any substantial improvement.

16.311. Symmetrical zero spin meson theory. As a possible way out of this difficulty, FERRETTI [41b] suggests an ingenious modification of the theory, which consists in assuming only the neutral mesons to be of the pseudoscalar type, while the charged mesons would be scalar ones; the "symmetrical" character of the theory is upheld, however, by maintaining the variables $\tau_j^{(i)}$ in the expressions for the source densities (8.31-5, 10) and assuming the source constants f_2 and f_1' to be equal in absolute value. The resulting nuclear potential is therefore (8.31-7)

$$\mathcal{V}_{\text{Ferr}} = f_2^2 [\tau_3^{(1)} \tau_3^{(2)} \left(\frac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \varphi + \mathcal{D}^{(12)} F \right) - 2 \mathcal{T}^{(12)} \varphi]; \quad (2)$$

it differs from that of the purely pseudoscalar theory (1) by a term

$$-2 f_2^2 \mathcal{T}^{(12)} \left[\left(\frac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} + 1 \right) \varphi + \mathcal{D}^{(12)} F \right] \quad (3)$$

which vanishes in ^1S states; the charge independence property is thus restricted to those states only (8.1). In the ground state of the deuteron, the potential due to the scalar field is repulsive; consequently, the axial dipole term must yield a stronger attraction than in the potential (1), and must thus be cut-off at a smaller distance. A reasonable value of the cut-off radius is obtained, as well as the right sign and order of magnitude of the quadrupole moment. Ferretti, however, does not insist further on this theory, which is admittedly rather artificial.

16.32. Neutral vector theory. It is somewhat paradoxical that the type of meson field which has most extensively been studied, should be just the *vector* field, for no choice could be more unfortunate; not only because it corresponds to a wrong value of the spin, but also because the right sign of the deuteron quadrupole moment, by the same argument as above (16.31), can only be given by the *neutral* variant, against which the gravest objections can be raised (8.31). Nevertheless, when BETHE [40] published a very thorough investigation of a special form of vector meson theory, this theory was uncritically adopted by various physicists * as the basis of further discussion. We shall here rapidly review the main facts brought to light by this ill-spent labour.

Bethe's potential is that corresponding to a tensor source only ($g_1 = 0$), i.e. [table 15.31 and formula (15.31-12)]:

$$V_{\text{Bethe}} = g_2^2 \left[\frac{2}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \varphi - \mathcal{D}^{(12)} F \right]. \quad (4)$$

In triplet states, this expression has, qualitatively, the same properties as Ferretti's potential (2) and leads, therefore, to plausible results for the ground state of the deuteron. Bethe adopts the value $M_m = 177 m$ for the meson mass: since we are dealing with neutral mesons, this choice, though of course inspired by the value then believed to apply to the observed charged mesons, is in effect entirely arbitrary. The cut-off radius depends only slightly on the particular way of cutting-off and turns out to be ≈ 0.4 or 0.3 of the range; further, one finds $g_2^2/\hbar \approx 0.08$, corresponding to $1j^{(0)} \approx 14.2$ MeV. The resulting proportion of D state is a little too high (6.6 or 6.8 %), but the quadrupole moment comes out remarkably well.

As regards the effective potentials in ${}^3P$ states, the situation in Bethe's theory is essentially different from the Rarita-Schwinger case (16.22): the ${}^3P_1$ potential is attractive, while the other two are repulsive. No general prediction can consequently be made about the anisotropy of the scattering of fast neutrons by protons. This problem has first been treated summarily by BRIIT† and KITTEL [39c] and later very extensively by FERRETTI [41a, b]. Numerical results are available for an energy $E = 17.4$ MeV of the incident neutrons, account being taken of the P -wave scattering only. In contrast with the Rarita-Schwinger theory, and in agreement with experiment, it is found that the P -waves do not substantially contribute to the *total* scattering cross-section. But the anisotropy

$$A_{\text{Bethe}} = 0.545 \quad (E = 17.4 \text{ MeV}) \quad (5)$$

indicates a strong predominance of forward scattering, quite at variance with the trend suggested by the Bristol experiments.

The *proton-proton scattering* is only slightly affected, at low energies, by the P -wave anomaly (7.12). Computations of this anomaly on

* As I mentioned my doubts about Bethe's theory to a distinguished American friend, he dismissed all objections with the remark: "Anyway, it is the latest!".

Bethe's theory have been carried out by BREIT *et al.* [40] and by THAXTON *et al.* [40b] for the range 0.67 ... 3 MeV of proton energies. The *P*-wave anomaly is larger than on the assumption of a central potential; in fact, it is too large to agree with observation, unless the meson mass be accorded a much larger value than is assumed in Bethe's theory. The cross-section for the *photo-disintegration of the deuteron*, calculated by RARITA, SCHWINGER and NYE [41a] for 17.5 MeV photons, gives the following results, similar to those of Rarita and Schwinger's neutral theory (16.23–13, 14):

$$\begin{aligned}\Phi_0 &= 0.26, & H &= 0.206 & (h\nu = 17.5 \text{ MeV}) \\ \Phi_{\text{Bethe}} &= 0.376 \cdot 10^{-27} \text{ cm}^2.\end{aligned}\quad (6)$$

16.33. *Unsymmetrical zero spin meson theory.* At the time that the Italian results about fast neutron scattering (6.42) seemed to point to a field theory of the neutral type, an interesting attempt was made by HULTHÉN [44a, b, 45b] to set up a theory of this type, using only scalar and pseudoscalar fields, and taking account in an essential way of the existence of charged mesons. Hulthén assumes that the meson field produced by nucleons is mostly of the *neutral scalar* type, but that a *charged pseudoscalar* field can also be produced with a much smaller intensity. The nuclear potential resulting from these assumptions may therefore be written

$$V_{\text{Hulth}} = -f_1'^2 \varphi_0 + 2 T^{(12)} f_2^2 \left[\frac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \varphi + \mathcal{D}^{(12)} F \right], \quad (7)$$

with

$$f_2^2 \ll f_1'^2; \quad (8)$$

the symbol φ_0 indicates that the mass of the scalar meson is not assumed to be identical with that of the pseudoscalar one, so that the range of the potential φ_0 is not necessarily the same as that of φ and F ; in fact, Hulthén adopts for the scalar meson the mass value $M_m^0 = 330 m$ indicated by the proton-proton scattering experiments (7.13), while the mass of the pseudoscalar mesons is naturally the observed one. The effective ${}^1S$ potential of the deuteron system differs by $-2f_2^2 \varphi$ from that of the like nucleon system $-f_1'^2 \varphi_0$: this is qualitatively in accordance with the conclusion drawn from the analysis of the scattering data (8.1); the figures of table 8.1 show that the ratio $f_2^2/f_1'^2$ should not exceed a few percent.

Obviously, in such a theory, the separation between ${}^3S$ and ${}^1S$ states of the deuteron arises from the non-central coupling due to the charged field; owing to the small value of the coupling constant, the cut-off radius must therefore be taken rather small, viz. $\approx |f_2/f_1'| \kappa^{-1}$, which is a satisfactory feature of the theory. On the other hand, a quadrupole moment of the right order of magnitude is obtained, as is verified with the help of the perturbation formula (16.12–12). Hulthén's theory predicts a predominantly forward scattering of fast neutrons by protons, in contradiction with the Bristol experiments (HULTHÉN [44a]).

16.4. Mixed theories

16.40. Let us finally turn our attention to mixed theories (15.32). The alternative here is between theories using mesons of different or equal masses. At first sight, the former possibility seems to afford a simpler picture, since it yields a static axial dipole coupling and one might thus expect to do without the non-static couplings, which, on the contrary, are essential in the case of a mixture of equal mass mesons. From the quantitative point of view, however, the static approximation of Schwinger's mixture proves insufficient. We cannot, therefore, dispense with the introduction of the velocity dependent forces. If, on the other hand, we treat the non-static axial dipole coupling by perturbation theory, we find that, with a reasonable assumption about the order of magnitude of the relevant source constants, it is just sufficient to account for the observed *D*-admixture of the ground state and its quadrupole moment. It would thus seem that the hypothesis of different masses for spin zero and spin one mesons — as long as no empirical evidence in its favour is forthcoming — is an unnecessary complication: this would bring us back to Møller and Rosenfeld's original proposal of a mixed theory using mesons of uniform mass. However, a closer examination shows that the results of perturbation calculations in this case are very unreliable, so that the assumption of equal masses does not lead to a simpler description of nuclear forces. The present section contains the elaboration of the preceding argument.

16.41. Schwinger's mixed theory. The mixed theory proposed by SCHWINGER [42] is of the *symmetrical* type; the nucleons are assumed to give rise to a *pseudoscalar* and a *vector* meson field; the source constants g_2, f_2 are connected by the relation (15.32–13) or

$$g_2^2 = f_2^2 \mu^2, \quad (1)$$

where μ is the ratio of the masses of vector and pseudoscalar mesons. Let us further put

$$g_1^2 = f_2^2 \gamma^2.$$

If κ denotes the inverse range of the pseudoscalar field (and, therefore, $\kappa\mu$ that of the vector field), the nuclear potential of Schwinger's theory may be written

$$V_{\text{Schw}} = f_2^2 \tau^{(1)} \tau^{(2)} \left[\frac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \frac{e^{-\kappa r} + 2\mu^2 e^{-\kappa\mu r}}{r} + \gamma^2 \frac{e^{-\kappa\mu r}}{r} + \mathcal{D}^{(12)} (F_\kappa - \mu^2 F_{\kappa\mu}) \right]. \quad (2)$$

The sign of the deuteron quadrupole moment, considering formula (15.32–14), requires

$$\mu > 1. \quad (3)$$

In order to determine the numerical value of μ , JAUCH and HU [44] have calculated, for different choices of this parameter, the value of f_2^2 required to yield, on the one hand, the binding energy of the deuteron, on the other the slow neutron scattering cross-section; they assumed $\gamma = 0$ and took for the mass of the pseudoscalar meson the value $177m$, a little too low according to the most recent estimates (1.331). The intersection of the two curves $f_2^2(\mu)$ gives

$$\mu = 1.6, \quad f_2^2/\hbar = 0.05. \quad (4)$$

Unfortunately, for these values of the parameters, the calculated value of the quadrupole moment, using only the static potential (2), comes out about 3 times too small. This, as stated above, spoils the economy of Schwinger's theory. For, instead of putting simply $f_1' = 0$ so as to cancel the velocity dependent coupling altogether, we should choose for this source constant some suitable non-vanishing value, in order to get an additional contribution of the required magnitude to the axial dipole potential. The question then arises whether one could not just take $\mu = 1$ and make the non-static potential responsible for the whole observed deviation from spherical symmetry in the ground state of the deuteron.

The problem of fast neutron scattering has been treated by JAUCH [45] on Schwinger's theory, at any rate so far as P -wave contributions are concerned. He finds an anisotropy ratio

$$A_{\text{Schw}} = 1.1 \quad \text{for } E = 14 \text{ MeV}, \quad (5)$$

of much the same order as in the symmetrical theory of Rarita and Schwinger (16.22-9). Compared with the results (table 8.33) due to a purely central meson potential, this figure indicates a tendency of the axial dipole coupling to make the angular distribution of the scattered particles more uniform.

Finally, it might be enquired whether the discrepancy between the value of the mass of the cosmic ray mesons and that deduced from the range of the meson potential required to explain the results of proton-proton scattering experiments (7.13), if real, could be brought into harmony with the point of view of Schwinger's theory: it would be taken to indicate the presence of a field corresponding to mesons of larger mass than those observed in cosmic radiation, and which would be identified with the vector mesons postulated by the theory. A closer examination of this point, carried out at my request by Dr. RAMSEY, leads, however, to a negative conclusion. Let us indeed compare the ^1S effective potential derived from the Schwinger interaction operator (2), viz.

$$\mathcal{V}_{\text{Schw}}(^1S) = -f_2^2 [e^{-\mu r} + (2\mu^2 - \gamma^2) e^{-\gamma\mu r}] \frac{1}{r}, \quad (2a)$$

with a simple meson potential of the form

$$\mathcal{V}_{\text{mes}}(^1S) = -G^2 \frac{e^{-\kappa r}}{r}. \quad (2b)$$

If we identify these two expressions and their derivatives for some value r_0 of r , we readily find that the range parameter K is a monotonic function of r_0 ; the sign of dK/dr_0 being that of $(\gamma^2 - 2\mu^2)$. Now, since one would expect that the proton-proton scattering is mainly determined by the behaviour of the nuclear interaction at larger distances than the neutron-proton scattering or binding, the trend of empirical evidence, pointing to a larger value of the range parameter for the proton-proton than for the neutron-proton potential, would be interpreted, from the present point of view, to require that K must be an *increasing* function of r_0 , i.e., that $\gamma^2 > 2\mu^2$. It is easily seen, however, that this condition, together with $\mu > 1$, would lead to such high values of the coupling constants that the static approximation would break down altogether. Moreover, the meson masses would have to be taken *larger* than the large value suggested by the proton-proton scattering experiments, in contradiction with the cosmic ray data.

16.42. Moller and Rosenfeld's mixed theory. The considerable simplification, in mixed theory, implied by the hypothesis of a unique meson mass is that the nuclear interaction is then reduced, in first approximation, to the form of a central potential. The choice of suitable mixtures can be based on the consideration of this static approximation only, as embodied in table 15.31. An easy discussion (MØLLER and ROSENFELD [40]), starting from the purely qualitative requirements that (a) the effective potentials in triplet and singlet even states must be attractive, (b) the triplet potential must yield a stronger attraction than the singlet one, leads to only two essentially distinct mixtures compatible with the charge independence property. So far as these requirements go, one may choose either a *symmetrical* type of mixture of a *pseudoscalar* and a *vector* field, or a mixture of *neutral scalar* and *pseudovector* fields. One could add to either mixture further fields of the other types, but most people will feel that a two-field mixture is about as much as they can stand. The evidence about two-nucleon systems has been extensively discussed, from the point of view of central interactions, in Part II of this work. From the values (6.432-8) derived from that analysis for the effective potential strengths ${}^3J^{(0)}$ and ${}^1J^{(0)}$ in terms of the meson mass, the expressions for the source constants entering into the static potential are readily obtained. In the case of the symmetrical mixed theory, e.g., for which (table 15.31)

$$\mathcal{V}_{MR} = \tau^{(1)} \tau^{(2)} [g_1^2 + g_2^2 \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}] \varphi(r) \quad (6)$$

with

$$f_2^2 = g_2^2, \quad (7)$$

one has

$$\frac{g_1^2}{\hbar} = \frac{1}{4M_m} ({}^3J^{(0)} - {}^1J^{(0)}), \quad \frac{g_2^2}{\hbar} = \frac{1}{4M_m} (\frac{1}{3} {}^3J^{(0)} + {}^1J^{(0)}); \quad (8)$$

the value of

$$\frac{g_1^2}{\hbar} \approx 0.03 \quad (9)$$

is approximately independent of the meson mass, whereas that of

$$\frac{g_2^2}{\hbar} \approx 3.06 \cdot 10^{-4} \frac{M_m}{m} + 0.004 \quad (9a)$$

varies nearly linearly with this mass;

$$\frac{g_2^2}{\hbar} \approx 0.065 \quad \text{for } M_m = 200 m. \quad (9b)$$

Proceeding to the examination of the velocity dependent potentials (15.34), we may at once eliminate the neutral variant (HOLMBERG [44], ROSENFELD [45b]), since in that case there would be no combination of *S* and *D* states to a first approximation. We are therefore left, as *the only acceptable mixture of two meson fields of equal masses, with the symmetrical pseudoscalar-vector theory*. On this theory, the velocity dependent potential, when rid of all contact interactions, consists of additional contributions to the central couplings

$$-\tau^{(1)}\tau^{(2)}\frac{M_m}{M}[g_1g_2+\frac{1}{3}(2g_1g_2-f_1f_2)\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}]\varphi(r) \quad (10)$$

and of the non-central interaction

$$\tau^{(1)}\tau^{(2)}\frac{M_m}{M}[(g_1g_2+f_1f_2)\mathcal{D}^{(12)}F(r)-\frac{2g_1g_2}{\hbar}\frac{\varphi'(r)}{x^2r}\mathcal{M}^{(12)}], \quad (11)$$

of which only the first term, representing an axial dipole coupling, needs to be taken into consideration (16.11). In view of the factor M_m/M , one is tempted to regard this coupling as "small" (16.21), in contrast to the static one occurring in Schwinger's mixed theory. When treated as a perturbation (16.12), the interaction (11) only gives rise to a second order displacement of the ground state of the deuteron. At the same time, it yields according to (16.12-12) an electric quadrupole moment Q proportional to $g_1g_2+f_1f_2$. Even a rough estimate of Q (MØLLER and ROSENFELD [40], HULTHÉN [43a]) shows that the calculated value can be adjusted to fit observation by giving the combination of source constants $g_1g_2+f_1f_2$ some positive value of the same order of magnitude as g_2^2 or g_1^2 . This implies that the only effect of the additional central potential (10) will be to modify slightly the determination (7), (8) of the source constants. As a matter of fact, if we also take the experimental value of the quadrupole moment into account, we get instead of these formulae a system of equations of the form

$$\begin{aligned}
 \frac{g_1^2}{\hbar} - \frac{M_m}{M} \frac{g_1 g_2}{\hbar} &= \alpha \\
 \frac{g_2^2}{\hbar} - \frac{1}{3} \frac{M_m}{M} \frac{2g_1 g_2 - f_1 f_2}{\hbar} &= \beta \\
 \frac{g_1 g_2 + f_1 f_2}{\hbar} &= \gamma \\
 \frac{g_2^2}{\hbar} &= \frac{f_2^2}{\hbar},
 \end{aligned}
 \tag{12}$$

α, β, γ being positive numerical quantities of comparable orders of magnitude. It is easy to see that this system yields two essentially distinct solutions*, according to the sign assumed for the product $g_1 g_2$.

It may be mentioned that the influence of the velocity dependent potential (11) on the scattering of fast neutrons by protons has been summarily discussed by HULTHÉN [43b], using Born's approximation. His formulae, however, do not correspond to the velocity dependent potential defined by (10), (11) above, but to a potential including in addition the contact interactions which we have systematically discarded.

16.43. "Large" or "small" axial dipole coupling. The conclusion of the preceding subsection would seem to be altogether in favour of the simpler solution afforded by the mixture of meson fields of equal masses. But the argument is essentially based on the application of the perturbation method to the non-static axial dipole coupling; and in view of the rather bad convergence to be expected in this case for perturbation calculations (15.34), owing to the peculiar distance dependence of the axial dipole coupling, it may be questioned** whether the contrast between "large" and "small" axial dipole coupling is really as marked as is suggested by the presentation adopted here for methodical reasons. With a view to settling this question, the numerical integration of the radial wave-equations (16.11-i) for the ground state of the deuteron has now been undertaken by Mr. GROSJEAN, assuming in (16.11-4) the forms of potentials $J(r)$ and $F(r)$ corresponding to meson theory, viz.

$$J(r) = {}^3J \varphi(r), \quad F(r) = F F(r). \tag{13}$$

with arbitrarily fixed ratio of the strengths ${}^3J$ and F , and, of course, some cut-off prescription. Actually, the zero cut-off is adopted, and for each given value of the ratio $F/{}^3J$, the value of ${}^3J$ corresponding to the empirical value of the binding energy is determined as the solution of an

* There subsists, of course, an arbitrariness in the signs corresponding to the invariance of the equations (12) for a simultaneous change of sign of g_1 and g_2 , or f_1 and f_2 , or both pairs. But this arbitrariness has no physical significance.

** I am indebted to Prof. Peierls for stimulating discussions on this point.

eigenvalue problem; the eigenfunctions χ_0, χ_2 give directly in each case the corresponding amount of D -admixture and electric quadrupole moment.

Although this investigation is not yet completed, the first results already allow some general conclusions to be drawn. In the first place, if we adopt for the ratio

$$\alpha = - \frac{av}{3J} \frac{|^{(12)} \cdot F}{\dots} \quad (14)$$

numerical values of the order of 10^{-1} , corresponding to the expression for α ,

$$\alpha = \frac{M_m}{M} \cdot \frac{g_1 g_2 + f_1 f_2}{g_1^2 + g_2^2}, \quad (15)$$

given by the Møller-Rosenfeld symmetrical mixed theory, we find much too small values for the admixture of D state and the quadrupole moment: this means that estimates based on perturbation calculations are not to be trusted in this case. In fact, assuming the central part of the potential to give rise to an *attraction* in the ground state, one has to take, in order to arrive at the right orders of magnitude for the two quantities just mentioned, a value of α of the order of $\frac{1}{2}$; in other words one has to consider the axial dipole coupling as "large". This result, combined with the other arguments developed in the present Chapter, would seem to point to the necessity of adopting some mixture of meson fields, since no single type of field will yield the required value of α ; moreover, this mixture must be such as to give rise to an axial dipole coupling even in the static approximation. This last requirement rules out the simple case considered by Møller and Rosenfeld; it might be satisfied by a Schwinger mixture, but we have seen that the prospects of such a theory are not very encouraging. It would therefore seem that the original purpose of introducing a mixed theory, viz. to remove the r^{-3} singularity, has to be abandoned, and that we are left in the awkward position of having to adopt a mixed theory (with equal or different meson masses), which also involves a cut-off prescription.

16.5. General conclusion

All theories involving a non-central term of nuclear interaction are adjusted in such a way as to account for the properties of the ground state and the ^1S virtual level of the deuteron. The more ambitious ones (the cut-off theories) predict such quantities as the quadrupole moment and amount of D -admixture of the ground state. Others (the Rarita-Schwinger and the mixed theories) can only make use of these quantities to fix the values of their adjustable parameters; even so, the possibility of such a determination, leading to reasonable results, affords in some measure a check on the theory.

The various theories that have been put forward differ widely, however, as regards the properties of states of higher orbital momenta, in particular the P states. Their predictions in this respect can be put to test, at least in principle, in the study of the scattering of fast neutrons by protons. Although no obvious general reason can be given for this result, it is found that neutral theories with "large" axial dipole coupling generally predict a predominance of forward scattering ($A < 1$) for neutron energies of the order of 15 MeV, whereas symmetrical theories lead to predominantly backward scattering ($A > 1$) *. Consequently, the Bristol results would condemn neutral theories with "large" axial dipole coupling; on the other hand, they do not allow us to discriminate between several possible forms of symmetrical theories, nor to eliminate the possibility of a neutral theory involving only "small" axial dipole coupling (8.33). The removal of the latter uncertainty, however, and with it the full confirmation of the *symmetrical character of the nuclear interaction* requires only an extension of the measurements to slightly higher energies. A further corroboration of this conclusion may be expected from the study of the disintegration of the deuteron by photon or electron impact, either with increased accuracy in the domain of relatively low energies (8.34), or for higher energies (16.23, 16.32).

Among the variants of meson theory that have been proposed, none is entirely satisfactory. It seems that one should assume some form of *symmetrical mixed theory*, but probably one involving also a cut-off prescription; whether the masses of the different kinds of mesons occurring in the theory should be taken equal or different cannot be decided. In fact, there is hardly any hope of removing these uncertainties of the meson field description of nuclear forces without more information about the properties of mesons, furnished either by a more refined analysis of relevant cosmic ray data, or by the artificial production of mesons. Evidence from ordinary nuclear processes could only be helpful if its accuracy could be substantially increased. It will not be superfluous, in order to complete the picture of the present uncertainties and prospects of nuclear theory, to recall the further alternative which underlies the whole problem of nuclear forces, viz. that between the assumption of strong or weak coupling of the meson fields with their sources (1.35). It is true that the strong coupling treatment leads to very grave difficulties; but here also it is the experiment that has to make a final pronouncement, by the detection of isobaric states of nucleons or the disproof of the occurrence of such states.

* HULTHÉN [44a] has calculated the differential scattering cross-section on Born's approximation for various theories: symmetrical pseudoscalar theory (16.31), symmetrical and neutral vector theories (16.32), Schwinger's mixed theory (16.41) and his own unsymmetrical theory (16.33). For the anisotropy ratio A , this procedure yields an upper limit if $A > 1$, a lower limit if $A < 1$.

CHAPTER XVII

NON-CENTRAL COUPLINGS AND PROPERTIES OF HEAVIER NUCLEI

17.0. The subject of this last Chapter, which closes the circle of our inquiry, is difficult to approach and has received as yet only sporadic treatment. Although potentially capable of yielding valuable and even decisive information about the properties of nuclear forces, the study of heavier nuclei cannot at present give us more than a certain amount of corroborative evidence regarding conclusions drawn from the analysis of the properties of two-nucleon systems. In this respect, the examination of the conditions imposed on the form of the law of nuclear force by the saturation properties of heavy nuclei is especially interesting. Following a course closely parallel to that of Part III, we shall begin with a study of this general problem, and afterwards turn to more specific questions pertaining to light nuclei: we shall successively discuss the influence of non-central couplings on the binding energy and on the fine structure of the ground states of these nuclei.

17.1. Non-central couplings and saturation properties

The occurrence of non-central couplings in the expression of the interaction of a pair of nucleons may be of considerable influence on the saturation properties of nuclear systems bound by such interactions. This important problem has not yet been fully investigated. Two papers by VOLKOFF [42a, b], following a suggestion of WIGNER and EISENBUD [39a] *, discuss the effect of *axial dipole couplings*. The outcome of this discussion is to exclude such theories as involve an extreme predominance of the axial dipole interaction (e.g. Bethe's theory, 16.32), but no clear-cut conclusion is reached concerning intermediate cases like the Rarita-Schwinger theory (16.2).

The gist of the argument consists in showing that a predominant axial dipole coupling will tend to favour collapsed configurations of a heavy nucleus, characterized by large values of the total spin and possibly the neutron excess, and by a shape strongly deviating from spherical symmetry. We prove, to this effect, that in such configurations the contribution of the axial dipole operator (16.11)

$$\frac{1}{2} \sum_{i,k} \mathcal{D}^{(ik)} F(\mathbf{r}^{(ik)}) \quad (1)$$

to the interaction terms of the ordinary type (11.11) is attractive: this

* See also BETHE [40], p. 270.

means, indeed, that the resulting binding energy will not exhibit any saturation (11.13). In the first place, it is always possible to give the same sign to all contributions from the isotopic factors $I^{(ik)}$: in a neutral theory, this is automatically the case, irrespective of the neutron or proton character of the constituent nucleons; in a symmetrical theory, we have only to choose an assembly composed exclusively of nucleons of like charge. Consider, further, a configuration in which all the spins are parallel. Since in such a configuration the average value of $\mathcal{D}^{(12)}$ is $\cos^2 \vartheta - \frac{1}{3}$, we can make the total axial dipole interaction of the ordinary type attractive by suitably choosing the spatial distribution of the particle density: according as $F(r^{(ik)}) \text{av} I^{(ik)}$ is positive or negative, we have just to take an oblate or a prolate distribution with respect to the total spin axis.

It is true that the kinetic energy will counteract this effect, and one might think that the potential energy would only prevail for systems with larger numbers of nucleons than actual nuclei. The quantitative discussion of this point, carried out by VOLKOFF [42a] both in the case of Bethe's neutral vector theory and in the less extreme case of the neutral Rarita-Schwinger theory, clearly shows that the critical mass number, for which the non-saturated attraction begins to dominate, is well within the region of the actual nuclei. Indeed, the collapse of the nuclear system could only be prevented if the character of the interaction were radically to change at very small distances; e.g. HULTHÉN [44a] has pointed out that in his unsymmetrical form of theory (16.33) the higher approximation terms of interaction between two like nucleons become repulsive at small distances: such a repulsion might lead to a saturation of the nuclear bindings, analogous to that of intermolecular forces in liquids rather than to that of the chemical bonds of the quantal "exchange" type (2.22) *.

If, on the other hand, the axial dipole couplings may be regarded as small perturbations, their total effect will not disturb the saturation properties of the binding due to the central interactions. In fact, the latter will be such as to favour configurations with saturated spins (and minimum neutron excess). Now,

$$\sum_{\sigma_z^{(1)}, \sigma_z^{(2)}} v_{\pm}(1) v_{\pm}(2) \mathcal{D}^{(12)} v_{\pm}(1) v_{\pm}(2) = \pm (\cos^2 \vartheta - \frac{1}{3}), \quad (2)$$

the sign on the right-hand side being + or — according as the sign indices of the spin eigenfunctions v are the same or opposite for the two nucleons.

* It has recently been suggested (FEENBERG and PRIMAKOFF [46]) that the collapsed configurations of heavy nuclei predicted by the theories with "large" axial dipole couplings might actually represent the stablest states of such nuclei, while their more loosely bound saturation configurations would be metastable: such a situation is termed "conditional" saturation. Besides axial dipole couplings, non-static forces such as short range, attractive many-body interactions would be liable to bring about conditional saturation. The collapsed forms of nuclei would constitute extremely rare isotopes of known elements, some curious properties of which are described in the note cited.

Consequently, in any spin-saturated configuration the operator (1) of axial dipole coupling will not give rise to any contribution of the ordinary type to the potential energy, since the integrals pertaining to pairs of nucleons with parallel and antiparallel spins will exactly compensate each other. The situation is just the same as with the central interactions involving the spin factor $\vec{\sigma}^{(l)} \vec{\sigma}^{(k)}$, for which such a compensation likewise occurs in spin-saturated configurations (cf. tables 11.11-1, 2).

17.2. Non-central couplings and binding energies of light nuclei

As soon as we deal with systems of more than two nuclei, the complexity of the nuclear interactions coming into play increases considerably. While in the interaction between two nucleons only the axial dipole coupling shows up, in dealing with more complex nuclear systems we should take into consideration the possible occurrence of spin-orbit couplings and of many-body forces. This aspect of the problem of nuclear forces has hitherto been entirely neglected; its approach is admittedly very uncertain at the present stage. Only one paper bearing on the question of the influence of non-central couplings on the binding energies of light nuclei has been published, by GERJUOV and SCHWINGER [42]. In this paper, the binding energies of ${}^3\text{H}$ and ${}^4\text{He}$ are calculated, by a variational method, on the assumption that the interaction potential has the form (16.22-4), given by Rarita and Schwinger's theory for states of even parity (the only ones occurring in this problem): this implies that only the influence of axial dipole forces on the binding energy is discussed (within the scope of the Rarita-Schwinger theory), while the possible occurrence of other interactions is disregarded.

In the case of three- or four-nucleon systems, the occurrence of non-central interactions implies a breakdown of the conservation of not only orbital momentum, as in the deuteron case, but also the spin: we now get, for ${}^3\text{H}$, a mixture of doublet and quartet states (14.11), for ${}^4\text{He}$ a mixture of singlet, triplet and quintet states. The ground state of ${}^3\text{H}$, e.g., in which $J = \frac{1}{2}$, is of the type ${}^2S_{\frac{1}{2}} + {}^2P_{\frac{1}{2}} + {}^4P_{\frac{1}{2}} + {}^4D_{\frac{1}{2}}$; the ground state of ${}^4\text{He}$ ($J = 0$) comprises ${}^1S_0$, ${}^3P_0$ and ${}^5D_0$ eigenfunctions. Moreover, the assignment of a definite parity to the eigenfunction does not, in many-nucleon systems, result in any reduction of the admixture of states of different orbital momenta. However, in first approximation, the axial dipole operator will only couple the ${}^2S_{\frac{1}{2}}$ state of ${}^3\text{H}$ with ${}^4D_{\frac{1}{2}}$ states and the ${}^1S_0$ state of ${}^4\text{He}$ with ${}^5D_0$ states*. Gerjuoy and Schwinger limit themselves to this approximation; adopting the constants (16.21-1) derived from the deuteron properties, they find that the D -admixture in the ground

* This is a simple consequence of the fact that the operator $\sum_{i,k} \mathcal{D}^{(ik)} F(r^{(ik)})$ transforms, when the spin or space coordinates *alone* are rotated, according to a representation of rank 2 of the rotation group.

states of either of the nuclei ^3H and ^4He would be about 4 %, just as in the case of the deuteron. However, while in the latter case the interference of this small amount of D -wave with the S -wave produces, on the Rarita-Schwinger theory, practically all the binding of the deuteron, the corresponding effect turns out to be much smaller in the heavier nuclei, with the result that the calculated binding energies of these nuclei become much too small, owing to the decrease of the central potential strength as compared with a purely central interaction (14.2). The authors' conclusion is accordingly that the assumptions underlying Rarita and Schwinger's theory, and especially that of equal ranges for the central and non-central interactions, are too restrictive to account successfully for the binding energies of the lightest nuclei *.

17.3. Fine structure of nuclear levels

The preceding discussions of this Part IV, while leaving many details unsettled, have at any rate made it clear that in nuclear systems quite appreciable forces are operative to couple the spins and orbital motions of the constituent nucleons. On any nuclear model such couplings will give rise to a *fine structure* of the stationary states of the system, due to the splitting of the multiplet states. Actually, while the order of magnitude of the separations between levels of low excitation in lighter nuclei is generally ** of a few MeV, there are some conspicuous exceptions, in which groups of levels occur, with intervals of the order of several tenths of a MeV only; and one is tempted to recognize in such close groupings the expected multiplet splittings. However, extreme caution is required in attempting identifications of neighbouring levels as members of a multiplet: for, leaving aside the possibility of mere chance coincidence of unrelated states, other types of fine structure than multiplet splitting are suggested by the α -particle model. In view of the present uncertain position regarding both the most adequate model to be applied in each case, and the exact form of the law of nuclear force, we cannot hope, therefore, to reach unambiguous conclusions about fine structures of nuclear levels, but at best to obtain additional arguments concerning the various possibilities.

In the domain of light nuclei, the available empirical material is rather scarce: the only well-studied instances of fine structures of the ground state are those of the related (unstable) nuclei ^5He and ^5Li and of the stable isotope ^7Li . They will be more closely examined in the following sections. Additional indications, in the case of β -active nuclei, might in principle be expected, as explained in A1.21, from the observation of composite β -spectra and of concomitant radiative transitions.

* On the possible bearing on this problem of the recent determination of the magnetic moment of ^3H , see A 2.251.

** See the recent survey of the evidence about energy levels of light nuclei by HORNYAK and LAURITSEN [48].

Indirectly connected with the problems of fine structure is the question of the total angular momentum of the ground state of a nucleus (also when no close-lying excited state has been observed). For the value of this quantity is ultimately determined by just the types of coupling responsible for the fine structures. In fact, it can be calculated, for any nucleus, by starting from a definite model and a definite type of non-central coupling: if the angular momentum has been observed, this provides a check on the assumptions made; if not, it gives us an indication, unfortunately very unreliable, of the value that might be expected. Examples of this kind of argument will be found in A2.252, A2.253, A2.26, in connexion with the calculation of magnetic moments; we shall just mention the interesting case of ${}^4\text{Li}$, whose configuration, on the quasi-atomic model, is characterized by the presence of a half-filled neutron p shell besides a p -proton (ROSE and BETHE [37]).

We shall now proceed to give an account of the two best known cases mentioned above: the five-nucleon systems ${}^5\text{He}$ and ${}^5\text{Li}$, and the nucleus ${}^7\text{Li}$.

17.4. Fine structure of ${}^5\text{He}$ and ${}^5\text{Li}$

17.40. Since no stable state exists for the nuclei ${}^5\text{He}$ and ${}^5\text{Li}$, the main source of information about their "virtual" levels is the study of nuclear reactions involving one of these nuclei either as a partner in the final stage of the reaction or as the compound nucleus. In the reaction ${}^7\text{Li}(d, \alpha)$, ${}^5\text{He}$ is obtained in a state unstable with respect to neutron emission, which is very probably the lowest one, with an energy of about 0.8 MeV above that of α -particle plus neutron. A sensitive method of investigating the properties of this ground state of ${}^5\text{He}$ is the study of the scattering of neutrons by ${}^4\text{He}$: the existence of the virtual level of the compound nucleus ${}^5\text{He}$ will be marked by a resonance effect in the dependence of the scattering cross-section on the energy of the impinging neutrons. As we shall see, it is this effect that has actually disclosed the doublet structure of the ground state. The properties of ${}^5\text{Li}$ may serve as a check on the conclusions arrived at with regard to ${}^5\text{He}$, since the ground states of the two nuclei, according to the symmetry of nuclear forces with respect to charge (3.3), should only differ by the Coulomb energy*, which would raise the ground state of ${}^5\text{Li}$ about 1 MeV above that of ${}^5\text{He}$: hence the interest of experiments on the scattering of protons by ${}^4\text{He}$. In this section, we shall therefore begin by establishing the necessary "dispersion formulae" which give the scattering cross-section of Helium for nucleons in terms of the energy of impact. We shall then survey the results of the

* An estimate of the Coulomb energy difference between ${}^5\text{Li}$ and ${}^5\text{He}$, based on Bethe's theory of the "last" nucleon binding (3.3), has been carried out by STEPHENS [40] and yields 0.88 MeV. By equating this quantity to the expression $\frac{3}{5} \frac{e^2}{R} (A-1)$, valid for a homogeneous charge distribution of radius R , one gets an "equivalent" nuclear radius $R \approx 4 \cdot 10^{-13}$ cm.

scattering experiments, and finally proceed to a theoretical discussion of the fine structure they reveal.

17.41. Theory of the scattering of nucleons by Helium. In investigating the elastic scattering of a nucleon by any nucleus of mass A and charge Z , the latter may to a first approximation be treated just as a point, its structure entering only indirectly through the phase-shifts of the asymptotic wave-functions of the scattered nucleon. Such a treatment neglects the polarization of the nucleus by the impinging nucleon, as well as the effect of the exchange of this nucleon with one of the constituent particles of the nucleus*. The case of proton scattering is the more general one: as for proton-proton scattering (7.11), except that here no complication arises from the exclusion principle, and that if we assume the scattering nucleus (A, Z) to have no angular momentum, we are dealing with doublet states of the total system, rather than with the singlet and triplet configurations of the two-proton system.

If we consider the scattering process in the barycentric system of reference, we have to use the reduced mass $M_{\text{red}} = M \frac{A}{A+1}$, and the connexion between the relative kinetic energy ε of the colliding particles, their relative velocity v , relative momentum p and wave-number k , is as usual

$$p = \hbar k, \quad \frac{v}{c} = \frac{p}{M_{\text{red}}}, \quad \varepsilon = \frac{p^2}{2M_{\text{red}}}. \quad (1)$$

The scattering angle ϑ in the barycentric system is simply related to the angle of recoil Θ' of the nucleus which is at rest in the laboratory system by the relation (6.212-20) $\Theta' = \frac{1}{2}(\pi - \vartheta)$. The transformation of scattering angle ϑ and energy ε to the corresponding quantities Θ, E in the laboratory system is different according as the nucleus (A, Z) or the proton is supposed to be at rest in the latter system (scattering of protons or of α -particles). In the former case, we have

$$\varepsilon = E \cdot \frac{A}{A+1} \quad (2)$$

and the formulae quoted in the first footnote of 14.12; in the latter case, one has just to replace A by $1/A$ in these formulae. Finally, the characteristic parameter a (7.11-2) becomes

$$a = \frac{Ze^2 c}{\hbar v}; \quad (3)$$

with this new meaning of a , the formulae (7.11-3, 4, 5, 6) for pure Coulomb scattering are applicable to our case.

Denoting, as usual, by v_+, v_- the spin eigenfunctions of the scattered nucleon, the spin and angular dependence of the doublet states of our

* Dr. Fröhlich and Dr. Humblet have independently pointed out to me that both effects may be important, especially in the case of scattering by Helium.

system may be written in the form

$$\begin{aligned} {}^2Z_{l+\frac{1}{2}}^{(l)m} &= \frac{1}{\sqrt{2l+1}} [\sqrt{l+m+\frac{1}{2}} v_+ Y_l^{m-1} - \sqrt{l-m+\frac{1}{2}} v_- Y_l^{m+1}] \\ {}^2Z_{l-\frac{1}{2}}^{(l)m} &= \frac{1}{\sqrt{2l+1}} [\sqrt{l-m+\frac{1}{2}} v_+ Y_l^{m-1} + \sqrt{l+m+\frac{1}{2}} v_- Y_l^{m+1}]. \end{aligned} \quad (4)$$

The general solution is therefore

$$\Psi = \sum_{l,j} \mathcal{R}_j^{(l)m} \cdot {}^2Z_j^{(l)m} \cdot \frac{1}{r} L_l(r), \quad (5)$$

the asymptotic form (17.11–11) of the radial functions L_l depending on the phase-shifts $\delta_j^{(l)}$ due to the proper nuclear interactions; in general, these phases will be different for the two substates $j = l \pm \frac{1}{2}$ corresponding to a given l . The $\mathcal{R}$'s are determined by comparing the expression (5) with the solution [cf. (17.11–5)]

$$\Psi_c = v_+ \Psi_c^{(+)} \quad (6)$$

corresponding to pure Coulomb scattering, and by expressing that $\Psi - \Psi_c$ asymptotically represents an outgoing spherical wave; this gives for the scattered wave

$$\begin{aligned} \Psi_{\text{scatt}} &\simeq v_+ (\Psi_c^{(+)})_{\text{scatt}} \\ &+ \frac{e^{ikr - i\alpha \log 2kr}}{kr} \sum_l e^{2i\gamma^{(l)}} \left[\left(\frac{l+1}{\sqrt{2l+1}} \mathcal{A}_{l+\frac{1}{2}}^{(l)} + \frac{l}{\sqrt{2l+1}} \mathcal{A}_{l-\frac{1}{2}}^{(l)} \right) v_+ \sqrt{4\pi} Y_l^0 \right. \\ &\quad \left. - \sqrt{\frac{l(l+1)}{2l+1}} (\mathcal{A}_{l+\frac{1}{2}}^{(l)} - \mathcal{A}_{l-\frac{1}{2}}^{(l)}) v_- \sqrt{4\pi} Y_l^1 \right], \end{aligned} \quad (7)$$

where

$$\mathcal{A}_j^{(l)} = \frac{1}{2i} (e^{2i\delta_j^{(l)}} - 1) = e^{i\delta_j^{(l)}} \sin \delta_j^{(l)} \quad (8)$$

and

$$(\Psi_c^{(+)})_{\text{scatt}} = -\frac{\alpha}{2kr \sin^2 \vartheta/2} e^{ikr - i\alpha \log 2kr \sin^2 \vartheta/2 + 2i\gamma^{(0)}}. \quad (9)$$

The existence of virtual levels of the compound system is expressed by the energy dependence of the phase-shifts $\delta_j^{(l)}$ in the following way. Suppose that for some value L of l , there is a virtual level with the doublet structure corresponding to $J = L \pm \frac{1}{2}$; let $\epsilon_j^{(L)}$ be the energy values, $\Gamma_j^{(L)}$ the widths of the two virtual substates. Upon the assumption that these widths are sufficiently small (compared with a suitably defined critical energy), BLOCH [40] has derived a general dispersion formula, from which

it follows that the phases $\delta_f^{(L)}$ have the form of a smoothly varying function of the energy, due to the so-called "potential scattering" (this we shall continue to denote by $\delta_f^{(L)}$), to which is added a resonance term *

$$*\delta_f^{(L)} = \arctg \frac{\frac{1}{2} \Gamma_f^{(L)}}{\epsilon_f^{(L)} - \epsilon}; \quad (10)$$

this means that the characteristic phase function $\Delta_f^{(L)}$, defined by (8), must be replaced by

$$\begin{aligned} \Delta_f^{(L)} + e^{2i\delta_f^{(L)}} * \Delta_f^{(L)}, \\ * \Delta_f^{(L)} = \frac{\frac{1}{2} \Gamma_f^{(L)}}{\epsilon_f^{(L)} - \epsilon - \frac{1}{2} i \Gamma_f^{(L)}}. \end{aligned} \quad (11)$$

In view of the possible influence of non-central couplings, we must, however, as emphasized by BECK and TSIEN [42], envisage a somewhat more general description of the virtual states, involving a mixture of the two states of different orbital momenta $L = J \pm \frac{1}{2}$ which correspond to a definite value of the total angular momentum J . The two substates Ψ_e ($e = I, II$) pertaining to a given J will conveniently be represented by the orthogonal eigenfunctions

$$\Psi_e = \sum_{L=J \pm \frac{1}{2}} c_{eL} i^{L-(J-\frac{1}{2})} \cdot {}^2Z_J^{(L)m} \cdot \frac{1}{r} L_L(r), \quad (12)$$

with a unitary matrix c_{eL} which may be written

$$(c_{eL}) = \begin{pmatrix} \cos \omega & \sin \omega e^{i\chi} \\ \sin \omega e^{-i\chi} & -\cos \omega \end{pmatrix}. \quad (13)$$

A straightforward extension of the preceding argument, again starting from Bloch's general dispersion formula, shows that in the expression (7) for the scattered wave, we have to replace the $\Delta_f^{(L)}$ by

$$\begin{aligned} \Delta_f^{(L)} + e^{2i\delta_f^{(L)}} \sum_e A_e^{(L)} c_{eL} * \Delta_f^{(e)}, \\ * \Delta_f^{(e)} = \frac{\frac{1}{2} \Gamma_f^{(e)}}{\epsilon_f^{(e)} - \epsilon - \frac{1}{2} i \Gamma_f^{(e)}}, \\ A_e^{(L)} = \sum_{L'=J \pm \frac{1}{2}} c_{L'e}^* e^{i[\delta_f^{(L')} - \delta_f^{(L)}] + i[\gamma_f^{(L')} - \gamma_f^{(L)}]}. \end{aligned} \quad (14)$$

Let us now write down the differential scattering cross-section in different cases of varying complexity. For an energy value ϵ of the relative kinetic energy not in the vicinity of any resonance energy, we may use directly the expression (7), in which the phases then practically reduce

* Formula (30) of Bloch's paper, giving this resonance term, is erroneous: but the correct expression is used throughout the paper.

to the potential scattering contributions; we get accordingly, with some obvious transformations *,

$$dS = \frac{d\Omega}{k^2} [|F_+(\vartheta)|^2 + \sin^2 \vartheta |F_-(\vartheta)|^2],$$

$$F_+(\vartheta) = \frac{\alpha}{2 \sin^2 \vartheta/2} e^{-i\alpha \log \sin^2 \vartheta/2} - \sum_l e^{2i \sum_{v=1}^l \arctg \alpha/v} [(l+1) A_{l+1}^{(l)} + l A_{l-1}^{(l)}] P_l(\cos \vartheta) \quad (15)$$

$$F_-(\vartheta) = \sum_l e^{2i \sum_{v=1}^l \arctg \alpha/v} [A_{l+1}^{(l)} - A_{l-1}^{(l)}] \frac{d}{d(\cos \vartheta)} P_l(\cos \vartheta).$$

For pure Coulomb scattering, we have, using (1) and (3), the Rutherford formula

$$dS_c = \left(\frac{Ze^2}{4\epsilon} \right)^2 \frac{d\Omega}{\sin^4 \vartheta/2}. \quad (16)$$

The ratio of the scattering cross-section (15) to pure Coulomb scattering may be written

$$\frac{dS}{dS_c} = |f_+(\vartheta)|^2 + \frac{4 \sin^4 \vartheta/2}{\alpha^2} \sin^2 \vartheta |F_-(\vartheta)|^2 \quad (17)$$

$$f_+(\vartheta) = 1 - \frac{2 \sin^2 \vartheta/2}{\alpha} e^{i\alpha \log \sin^2 \vartheta/2} \sum_l e^{2i \sum_{v=1}^l \arctg \alpha/v} [(l+1) A_{l+1}^{(l)} + l A_{l-1}^{(l)}] P_l(\cos \vartheta).$$

In the case of neutron scattering, we have

$$\begin{aligned} dS &= \frac{d\Omega}{k^2} [|F_+^0(\vartheta)|^2 + \sin^2 \vartheta |F_-^0(\vartheta)|^2] \\ F_+^0(\vartheta) &= \sum_l [(l+1) A_{l+1}^{(l)} + l A_{l-1}^{(l)}] P_l(\cos \vartheta) \\ F_-^0(\vartheta) &= \sum_l [A_{l+1}^{(l)} - A_{l-1}^{(l)}] \frac{d}{d(\cos \vartheta)} P_l(\cos \vartheta). \end{aligned} \quad (18)$$

In order to take account of a virtual state consisting of two neighbouring levels $^2P_{1/2}$, $^2P_{3/2}$, we should add to the F 's in (15) the resonance amplitudes

$$\begin{aligned} R_+(\vartheta) &= e^{2i \arctg \alpha} [2 e^{2i \vartheta_{1/2}^{(1)}} * A_{3/2}^{(1)} + e^{2i \vartheta_{3/2}^{(1)}} * A_{1/2}^{(0)}] \cos \vartheta \\ R_-(\vartheta) &= e^{2i \arctg \alpha} [e^{2i \vartheta_{1/2}^{(1)}} * A_{3/2}^{(1)} - e^{2i \vartheta_{3/2}^{(1)}} * A_{1/2}^{(1)}]; \end{aligned} \quad (19)$$

e.g., neglecting all potential scattering except the S -contribution, we get for the ^2P resonance scattering of neutrons

$$dS = \frac{d\Omega}{k^2} [|e^{i\vartheta^{(0)}} \sin \vartheta^{(0)} + (2 * A_{3/2}^{(1)} + * A_{1/2}^{(1)}) \cos \vartheta|^2 + |* A_{3/2}^{(1)} - * A_{1/2}^{(1)}|^2 \sin^2 \vartheta]. \quad (20)$$

If, however, we have a "mixed" resonance of the type (12), (13), (14),

* See, in particular, the first footnote of 13.11.

we must take

$$R_+(\vartheta) = (J + \frac{1}{2}) \sum_e^* A_f^{(e)} \sum_{L=J \pm \frac{1}{2}} e^{2i\vartheta f^{(L)}} A_e^{(L)} c_{eL} P_L(\cos \vartheta) \quad (21)$$

$$R_-(\vartheta) = \sum_e^* A_f^{(e)} \sum_{L=J \pm \frac{1}{2}} e^{2i\vartheta f^{(L)}} A_e^{(L)} c_{eL} (-1)^{L-(J-\frac{1}{2})} \frac{d}{d(\cos \vartheta)} P_L(\cos \vartheta).$$

In particular, if we may, in first approximation, disregard all potential and Coulomb scattering (the latter becomes small at large scattering angles ϑ) and if the separation of the two p substates is sufficiently large, we get a simple estimate of the resonance cross-section:

$$dS_e = \frac{d\Omega}{k^2} |A_f^{(e)}|^2 |A_e|^2 [|R_+(\vartheta)|^2 + \sin^2 \vartheta |R_-(\vartheta)|^2] \quad (22)$$

$$A_e \approx \cos \omega \pm \sin \omega e^{i\chi}$$

$$R_+(\vartheta) \approx [(l+1)(P_l \cos \omega \pm P_{l+1} \sin \omega e^{i\chi})]_{l=J-\frac{1}{2}}$$

$$R_-(\vartheta) \approx [P'_l \cos \omega \mp P'_{l+1} \sin \omega e^{i\chi}]_{l=J-\frac{1}{2}},$$

i.e., making use of a well-known relation between Legendre functions,

$$dS_e \approx \frac{d\Omega}{k^2} |A_f^{(e)}|^2 [1 \pm \sin 2\omega \cos \chi] \cdot [S(\vartheta) \pm \sin 2\omega \cos \chi S'(\vartheta)] \quad (23)$$

$$S(\vartheta) = [(l+1)^2 P_l^2 + \sin^2 \vartheta (P'_l)^2]_{l=J-\frac{1}{2}}$$

$$S'(\vartheta) = [(l+1)^2 P_l P_{l+1} - \sin^2 \vartheta P'_l P'_{l+1}]_{l=J-\frac{1}{2}}.$$

The last relation takes an especially simple form in the case of backward scattering, i.e. for $\vartheta = \pi$. From the well-known behaviour of the Legendre polynomials, one derives immediately

$$\left(\frac{dS_e}{d\Omega} \right)_{\vartheta=\pi} \approx \frac{1}{k^2} |A_f^{(e)}|^2 (J + \frac{1}{2})^2 [1 - \sin^2 2\omega \cos^2 \chi]. \quad (24)$$

On this estimate of the backward scattering cross-section is based a procedure, the principle of which goes back to P. WENZEL [34], to ascertain the angular momentum and even the degree of mixture, of a resonance level (BECK and TSIEN [42]). In fact, putting $|A_f^{(e)}|^2 = 1$ for the resonance maximum, we obtain from (24), by varying both J and the mixture parameter $a^2 \equiv \sin^2 2\omega \cos^2 \chi$, a family of curves representing the maximum backward cross-section in terms of the energy of the impinging particles for different J 's and a^2 's: the curve on which the observed maximum falls gives us a reliable determination of the J -value and an approximate indication of the value of a^2 ; from the latter, we can only deduce an upper and a lower limit for the proportion of admixture $\sin^2 \omega$ of the substate $L = J + \frac{1}{2}$:

$$\frac{1}{2} - \frac{1}{2} \sqrt{1-a^2} < \sin^2 \omega < \frac{1}{2} + \frac{1}{2} \sqrt{1-a^2}. \quad (25)$$

17.42. Analysis of scattering experiments. The scattering of neutrons by Helium has been investigated in the domain of energies of the impinging neutrons extending from the thermal region up to about 3 MeV; the most interesting feature is the occurrence of a resonance at about 1 MeV: this corresponds to an energy of about $\frac{4}{5} \cdot 1$ MeV or 0,8 MeV for the resonance level, in nice agreement with the energy of the virtual ground state as derived from the $^7\text{Li}(d, \alpha)$ reaction (17.40). The cross-section for thermal neutrons, determined by CARROLL and DUNNING [38], is

$$\mathcal{S}_{\text{therm}} = 1,5 \cdot 10^{-24} \text{ cm}^2. \quad (26)$$

By the method previously described (6.42), BARSCALL and KANNER [40] studied the angular distribution of the recoil α -particles on impact of neutrons of 2,5 MeV and 3,1 MeV energy. A careful study of the backward scattering ($\vartheta = \pi \pm 24^\circ$) in the interesting energy region $\approx 0,5 \dots 2,5$ MeV has been made by STAUB and TATEL [40], using a cloud chamber method; by suitably filling the chamber, they obtained the He-scattering cross-sections relative to those of hydrogen, and took for the latter the values given by the theoretical formula (6.431-2) (with a well width d). If Wenzel's criterion, in the form (24) as just explained, is applied to their results, the maximum of the backward scattering cross-section at ≈ 1 MeV is found to indicate a nearly pure $^2P_{\frac{3}{2}}$ resonance level.

This maximum cross-section, however, exhibits a fine structure, the main peak at 1 MeV neutron energy being accompanied by a somewhat lower and apparently broader one at $\approx 1,35$ MeV neutron energy. In this case, Wenzel's criterion in the simple form (24) would not be applicable. Staub and Tattel try to analyze their experimental curve for the backward scattering cross-section in terms of neutron energy on the assumption of a doublet resonance level of the type $^2P_{\frac{3}{2}} - ^2P_{\frac{1}{2}}$. In using formula (20), which covers this case, one is chiefly hampered by the uncertainty prevailing with respect to the sign and magnitude of the potential scattering phase $\delta^{(0)}$. A clue to the order of magnitude of $\sin^2 \delta^{(0)}$ is provided by the value (26) of $\mathcal{S}_{\text{therm}}$, but there would seem to be no basis for any conjecture about the sign* of $\delta^{(0)}$. According to the choice of this sign, there are two equally acceptable solutions, corresponding to different orders of the doublet levels:

Sign of $\delta^{(0)}$ (assumed)	$\delta^{(0)}$ (approxim.)	$2\varepsilon_{\frac{3}{2}}^{(1)}$ MeV	$2\varepsilon_{\frac{1}{2}}^{(1)}$ MeV	$ 2\varepsilon_{\frac{3}{2}}^{(1)} - 2\varepsilon_{\frac{1}{2}}^{(1)} $ MeV	$2I_{\frac{1}{2}, \frac{3}{2}}^{(1)}$ MeV	Order of levels
+	$+24,9^\circ$	0,76	1,08	$0,32 \pm 0,10$	0,32	inverted
-	$-37,8^\circ$	1,08	0,84	$0,24 \pm 0,08$	0,32	regular

(27)

* Staub and Tattel's remarks on this point at the end of their paper are not substantiated by the work of Wheeler and Barschall, mentioned below.

More recent experiments by HALL and KOONTZ [47] fail to bring any further elucidation. These authors use as scattering chamber a He-filled proportional counter which allows them to determine the distribution of the recoil He nuclei in terms of the recoil energy E' . Since the latter quantity is proportional to $\cos^2\theta'$, i.e. to $1 - \cos\theta$, this distribution curve has the same shape as that giving the differential cross-section in terms of $\cos\theta$. The distribution curves obtained for neutron energies in the range 0.8 ... 1.6 MeV have the parabolic shape expected, according to formula (20), for P scattering. Unfortunately, the parabolic fits are not sufficiently precise to lead to a unique determination of the parameters characterizing the resonance levels. One can only say that the new data are not incompatible with the above values indicated by Staub and Tatel.

It must be noted that the other available evidence casts some doubt on Staub and Tatel's interpretation of the resonance doublet at 1 MeV neutron energy. In the first place, this interpretation can be tested with the help of Barschall and Kanner's above-mentioned measurement of the differential scattering cross-section for neutrons of 2.5 MeV energy. WHEELER and BARSCHALL [40] attempted to analyze these data by using graphical methods of phase-shift determination similar to that applied by Wheeler to He-He scattering (13.11). For such an energy value, far from any resonance region, we may start from the simple expression (18) for the differential cross-section, in which only the terms corresponding to the first few values of l are retained. If we try to fit the empirical curve by a superposition of the ${}^2S_{1/2}$, ${}^2P_{1/2}$ and ${}^2P_{3/2}$ waves only, we find values of the P -phases differing by as much as 60° . This is a much larger difference than can be accounted for by the narrow resonance doublet at 1 MeV: for the eigenfunctions of doublet components of such small separation would be very similar and their phases much nearer equality. This discrepancy might be taken to indicate a considerable contribution from D -waves; it is indeed found that ${}^2D_{3/2}$ and ${}^2D_{5/2}$ waves might be present with a phase difference of about 40° , but it is not possible to estimate with any accuracy the resulting modification of the P -phases.

Another check on Staub and Tatel's results is provided by the experiments on scattering of protons by Helium, carried out especially for this purpose by HEYDENBURG and RAMSEY [41], with the apparatus represented by fig. 7.12-1. The resonance of the ${}^5\text{Li}$ compound nucleus expected (17.40) in the neighbourhood of 2 MeV kinetic energy of the incident protons* was actually found, but the maximum of the scattering cross-section at $\Theta = 140^\circ$ (nearest approach to backward scattering) was flatter and broader than in the case of the ${}^5\text{He}$ system and did not exhibit any fine structure. The application of Wenzel's criterion (TSIEN [44]) yields $J = \frac{3}{2}$ and $0.07 < \sin^2\omega < 0.93$, which is compatible with the description

* Heydenburg and Ramsey's experiments cover the range 1.2 ... 3 MeV of proton energies. Data for 4.2 MeV protons are supplied by POWELL *et al.* [47b].

of the resonance level as a nearly pure ${}^2P_{\frac{3}{2}}$ state. A closer analysis in terms of ${}^2S_{\frac{1}{2}}$, ${}^2P_{\frac{1}{2}}$ and ${}^2P_{\frac{3}{2}}$ waves has been attempted by LANDAU and SMORODINSKY [44]: they state that the phases of the P -components are found to differ strongly from each other*.

On the whole, it would seem that Staub and Tatel's interpretation of the ${}^5\text{He}$ resonance level, while in serious need of further check, may provisionally be retained, at any rate as a first approximation. It will possibly turn out, on closer study**, that we shall have to deal with mixtures of the two P levels with ${}^2S_{\frac{1}{2}}$ and ${}^2D_{\frac{3}{2}}$ states, respectively; and since such admixtures give rise to *two* states of each of the types ${}^2S_{\frac{1}{2}} + {}^2P_{\frac{1}{2}}$, ${}^2P_{\frac{3}{2}} + {}^2D_{\frac{3}{2}}$, it may be asked whether the observed fine structure, instead of being primarily the P -doublet splitting, would not rather correspond to either one of these pairs of mixed states. Wenzel's criterion is useful in eliminating such possibilities: for it shows, on the one hand, that the resonance levels cannot both have the angular momentum $J = \frac{1}{2}$, and on the other hand, that the degree of admixture associated with the value $J = \frac{3}{2}$ is too small to be compatible with the occurrence of a close pair of states with this quantum number. In conclusion, we shall confine ourselves, in the theoretical discussion which follows, to the consideration of a simple P -doublet splitting.

17.43. Fine structure of ${}^5\text{He}$ and spin-orbit coupling. The properties of the deuteron give us information on the axial dipole interaction between nucleons only, but cannot tell us anything about the possible occurrence of a spin-orbit coupling (16.11). The ${}^5\text{He}$ fine structure, on the other hand, if interpreted as a P -doublet splitting, should primarily depend just on the latter coupling; herein lies the special interest of this case. In fact, it is easily seen*** that with a non-central interaction of the axial dipole type a separation of the P -doublet components will be obtained only as a second order perturbation effect. This effect has been discussed by DANCOFF [40]. Let ϵ_0 be the energy of the unsplit ground state; the displacements $\Delta\epsilon_{0,J}$ of the substates with $J = \frac{1}{2}, \frac{3}{2}$ due to the axial dipole operator $\mathcal{W} = \sum_{i,k} \mathcal{D}^{(ik)} F(r^{(ik)}) \mathbf{1}^{(ik)}$ are of the form

$$\Delta\epsilon_{0,J} = \sum_n \frac{\langle 0, J | \mathcal{W} | n \rangle \langle n | \mathcal{W} | 0, J \rangle}{\epsilon_0 - \epsilon_n}, \quad (28)$$

* As regards the ${}^5\text{Li}$ nucleus, the issue is further obscured by the conflicting results of the experiments on scattering of α -particles in hydrogen. Some authors, most recently TSIEH [40, 44] (see also BECK and TSIEH [42]), claim to have disclosed a complex resonance at lower energies than Heydenburg and Ramsey: in fact Tsien describes as many as six equidistant resonance levels of ${}^5\text{Li}$ between 0.56 and 1 MeV. Such a structure would not only be very hard to understand by itself, but it could not be reconciled with the empirical and theoretical (14.21) evidence tending to show that the virtual state of 0.8 MeV energy of ${}^5\text{He}$ is the lowest one.

** Especially in view of the existence of a ${}^2S$ virtual level close to the ${}^2P$ state, as revealed by Tyrrell's work (14.21).

*** Cf. the footnote of 17.2.

the summation extending to all intermediate states, of energy ε_n , which give rise to non-vanishing matrix-elements $(0, J | \mathcal{W} | n)$ coupling them to the substates $0, J$ of the ground level. Since the intermediate states involve configurations in which the " α -cluster" of the ground state is broken up, the excitation energies ε_n are at least of the order of 15 MeV. In order to estimate the difference $\Delta\varepsilon_{0,1} - \Delta\varepsilon_{0,2}$ Dancoff starts from the identity

$$\frac{1}{\varepsilon_0 - \varepsilon_n} = \frac{1}{\varepsilon_0 - \bar{\varepsilon}} + \frac{\varepsilon_n - \bar{\varepsilon}}{(\varepsilon_0 - \bar{\varepsilon})^2} + \frac{(\varepsilon_n - \bar{\varepsilon})^2}{(\varepsilon_0 - \bar{\varepsilon})^2 (\varepsilon_0 - \varepsilon_n)}, \quad (29)$$

which, inserted in (28), yields

$$\Delta\varepsilon_{0,J} = \frac{(0, J | \mathcal{W}^2 | 0, J)}{\varepsilon_0 - \bar{\varepsilon}} \left(1 - \frac{\bar{\varepsilon}}{\varepsilon_0 - \bar{\varepsilon}} \right) + \frac{(0, J | \mathcal{W} \mathcal{H} \mathcal{W} | 0, J)}{(\varepsilon_0 - \bar{\varepsilon})^2}, \quad (30)$$

provided the mean energy $\bar{\varepsilon}$ is determined by the condition that the contribution to $\Delta\varepsilon_{0,J}$ arising from the third term on the right-hand side of (29) vanishes. In (30), $\mathcal{H}$ represents the unperturbed Hamiltonian. A closer consideration shows that the expectation values $(0, J | \mathcal{W}^2 | 0, J)$ of $\mathcal{W}^2$ in the two substates will differ very little from each other; in fact, taking for $\mathcal{W}$ the Rarita-Schwinger form* with the constants determined by the deuteron properties (16.21), Dancoff estimates that the contribution of the first term of (30) to the doublet splitting will hardly exceed 0.01 MeV. Likewise, the contribution to the splitting from the part of the operator $\mathcal{W} \mathcal{H} \mathcal{W}$ involving the unperturbed potential energy will be very small. The kinetic part of $\mathcal{H}$, on the other hand, leads to a significant term in $\mathcal{W} \mathcal{H} \mathcal{W}$ (i.e. one having different expectation values in the two substates), which is of the form

$$\frac{1}{2} \sum_{i,k} \frac{\hbar^2}{M} \left[\frac{1}{r^{(ik)}} F(r^{(ik)}) \right]^2 \mathcal{M}^{(ik)}, \quad (31)$$

with the spin-orbit operator $\mathcal{M}^{(ik)}$ (type III of table 15.21). The effect of the axial dipole coupling is thus reduced to that of an "equivalent spin-orbit coupling". The resulting order of the doublet components is seen to be *regular*. As to the order of magnitude of the separation, it is very sensitive to the form of distance dependence $F(r)$ adopted; Dancoff's estimates vary from 0.014 MeV to 0.13 MeV, the latter estimate corresponding to the potential of Bethe's neutral meson theory (16.32). Even in this rather extreme case, the axial dipole coupling would thus prove to be insufficient.

There is, of course, an interaction of the spin-orbit coupling type which is always present in any nuclear system: it is the relativistic coupling (of the second order in the nucleon velocities) discussed in 15.22. Let us estimate the importance of this effect for the ${}^5\text{He}$ doublet. For this purpose,

* Dancoff adopts the neutral theory; but another choice would not essentially modify his estimates.

it will suffice — again following DANCOFF [40] — to consider the odd neutron as moving in the nuclear field $\mathcal{V}_n$ of the α -particle; neglecting the Larmor effect due to the neutron's magnetic moment, we are left with a Thomas coupling of the form (15.22–11)

$$-\left(\frac{\hbar}{2M}\right)^2 \frac{1}{r} \frac{d\mathcal{V}_n}{dr} \cdot 2\vec{S}\vec{L}. \quad (32)$$

The expectation values of $2\vec{S}\vec{L}$ for the substates $J = L \pm \frac{1}{2}$ are $+L$ and $-(L+1)$, respectively. Since $\text{av}\left(\frac{1}{r} \frac{d\mathcal{V}_n}{dr}\right)$ is obviously positive, the operator (32) gives rise to an *inverted* doublet, with a splitting

$$|{}^2\varepsilon_{\frac{1}{2}}^{(1)} - {}^2\varepsilon_{\frac{3}{2}}^{(1)}| = \frac{3}{4} \left(\frac{\hbar}{M}\right)^2 \text{av}\left(\frac{1}{r} \frac{d\mathcal{V}_n}{dr}\right). \quad (33)$$

If we assume for $\mathcal{V}_n$ the simple form of a well of depth J_n and width D , we get

$$\text{av}\left[\frac{1}{r} \frac{d\mathcal{V}_n}{dr}\right] = \frac{J_n}{D} [R^{(1)}(D)]^2, \quad (34)$$

where $R^{(1)}(r)$ denotes the radial wave-function of the ground state. For the purpose of our estimate, we may take the critical case of no binding energy; with the notations

$$z = \frac{1}{\hbar} \sqrt{2MJ} \cdot r \quad , \quad z_D = \frac{1}{\hbar} \sqrt{2MJ_n} \cdot D, \quad (35)$$

this corresponds to the radial equation

$$\begin{aligned} R'' - \frac{2R}{z^2} + w(z)R &= 0 \\ w(z) &= 1 \text{ for } z \leq z_D \\ &= 0 \text{ for } z > z_D, \end{aligned} \quad (36)$$

with the solution

$$\begin{aligned} R^{(1)} &\sim \sqrt{z} J_{\frac{1}{2}}(z) & (z \leq z_D) \\ &\sim z^{-1} & (z > z_D). \end{aligned} \quad (37)$$

The conditions of continuity for $z = z_D$ lead in the first place to

$$\frac{d}{dz} \log [z^{\frac{1}{2}} J_{\frac{1}{2}}(z)] = 0 \quad \text{for } z = z_D,$$

i.e. to $J_1(z_D) = 0$, or

$$z_D = \pi, \quad (38)$$

which, by (35), fixes the critical well depth in terms of its width. In the

second place, combining the continuity conditions with the condition of normalization, we readily obtain

$$[R^{(1)}(D)]^2 = \frac{2}{3D}. \quad (39)$$

Inserting (34), (39) into (33) and replacing J_n by its value deduced from (35), (38), we finally get

$$|{}^2\epsilon_{\frac{3}{2}}^{(1)} - {}^2\epsilon_{\frac{1}{2}}^{(1)}| = \frac{\pi^2}{4} \left(\frac{\hbar}{DM} \right)^4 M. \quad (40)$$

Taking, e.g. $D \approx 4 \cdot 10^{-13}$ cm (as suggested by the equivalent nuclear radius, cf. footnote of 17.40), equation (40) gives a doublet splitting of about 0.02 MeV, one order of magnitude too small*.

Clearly, therefore, we need some larger spin-orbit coupling of other origin. Now, it will be recalled that a *mixed meson theory* of nuclear interactions may provide such a coupling. In fact (16.42–11), it is found that the non-static interaction of the *first* order in the nucleon velocities contains a spin-orbit term, which may be written

$$-\frac{1}{2} \sum_{i,k} \frac{\hbar}{MM_m} \cdot \frac{1}{r^{(ik)}} \cdot 2g_1 g_2 \varphi'(r^{(ik)}) |^{(ik)} \mathcal{M}^{(ik)}. \quad (41)$$

Irrespective of the question as to the values of the masses of the mesons of different spins, this term is seen to arise from the *vector meson field* alone. It is only present when both the vector and the tensor source of this field contribute to its production; and if the source constants g_1 , g_2 are of the same order of magnitude, a comparison of the expression (41) with that derived from (15.22–19, 20) for the Thomas coupling corresponding to the static meson potential (which involves g_1^2 and g_2^2 instead of $g_1 g_2$) shows that the coupling (41) should produce effects roughly M/M_m larger than the latter, which is just the required order of magnitude. The order of the levels would thus be conditioned by the sign of $g_1 g_2$. From this point of view, the occurrence of a rather large spin-orbit coupling in ${}^5\text{He}$ may be regarded as an indication of the *existence of mesons of spin one*, not yet detected in cosmic radiation (15.35).

* A more exact treatment of the relativistic effect, taking the interactions between all pairs of nucleons into account, would have to start from the general expressions (15.22–19, 20) for the second order spin-orbit coupling between a pair of nucleons. We shall discuss this treatment in the case of ${}^7\text{Li}$ (17.5). For ${}^5\text{He}$, the calculation has been carried out under the same conditions by KITTEL [42], but in this case, the inclusion in the odd neutron eigenfunction of a radial factor involving an exponential $\exp(-r^2/R_v^2)$ represents a very bad approximation in view of the virtual character of the ground state. Since the resulting expression for the doublet splitting is very sensitive to the value of the "mean distance" R_v of the neutron from the centre of gravity of the α -cluster, it is unfortunately of little significance.

17.5. Fine structure of ${}^7\text{Li}$

17.51. Empirical data. An excited state of about 0.45 MeV energy of the ${}^7\text{Li}$ nucleus has been produced in a variety of ways. The results of the most important experiments are collected in table 17.51 below, in which the diverse reactions employed have been classified according to the compound nucleus (9.11) whose disintegration leads, in one or two steps, either to the excited or to the normal state of ${}^7\text{Li}$. We shall first give some brief comment on these nuclear reactions:

(1) The reactions ${}^6\text{Li}(d, n)$ and ${}^{10}\text{B}(p, \alpha)$ both lead to an unstable ${}^7\text{Be}$ nucleus, which transforms by K -capture either into the normal state of ${}^7\text{Li}$, or into an excited state of this nucleus, the energy of which is directly given by that of the γ -ray which accompanies the decay process.

(2) Likewise, the production of the nucleus ${}^7\text{Li}$ in two different states by the reactions ${}^6\text{Li}(d, p)$ and ${}^{10}\text{B}(n, \alpha)$, ${}^9\text{Be}(d, \alpha)$ is revealed by the occurrence of two groups of protons or α -particles of different energies: the difference of these energies yields the energy of the excited state of ${}^7\text{Li}$. In the case of the ${}^6\text{Li}(d, n)$ reaction, a check on this interpretation has been obtained by the study of the γ -radiation emitted in the course of the reaction: it was found to consist of quanta corresponding, within the limits of the available accuracy, to the transition of the excited ${}^7\text{Li}$ nuclei to their normal state.

(3) Finally, the inelastic scattering of protons or α -particles by ${}^7\text{Li}$ leaves the nucleus in an excited state, the energy of which is again determined by the measurement of the accompanying γ -radiation.

17.52. Discussion of data. The first question to be examined is whether the various experimental approaches just summarized all lead to the same excited level, or whether we are in reality dealing with different levels of neighbouring energies:

(1) It is fairly certain that both the neutron decay of the compound nucleus ${}^8\text{Be}$ and the α -decay of ${}^{11}\text{C}$ give rise to the same state — in fact, the ground state — of the ${}^7\text{Be}$ nucleus, with a definite life-time with respect to transmutation into ${}^7\text{Li}$. The two modes of K -capture effecting this transmutation correspond to a definite fine structure of the ${}^7\text{Li}$ ground state, with very nearly 0.45 MeV separation. The various measurements on ${}^7\text{Be}$ prepared by deuteron bombardment of ${}^6\text{Li}$ show that within a wide range of excitation energies of the ${}^8\text{Be}$ compound nucleus*, the same state of the ${}^7\text{Be}$ nucleus is reached by the α -decay of this compound nucleus. One might still desire a more accurate measurement of the γ -ray from ${}^7\text{Be}$ prepared by the ${}^{10}\text{B}(p, \alpha)$ reaction.

(2) The measurements pertaining to the proton decay of the ${}^8\text{Be}$ compound nucleus may most readily be interpreted as involving a definite

* An idea of the excitation of the compound nuclei may be gained from the kinetic energies of the impinging particles, indicated in the second column of table 17.51.

fine structure of the ${}^7\text{Li}$ residual nucleus, identical with that disclosed by the ${}^7\text{Be}$ decay. The close coincidence of the accurately determined excitation energies derived from the proton groups of the ${}^6\text{Li}(d,n)$ reaction and the γ -ray due to inelastic proton scattering on the one hand, and from the ${}^7\text{Be}$ decay on the other, speak strongly for this identification.

(3) As regards the ${}^7\text{Li}$ fine structure connected with the α -decay of the ${}^{11}\text{B}$ compound nucleus, the situation is less clear, especially in view of the discrepancy between Miss Graves' α -particle group measurements and the precise γ -ray determination by Siegbahn and Slätis. However, the concordance of the latter result with those of the two first groups of experiments again points to the identification of the ${}^7\text{Li}$ fine structures met with in the various cases.

Taking it for granted, therefore, that all the available data pertain to the same fine structure of the ground state of ${}^7\text{Li}$, we may proceed to inquire about the possible character of the two-component stationary states. The first interpretation that suggests itself is, of course, that we are dealing with a P doublet (which would be inverted, since the angular momentum of the ground state of ${}^7\text{Li}$ is known to be $\frac{3}{2}$); this would indeed be in keeping with the quasi-atomic model (10.21). The properties of the K -capture decay of ${}^7\text{Be}$ would not be incompatible with this description (A1.21); but the variation of the relative yield of excited ${}^7\text{Li}$ nuclei with the excitation energy of the compound nucleus ${}^8\text{Be}$ or ${}^{11}\text{B}$ definitely forces us to abandon it: for, if the two states under discussion — apart from a very small energy difference — are only distinguished from each other by different values of the quantum number J , one would expect the numbers of decays of the compound nuclei leading to these states to be in the constant ratio of the statistical weights $2J + 1$, i.e. ${}^7\text{Li}^* : {}^7\text{Li} = \frac{1}{2}$.

It has been pointed out by HAFSTAD and TELLER [38] that the α -particle model offers another possibility of interpretation not subject to such a difficulty. According to this model, the states of lowest energy of the ${}^7\text{Li}$ nucleus are characterized by the wave-function of the loose proton "hole" (13.22). Let ψ_1 , ψ_2 represent the wave-function of a nucleon in the field of the first or the second α -particle of the ${}^8\text{Be}$ system; the proton hole eigenfunction may then be either $\psi_1 - \psi_2$ or $\psi_1 + \psi_2$, with one or no node, respectively, at the centre of gravity. The energy difference between the two corresponding stationary states (of which the lower corresponds to the former eigenfunction) will be given by twice the exchange integral $\int \psi_1 \mathcal{H} \psi_2$ ($\mathcal{H}$ denoting the Hamiltonian of a nucleon in the field of the two α -particles). It is difficult to arrive at anything more than a very rough estimate of this quantity, but it is certainly not unreasonable to envisage the possibility of identifying the excited state of ${}^7\text{Li}$ found empirically with the nodeless state $\psi_1 + \psi_2$ predicted by the α -particle model. A variation of the relative yield of two such states of different symmetry with the excitation of the compound nucleus would not be unexpected. On the other hand, it is necessary for the validity of this interpretation that the

17.51. *Excited state of ${}^7\text{Li}$*

Nuclear reaction	Energy of impinging particles MeV	Yield ${}^7\text{Li}^*/{}^7\text{Li}$	Excitation energy of ${}^7\text{Li}^*$		Reference
			from proton or α -particle groups MeV	from γ -ray MeV	
Compound nucleus ${}^8\text{Be}$ ${}^6\text{Li}(d,p)$ ${}^7\text{Li}(p,p)$	0.1...0.9	0.25...0.74	0.455 ± 0.015	0.400 ± 0.025 0.495 ± 0.025 0.459	WILLIAMS <i>et al.</i> [37], RUMBAUGH <i>et al.</i> [38] FOWLER and LAURITSEN [39]* HUDSON, HERB and PLAIN [40]
	1.080; 1.290 1.032; 1.214; 1.639				
${}^6\text{Li}(d,n){}^7\text{Be} \xrightarrow{K} {}^7\text{Li}$	1			0.425 ± 0.025 0.485 ± 0.005 0.453 ± 0.005 0.465 ± 0.010	RUMBAUGH <i>et al.</i> [38] ZLOTOWSKI and WILLIAMS [42]** SIEGBAHN [46] RUBIN [41]
	2.7 6 ?				
Compound nucleus ${}^{11}\text{C}$ ${}^{10}\text{B}(p,\alpha){}^7\text{Be} \xrightarrow{K} {}^7\text{Li}$	0.95 0.85			0.425 ± 0.025 0.425 ± 0.020	RUMBAUGH <i>et al.</i> [38] MAIER-LEIBNITZ [38]
Compound nucleus ${}^{11}\text{B}$ ${}^{10}\text{B}(n,\alpha)$ ${}^9\text{Be}(d,\alpha)$ ${}^7\text{Li}(\alpha,\alpha)$	slow 0.2...0.35 5.29	15 1.7...1.115	0.42 0.494 ± 0.016	0.462	BØGGILD [45]*** MISS GRAVES [40]*** SIEGBAHN and SLÄTIS [47]†

* According to HUDSON, HERB and PLAIN [40], this value should be lowered by applying a correction to the measured absorption coefficient of the γ -ray, and should then agree with theirs.

** According to SIEGBAHN [46], a lower value might be derived from the data.

*** These papers contain reviews of older work.

† This measurement disproves the occurrence of more than one γ -ray in this reaction, as claimed in an early work by Bothe.

splitting of either of the states $\psi_1 - \psi_2$, $\psi_1 + \psi_2$ due to the interaction between the spin of the "hole" and its orbital momentum be of a smaller order of magnitude than the observed energy difference of 0.45 MeV; and in view of the conclusion derived from the ${}^5\text{He}$ case (17.43), this might appear rather doubtful. For ${}^7\text{Li}$, only discussions of the small ${}^2P_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$ splittings caused by the relativistic spin-orbit couplings or the axial dipole interactions* are available, and these are clearly insufficient to settle the issue.

17.53. Relativistic doublet splitting of ${}^7\text{Li}$. Although the relativistic effects are probably of little significance for the actual state of affairs, their discussion offers some interest in affording an illustration of the considerable differences which may arise between the predictions of different models of nuclear structure. The effect of relativistic spin-orbit coupling has been extensively treated on the quasi-atomic model by BREIT and STEHN [38c], in a paper containing also important results of general validity for any type of spin-orbit coupling; the α -particle model is examined in a paper by KITTEL [42], who lays special stress on the importance of the exchange integrals in the expression for the average interaction energy (11.11). Without entering into such involved calculations, we may, like INGLIS [39c, a], easily compare the orders of magnitude of the Larmor and Thomas terms on the two models.

These models may be said to differ in two respects; in the first place, by a different subdivision into "clusters" of nucleons assumed in first approximation to move as a whole: on the α -particle model, we have an α -particle and a triton; on the quasi-atomic model, we may in a rough way envisage the motion of the p -shell proton with respect to the residual ${}^6\text{He}$ cluster. The second difference concerns the forces acting between the clusters: these are weaker on the α -particle model than in the other case. While these differences do not very much affect the Larmor splitting, they have a considerable influence on the Thomas term. If M' denotes the reduced mass of the clusters with respect to the centre of gravity of the total system, the Larmor ${}^2P$ splitting may in fact be written

$$[{}^2\varepsilon_{\frac{1}{2}}^{(1)} - {}^2\varepsilon_{\frac{3}{2}}^{(1)}]_{\text{Larmor}} \approx -\frac{3}{4} \frac{\hbar^2}{MM'} g_s \frac{2e^2}{R^3}, \quad (1)$$

R denoting the average distance of the clusters, for which we may take the nuclear radius $R \approx 2r_0 \approx d$. Since $M' = \frac{1}{2}M$ on the α -particle model and $\frac{1}{3}M$ on the quasi-atomic model, the Larmor splitting is of the same order of magnitude in both cases, viz. $\approx 10^{-2}$ MeV. As regards the Thomas splitting, we can also write down in the same form in both cases the contribution arising from the acceleration due to the orbital motion of the clusters. Denoting by ω the angular velocity, divided by c , of this orbital motion, we have, for a P state,

$$\hbar \sqrt{2} \approx M' R^2 \omega; \quad (2)$$

on the other hand, the corresponding Thomas precession velocity

$$\omega_T \approx \frac{1}{2} \omega^3 R^2, \quad (3)$$

* Using the α -particle model of the ${}^7\text{Li}$ nucleus, INGLIS [39c], by a very crude estimate, concludes that the (second order) splitting due to the axial dipole interaction of meson theory will be less than 0.015 MeV.

whence for the ${}^2P$ splitting

$$[{}^2\epsilon_{\frac{3}{2}}^{(1)} - {}^2\epsilon_{\frac{1}{2}}^{(1)}]_{\text{Thomas}} \approx \frac{3}{\sqrt{2}} \frac{\hbar^4}{M'^3 R^4}. \quad (4)$$

On the α -particle model, we should take account of a further radial acceleration, due to the virtual processes of exchange of nucleons between the clusters, which contribute to their interaction; but, as shown by Inglis, the resulting correction to the Thomas precession (3) is rather small. Keeping, therefore, to formula (4), we see that the occurrence of the reduced mass M' to a large power has the effect of reducing the Thomas effect by a factor 8 on the α -particle model. The stronger nuclear field acting on the proton in the quasi-atomic picture would tend to lower the value of R , which again would work in the same direction, viz. to reduce the ratio of the Thomas splittings predicted by the two models. Since for $M' \approx M$ the ratio

$$\left| \frac{\text{Larmor splitting}}{\text{Thomas splitting}} \right| \approx \left(\frac{e^2}{\hbar} \right)^2 \frac{M}{m} \approx \frac{1}{10}, \quad (5)$$

we see that while the whole effect is dominated, on the quasi-atomic model, by the Thomas contribution, this is no longer true on the α -particle model; in this case, both the Thomas and the Larmor contributions become of the same order of magnitude, much smaller than in the former case, or the Larmor effect even becomes preponderant.

APPENDICES

APPENDIX I

ALLOWED β -TRANSITIONS

A1.0. The theory of β -radioactivity is only indirectly connected with the main subject of this book. We shall here only develop the treatment of allowed β -transitions on the basis of Wigner's theory of supermultiplets (10.14). For other aspects of the theory and a survey of the empirical material, we can refer the reader to the up-to-date and authoritative account of KONOPINSKI [43].

A1.1. Allowed transitions on Wigner's approximation

A1.11. *General expression of life-time.* If a light nucleus goes over by a $\beta^\pm$ -transition or a K -capture process from a state Ψ_i into a state Ψ_f , its life-time t_{if} in the initial state Ψ_i will be given in first approximation by a formula of the type

$$\frac{\log 2}{t_{if}} = \frac{1}{\tau_0} |G_{if}|^2 I(W_{if}), \quad (1)$$

in which τ_0 is a characteristic time-constant, G_{if} the matrix-element

$$G_{if} = \int \Psi_i^* \mathcal{V}_\beta \Psi_f \quad (2)$$

of a certain operator $\mathcal{V}_\beta$ characterizing the interaction between the nucleons and the lepton field, and $I(W_{if})$ a function of the total energy W_{if} (in units m) emitted in the transition:

$$\begin{aligned} \beta^\pm\text{-transition:} \quad W_{if} &= 1 + \frac{K_\beta}{m} \\ K\text{-capture:} \quad W_{if} &= \frac{K_\nu}{m}; \end{aligned} \quad (3)$$

K_β is the maximum kinetic energy of the emitted electron, while K_ν is given (1.131-5) by the difference of the atomic weights of initial and final nuclei. The function $I(W)$, which also depends on the charge number Z of the decaying nucleus, is a complicated expression, which need not be given in full here; we shall only mention the approximate formulae valid for small Z :

$$\begin{aligned} \beta^\pm\text{-transition:} \quad I(W) &= \sqrt{W^2-1} \left(\frac{1}{80} W^4 - \frac{3}{80} W^2 - \frac{2}{15} \right) \\ &\quad + \frac{1}{4} W \log(W + \sqrt{W^2-1}) \\ K\text{-capture:} \quad I(W) &= 2\pi (Z_{\text{eff}}/137)^3 W^2; \end{aligned} \quad (4)$$

Z_{eff} denotes the effective charge number of the nucleus for the K electrons (e.g. $Z_{\text{eff}} = 3.7$ for ${}_4\text{Be}$). When the matrix-element G_{if} does not vanish, one speaks of an *allowed* transition.

In this approximation, there are only two essentially different possibilities for the interaction operator $\mathcal{V}_\beta$, viz. the *Fermi interaction*

$$\mathcal{V}_\beta = \sum_i \tau_1^{(i)} \quad (5)$$

and the *Gamow-Teller interaction*

$$\mathcal{V}_\beta = \frac{1}{\sqrt{3}} \sum_i \tau_1^{(i)} \vec{\sigma}^{(i)}. \quad (6)$$

The integral sign in (2) implies summation over spin and isotopic variables as well as integration over space coordinates of the nucleons. In the case of Gamow-Teller interaction, $|G_{if}|^2 = \sum_x |G_x|^2$. The operators $\tau_1^{(i)}$, or more generally (4.12) $\tau_1^{(i)}$, occur in (5) and (6) as the sums of the operators Π and $\Pi^\dagger$ effecting the transition from a neutron to a proton state and vice versa. To each type of interaction belongs a specific time-constant τ_0 ; one might, of course, more generally take a linear combination of the operators (5) and (6); the right hand side of (1) would then be replaced by a sum of similar expressions*.

If there is only one possible transition $i \rightarrow f$, the resulting β -spectrum will be called *simple*. In general, we may expect that there will be several allowed final states, of different energies. The observed β -spectrum will in such cases be a superposition of simple spectra with different upper energy limits W_{if} and different half-lives t_{if} ; the observed half-life t is then given by

$$\frac{1}{t} = \sum_f \frac{1}{t_{if}} = \frac{1}{\tau_0 \log 2} \sum_f |G_{if}|^2 I(W_{if}). \quad (7)$$

Even if the separate $|G_{if}|^2$ may be difficult of evaluation, their sum $\sum_f |G_{if}|^2$ over all possible final states has a simple meaning: since $\mathcal{V}_\beta$ is Hermitian, it is just the expectation value of $\mathcal{V}_\beta^2$ in the initial state.

A1.12. Selection rules. Thanks to the connexion of the operators (5) and (6) with infinitesimal rotations in ordinary or symbolic space, it is possible, as emphasized by WIGNER [39b], to discuss the matrix-element (2) to a large extent without any explicit knowledge of the wave-functions. We may first enunciate selection rules of universal validity, pertaining to the parity and total angular momentum of the levels between which the

* Strictly speaking, the Gamow-Teller operator (6) itself arises from the linear combination of two distinct operators, viz. (6) and that obtained by substituting $\vec{\sigma}_3^{(i)} \vec{\sigma}^{(i)}$ for $\vec{\sigma}^{(i)}$, which become identical in non-relativistic approximation for the nucleons. This gives rise, by interference in the calculation of the total $|G_{if}|^2$, to a factor of the form $(1 + \frac{\lambda}{W})$, with $|\lambda| \leq 1$, multiplying the matrix-element $|G_{if}|^2$ defined by (2) and (6); we shall, however, disregard this correction. A linear combination of interactions (5) and (6), on the other hand, does not give rise to interference terms. See FIERZ [37].

transition takes place. It is clear that both levels must have *the same parity*. Further, since the interaction operators (5) and (6) behave as a scalar and a vector, respectively, with respect to rotations, the changes ΔJ , Δm of the quantum numbers J , m of the total angular momentum and its z -component are subject to the restrictions:

$$\text{Fermi interaction:} \quad \Delta J = \Delta m = 0 \quad (8)$$

$$\text{Gamow-Teller interaction:} \quad \Delta J \text{ and } \Delta m = 0 \text{ or } \pm 1. \quad (9)$$

Assuming now the validity of Wigner's approximation, we see that allowed transitions can only occur between states belonging to the same supermultiplet: *all transitions between states of different supermultiplets are forbidden*. In general, the initial state will be the ground state of the nucleus; its quantum number T_3 will therefore (10.311-18) be the largest in absolute value occurring in the supermultiplet. If $T_3 > 0$, any β^+ -transition (or K -capture), $T_3 \rightarrow T_3 + 1$, will then be forbidden, since it would lead to another supermultiplet; but for $T_3 < 0$ β^+ -transitions will be allowed. As regards β^- -transitions, $T_3 \rightarrow T_3 - 1$, they will generally be impossible because, on account of the Coulomb interaction, the energies of the states of a supermultiplet increase with decreasing T_3 ; the chief* exception to this situation arises from the fact (10.311-22) that the ground states of the nuclei of mass number $4a + 2$ with $T_3 = 0$ as well as $T_3 = \pm 1$ both belong to the same supermultiplet (1,0,0): if, then, the odd isobar ($T_3 = 0$) is more stable than the even one with $T_3 = 1$, the latter will transmute into the former with allowed emission of a negaton. If the odd isobar is unstable, it will undergo an allowed β^+ -decay. We thus arrive at the general conclusion that *most β -transitions from light nuclei are forbidden; there are, however, two exceptional classes of β -active nuclei*:

- (a) positron emitters with negative neutron excess,
- (b) negaton or positron emitters of mass number $4a + 2$, the ground state of which belongs to the supermultiplet (1,0,0).

To class (a) belongs, in particular, the well-known series with $T_3 = -\frac{1}{2}$ (3.3); the corresponding supermultiplets are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, according as A is of the form $4a + 1$ or $4a + 3$ (10.311-19, 21).

In addition to the general selection rules (8), (9), we now get, on Wigner's approximation, the analogous ones for the spin quantum number:

$$\text{Fermi interaction:} \quad \Delta S = 0 \quad (10)$$

$$\text{Gamow-Teller interaction:} \quad \Delta S = 0 \text{ or } \pm 1. \quad (11)$$

Moreover, it follows from the scalar character of both types of interaction (5) and (6) with respect to spatial rotations (without rotation of spins)

* Another exceptional case of allowed β^- -decay is that of ^3H ; see A1.212.

that no change of orbital momentum can take place:

$$\Delta L = 0. \quad (12)$$

Combining (8), (10) and (12), we see that on *Fermi interaction* there is at most one transition of definite (β^+ or β^-) type from any initial state: *the β -spectrum is simple*. Clearly, this conclusion remains strictly valid even when the spin dependent nuclear forces are not neglected and mix up different supermultiplets. But if the interaction is of the Gamow-Teller type, the β -spectrum will generally be complex and when Wigner's approximation is abandoned, no regular relation between life-time and energy can any more be expected. Since the number of possible transitions from a given initial state will become larger, while the sum $\sum_f |G_{if}|^2$ remains bounded, each $|G_{if}|^2$ will then on the average be much smaller than on Wigner's approximation.

A1.13. Calculation of matrix-elements. The interpretation of $\sum_f |G_{if}|^2$ as expectation value of $\mathcal{V}_\beta^2$ makes its calculation possible for any state belonging to a supermultiplet of either of the above classes (a) and (b) giving rise to allowed transitions. Owing to the property of $\mathcal{V}_\beta^2$, of representing an infinitesimal rotation operator, its expectation value will depend only on the rotation quantum numbers of the state i , but not on the number of particles of the nuclear system: we can therefore simplify its calculation by performing it for the smallest value of A for which the supermultiplet containing state i exists. In the case of Fermi interaction, this gives at once the only $|G_{if}|^2$ occurring. On Gamow-Teller interaction, the relative values of the various $|G_{if}|^2$ may be derived by the usual procedures of atomic spectroscopy; in this case, a complete determination of all $|G_{if}|^2$ can thus also be achieved.

A1.131. Matrix-elements for class (a) nuclei of neutron excess -1 . As regards the nuclei of class (a), we shall confine the discussion to those of neutron excess -1 , the ground state of which belongs to one of the supermultiplets $(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2})$. An example of the supermultiplet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is furnished by a single nucleon. Then $\sum_f |G_{if}|^2 = 1$ for all states i of the supermultiplet. The same holds for the supermultiplet $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$; for we may, to discuss this case, consider a system of 3 nucleons held together by such forces as to allow of a description by a quasi-atomic model: if they occupy an s shell, any matrix-element $|G_{if}|$ can be different from zero only if the final state f differs from the initial one i by just one individual wave-function ψ_i being replaced by another, ψ_f , in which case

$$|G_{if}| = |\int \psi_i^* \tau_1 \psi_f| \quad \text{or} \quad |\int \psi_i^* \tau_1 \vec{\sigma} \psi_f|,$$

exactly as in the one-nucleon problem (we may say that the matrix-element refers to the "hole" in the closed s shell).

To proceed further in the case of Gamow-Teller interaction, we first observe that, since $S = \frac{1}{2}$, only transitions with $\Delta S = 0$ are possible. If $L = 0$, there is only one such transition; hence, *for any interaction, if the ground state of the decaying nucleus is a $^{22}S_{\frac{1}{2}}$ state, the β^+ -spectrum is simple.* For other L -values, we get a compound spectrum; the $|G_{if}|^2$ are then given by expressions similar to the well-known Hönl-Kronig formulae, in which, however, the roles of S and L are interchanged:

Ground state of decaying nucleus	Transition $J \rightarrow J'$	$ G_{if} ^2$ (Gamow-Teller int.)
$^{22}(L)_{L+\frac{1}{2}}$	$L + \frac{1}{2} \begin{matrix} \nearrow L + \frac{1}{2} \\ \searrow L - \frac{1}{2} \end{matrix}$	$\frac{2L+3}{3(2L+1)}$ $\frac{4L}{3(2L+1)}$
$^{22}(L)_{L-\frac{1}{2}}$	$L - \frac{1}{2} \begin{matrix} \nearrow L + \frac{1}{2} \\ \searrow L - \frac{1}{2} \end{matrix}$	$\frac{4(L+1)}{3(2L+1)}$ $\frac{2(L-1)}{3(2L+1)}$

(13)

A1.132. Matrix-elements for class (b) nuclei. The discussion of case (b) proceeds on the same lines. The simplest example of the supermultiplet (1,0,0) is given by a system of two nucleons. The symmetry character of the spatial wave-functions is then (10.14-8) $A(1+1)$ or $S(2)$, i.e. the eigenfunctions are symmetrical in the space coordinates. Their dependence on spin and isotopic variables is therefore either $^1(\tau)^3(\sigma)$ or $^3(\tau)^1(\sigma)$. Writing

$$(\tau_1^{(1)} + \tau_1^{(2)})^2 = 2(1 + \tau_1^{(1)} \tau_1^{(2)}),$$

$$\frac{1}{3}(\tau_1^{(1)} \overset{\rightarrow}{\sigma}^{(1)} + \tau_1^{(2)} \overset{\rightarrow}{\sigma}^{(2)})^2 = 2(1 + \frac{1}{3} \tau_1^{(1)} \tau_1^{(2)} \overset{\rightarrow}{\sigma}^{(1)} \overset{\rightarrow}{\sigma}^{(2)}),$$
(14)

we easily get for the sums $\sum_f |G_{if}|^2$:

Initial state	$\sum_f G_{if} ^2$	
	Fermi interaction	Gamow-Teller interaction
$^1(\tau)_0^3(\sigma)$	0	$\frac{4}{3}$
$^3(\tau)_{\pm 1}^1(\sigma)$	2	2
$^3(\tau)_0^1(\sigma)$	4	0

(15)

There are two types of transitions involving a change $\Delta T_3 = \pm 1$:

$$^1(\tau)_0^3(\sigma) \leftrightarrow ^3(\tau)_{\pm 1}^1(\sigma) \quad (16)$$

$$^3(\tau)_0^1(\sigma) \leftrightarrow ^3(\tau)_{\pm 1}^1(\sigma). \quad (17)$$

Transitions (16) from or to triplet states involve a change of spin: they

are therefore forbidden on Fermi interaction, allowed on Gamow-Teller interaction. For transitions (17) between singlet states (with the additional restriction $\Delta L = 0$), the situation is just reversed: as shown by (15), they are forbidden on Gamow-Teller interaction, allowed on Fermi interaction.

For the allowed transitions, the $|G_{if}|^2$ are readily found. On Fermi interaction, we have, for any transition (17),

$$|G_{if}|^2 = 2, \quad (18)$$

as directly indicated by (15). (The value $\sum_f |G_{if}|^2 = 4$ for the transitions from ${}^3(\tau)_0 {}^1(\sigma)$ corresponds in fact to two possibilities, viz. ${}^3(\tau)_{\pm 1} {}^1(\sigma)$, with equal $|G_{if}|^2$.) For Gamow-Teller interaction, the result is as follows:

A1.132. Allowed β -transitions of class (b) nuclei, on Gamow-Teller interaction					
Transition	$T_3 \rightarrow T'_3$	$S \rightarrow S'$	$J \rightarrow J'$	$ G_{if} ^2 (L \neq 0)$	$ G_{if} ^2 (L = 0)$
${}^1(\tau)_0 {}^3(\sigma) \rightarrow {}^3(\tau)_{\pm 1} {}^1(\sigma)$	$0 \rightarrow \pm 1$	$1 \rightarrow 0$	$L+1 \rightarrow L$ $L \rightarrow L$ $L-1 \rightarrow L$	$\frac{2}{3}$ $\frac{2}{3}$ $\frac{2}{3}$	$\frac{2}{3}$ $—$ $—$
${}^3(\tau)_{\pm 1} {}^1(\sigma) \rightarrow {}^1(\tau)_0 {}^3(\sigma)$	$\pm 1 \rightarrow 0$	$0 \rightarrow 1$	$\nearrow L+1$ $L \rightarrow L$ $\searrow L-1$	$\frac{2}{3} \cdot \frac{2L+3}{2L+1}$ $\frac{2}{3}$ $\frac{2}{3} \cdot \frac{2L-1}{2L+1}$	2 $—$ $—$

A1.2. Discussion of empirical data

A1.21. *Class (a) nuclei of neutron excess -1.* As already mentioned (3.3), class (a) nuclei of neutron excess -1 have been the object of a systematic investigation. The evidence concerning them is presented in table A1.21. Columns 4 and 5 contain the observed half-lives t and maximum kinetic energies K_β of the emitted positons. In column 6, the value of the latter quantity, as calculated by formula (3.3-1) together with (2.1-1,2), is given: the good agreement with the observed value is of some theoretical significance, as explained in 3.3. Column 7 gives the W derived by (A1.11-3) from the observed K_β (except for $A = 21$, where this quantity is not known experimentally). Column 8 lists the quantity

$$\tau'_0 = \frac{1}{\log 2} I(W) t, \quad (1)$$

expressed in 10^3 sec; according to (A1.11-1), this quantity would be identical with the time-constant τ_0 , if the β -spectrum were simple and the matrix-element $|G_{if}|^2$ equal to unity. Finally, column 3 gives the observed quantum number J' of the total angular momentum of the product nucleus in its ground state: this is the only certain element we can use at present to speculate on the possible transitions contributing to the observed spectrum.

The first case in the table (${}^7_4\text{Be} \rightarrow {}^7_3\text{Li}$) is not one of β^+ -emission, but

of K -capture; accordingly, the value listed under K_β is in reality K_γ , and the W and $I(W)$ must be calculated by the appropriate formulae (A1.11-3, 4). The next two positron emitters (^{11}C and ^{13}N) have recently been studied with great care by K. SIEGBAHN and his collaborators [44; 45a, b] and we use their results. All other data concerning t , K_β (and τ'_0) are taken from KONOPINSKI [43], while the J' are given according to M&F.

A1.21. Positron emitters of neutron excess — 1.							
A	Isobaric pair	J'	t sec	K_β		W	τ'_0 10 ⁸ sec
				observed MeV	calculated MeV		
7	$^4\text{Be} \rightarrow ^3\text{Li}$	$\frac{3}{2}$	$5 \cdot 10^6$	0,87	—	1,7	2,58
11	$^6\text{C} \rightarrow ^5\text{B}$	—	1230	0,98	1,01	2,92	4,82
13	$^7\text{N} \rightarrow ^6\text{C}$	$\frac{1}{2}$	608	1,24	1,29	3,43	7,34
15	$^8\text{O} \rightarrow ^7\text{N}$	$\frac{1}{2}$	125	1,72	1,62	4,37	5,99
17	$^9\text{F} \rightarrow ^8\text{O}$	—	64	1,93	1,95	4,78	4,82
19	$^{10}\text{Ne} \rightarrow ^9\text{F}$	$\frac{1}{2}$	20,3	2,20	2,27	5,31	2,65
21	$^{11}\text{Na} \rightarrow ^{10}\text{Ne}$	—	23	—	2,56	6,00	7,68
23	$^{12}\text{Mg} \rightarrow ^{11}\text{Na}$	$\frac{3}{2}$	11,6	2,82	2,87	6,52	4,45
25	$^{13}\text{Al} \rightarrow ^{12}\text{Mg}$	—	7	2,98	3,17	6,85	3,46
27	$^{14}\text{Si} \rightarrow ^{13}\text{Al}$	$\frac{5}{2}$	4,9	3,54	3,42	7,93	5,11
29	$^{15}\text{P} \rightarrow ^{14}\text{Si}$	—	4,6	3,63	3,68	8,16	5,47
31	$^{16}\text{S} \rightarrow ^{15}\text{P}$	$\frac{1}{2}$	3,2	3,86	3,94	8,57	4,69
33	$^{17}\text{Cl} \rightarrow ^{16}\text{S}$	$\frac{5}{2}$	2,4	4,13	4,21	9,14	4,92
35	$^{18}\text{A} \rightarrow ^{17}\text{Cl}$	—	1,9	4,37	4,46	9,57	4,97
37	—	—	—	—	—	—	—
39	—	—	—	—	—	—	—
41	$^{21}\text{Sc} \rightarrow ^{20}\text{Ca}$	—	0,9	4,94	5,18	10,68	3,57

The question as to the complexity of the β -spectra is of course of considerable theoretical interest, as the existence of a compound spectrum would be a very strong argument in favour of Gamow-Teller interaction. Unfortunately, direct evidence in this respect is very scarce and inconclusive. In contradiction to previous belief, it has recently been established in a convincing way by K. SIEGBAHN *et al.* [45a, b], and by COOK *et al.* [47], that the β -spectrum of ^{13}N is simple. The latter authors state that they have been unable to detect any γ -ray in the interval 0,135 ... 0,700 MeV; they conclude that either the interaction is of the Fermi type, or the energy separation of the P doublet expected as the ground state of ^{13}C is outside the above interval (smaller than 0,135 MeV or larger than 0,700 MeV). Apart from this reliable case, we have only some evidence, by no means so secure, on the complexity of the transition $^7\text{Be} \rightarrow ^7\text{Li}$. The ^7Li nucleus has a well-known stationary state of about 0,45 MeV excitation, and it seems that about 1/10 of the nuclei resulting from the decay of ^7Be are formed in this excited state: this can be concluded from the intensity of γ -rays of the same energy which accompany the disintegration. It is

rather pointless, however, to try to analyse this process with the help of the formulae (A1.131–13), for it is very doubtful (17.52) whether the ground state and the low excited state of ${}^7\text{Li}$ can be described as the ${}^2P_{1/2}$ and ${}^2P_{3/2}$ components of a doublet.

Should we do so all the same, we should first have to make some assumption on the quantum numbers (either ${}^2P_{3/2}$ or ${}^2P_{1/2}$) of the initial state of ${}^7\text{Be}$. The ground state of ${}^7\text{Li}$ is taken to be the ${}^2P_{1/2}$ component of the doublet, in conformity with the observed angular momentum. The $I(W_{if})$ and $|G_{if}|^2$ being then determined, we get the fraction of transitions to the excited state of ${}^7\text{Li}$ as the ratio of the products $|G_{if}|^2 I(W_{if})$. By means of (A1.11–7) we can also derive the value of τ_0 from the observed half-life. The following results are obtained:

Transitions assumed	$ G_{if} ^2$	W_{if}	$I(W_{if})$	% transitions to excited state	τ_0 10 ³ sec
$P_{3/2} \nearrow P_{3/2}$ $\searrow P_{1/2}$	$\frac{5}{9}$	1.7	$3.57 \cdot 10^{-4}$	19	1.69
	$\frac{4}{9}$	0.82	$0.83 \cdot 10^{-4}$		
$P_{1/2} \nearrow P_{3/2}$ $\searrow P_{1/2}$	$\frac{8}{9}$	1.7	$3.57 \cdot 10^{-4}$	3	2.35
	$\frac{1}{9}$	0.82	$0.83 \cdot 10^{-4}$		

Although the lack of data about the nature of the states involved in the process prevents us from submitting the theory to a closer test, we can, following WIGNER* and KONOPINSKI [43], draw some general conclusions from the consideration of the constant τ'_0 listed in the above table. If we had to do with a Fermi interaction, this constant would be identical with τ_0 and should thus have the same value throughout. Considering the poor accuracy of the determinations of most life-times and upper limit energies, and the sensitiveness of $I(W)$ to even small changes of W , the relatively small fluctuations of the τ'_0 values might be taken as supporting the theory based on Fermi interaction, were it not that weighty arguments can be adduced in favour of a predominance of Gamow-Teller interaction**.

In the latter case, we have generally to expect complex β -spectra. Since the value of W given in the table is the largest of the W_{if} and $I(W)$ is an increasing function of its argument, it follows from the formula (A1.11–7) for the observed life-time and the definition (1) of τ'_0 that

$$\tau'_0 \geq \frac{\tau_0}{\sum_f |G_{if}|^2}, \quad (2)$$

the sign of equality corresponding to the special case of a simple spectrum.

* See WIGNER [39b] and a discussion by WIGNER at the end of the paper of WHITE *et al.* [41].

** These arguments are summarized by KONOPINSKI [43].

For the class of nuclei under consideration, this general formula reduces to (A1.131)

$$\tau'_0 \geq \tau_0. \quad (3)$$

In view of this relation, it is interesting to observe that the states involved in the transition $^{19}_{10}\text{Ne} \rightarrow ^{19}_9\text{F}$, which is one of those yielding the smallest τ'_0 values, are probably S states (10.32), while the angular momentum of the product nucleus is known to be $\frac{1}{2}$: this is therefore probably a $^2S_{\frac{1}{2}} \rightarrow ^2S_{\frac{1}{2}}$ transition, which (A1.131) indeed gives a simple spectrum. This argument would point to a value of

$$\tau_0 \approx 3 \cdot 10^3 \text{ sec.} \quad (4)$$

The association of a simple spectrum with a larger τ'_0 value in the case of ^{13}N would then imply a failure of Wigner's approximation for the states playing a part in this decay process. On the other hand, the case of ^7Be seems to afford an example of a compound transition with exceptionally small τ'_0 ; the above calculation shows that this is at any rate compatible with a value of τ_0 of the order of magnitude given by (4), provided the initial state is assumed to have the angular momentum $\frac{1}{2}$.

A1.211. *Life-time of the neutron.* Formally, the neutron belongs to the class of nuclei just discussed. Disregarding the kinetic energy of the proton after the process, we have

$$W = \frac{M_n - M_p}{m} = 2,47$$

(1.22-5) and the corresponding $I(W)$ is given by (A1.11-4). We have here to do with a simple transition, for which $|G_{if}|^2 = 1$; taking the value (4) of τ_0 , we compute by means of (A1.11-1) a half-life of about 25 minutes.

A1.212. *The β -decay of ^3H and the mass of the neutrino.* The process $^3\text{H} \xrightarrow{\beta^-} ^3\text{He}$, in which the states of both nuclei involved likewise belong to the supermultiplet $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, is an exceptional example of allowed β -transition, due to the fact that the increase in Coulomb energy in this case is just small enough to make it possible; the maximum kinetic energy of the electron emitted is in fact extremely small: according to the latest estimate (WATTS and WILLIAMS [46]), $K_\beta < 0,03 m$. On the other hand, the life-time of the triton has a surprisingly low value of about * 12 years; much too low, in fact, as emphasized by KONOPINSKI [47], to fit in with the evidence just discussed on class (a) nuclei, and especially with the estimate (4) for the constant τ_0 . Since for the process under consideration, in which S states are involved, one expects (A1.131) $|G_{if}|^2 = 1$, this constant should be directly given by the quantity τ'_0 defined by formula

NOVICK [47] gives $12,1 \pm 0,5$ years, GOLDBLATT *et al.* [47] $10,7 \pm 2$ years.

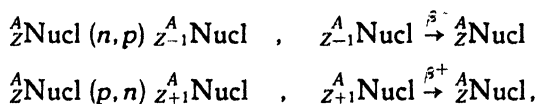
(1). Now, for very small values of $W-1$, the expression (A1.11-4) for the factor $I(W)$ occurring in (1) reduces to

$$I(W) \approx 0,216(W-1)^{1/2}.$$

The above data then yield $\tau'_0 \lesssim 0,55 \cdot 10^3$ sec. In order to remove this discrepancy, one would have to assume a maximum energy K_β as large as $0,05 m$; this figure is perhaps not entirely ruled out in view of the difficulty of the measurements.

Konopinski attempted to link the discrepancy to the question of the mass of the neutrino. In fact, the above expression for $I(W)$ is only valid, strictly speaking, for a vanishing neutrino mass, or at any rate for values of $W-1$ sufficiently large compared with the ratio μ of this mass to that of the electron. In this respect, the case of ${}^3\text{H}$, in view of the exceptionally small value of $W-1$, might require a quite different treatment. It is also necessary, as pointed out by PRUETT [48], to take into account the influence of the nuclear Coulomb field on the electron and a relativistic correction arising from the finite neutrino mass when summing over the spins of the emitted leptons. Pruett's conclusion is that the value of μ needed to account for the experimental data is in any case larger than $0,05$ and might be as high as $\approx 0,3$ (if $K_\beta \approx 0,02 m$).

However, such a large value of μ would definitely conflict with the evidence about the mass of the neutrino that can be derived from energy balances of cycles of nuclear reactions, mainly of the types



or from the study of the shape of the β -spectrum near its end-point. In a comprehensive and critical survey of the relevant reaction cycles, FRANK [47] obtains the estimate $\mu = 0,0631 \pm 0,0701$; this means, allowing for any possible systematic error, that the neutrino mass is certainly less than $\frac{1}{10}$ of the electron mass, and is probably zero. A recent investigation by HUGHES and EGGLER [48] leads to even more precise conclusions. From a cloud chamber study of the reactions



combined with the K_β values of the ${}^3\text{H}$ and ${}^{14}\text{C}$ β -spectra, they derived values of the neutrino mass amounting to a few keV, with a total margin of error of about 25 keV. This result puts the upper limit of μ at about $0,05$, while pointing to a probably much smaller, or even vanishing, value. Besides, it shows that the neutrinos involved in the two processes studied have the same mass within about 5 keV; this information is interesting, as we shall see presently, in view of the anomalous character of the ${}^{14}\text{C}$ decay (A1.22).

The other method of ascertaining the mass of the neutrino necessitates

for its application a very accurate determination of the shape of the β -spectrum right up to its end-point. Quite recently, it has been possible for the first time to derive from such measurements on the spectrum of ^{35}S (COOK, LANGER and PRICE [48], an upper limit for the neutrino mass $\mu < 0.01$.

A1.22. Class (b) nuclei. All known data on class (b) nuclei are collected in the following table (KONOPINSKI [43]*; the last entry according to M & F). The third column indicates the type of decay, as defined by the transition $T_3 \rightarrow T'_3$; the last one again gives the quantity τ'_0 defined by (1).

A1.22. Decay processes of nuclei of mass $4a+2$, involving an odd isobar without neutron excess						
A	Isobaric pair	Type of decay ($T_3 \rightarrow T'_3$)	t sec	K_β MeV	W	τ'_0 10^8 sec
6	$^2\text{He} \rightarrow ^3\text{Li}$	$\beta^- (1 \rightarrow 0)$	0.8	3.7	8.25	1.67
10	$^4\text{Be} \rightarrow ^5\text{B}$	$\beta^- (1 \rightarrow 0)$	$9.1 \cdot 10^{13}$	0.56	2.1	$8.2 \cdot 10^{10}$
10	$^6\text{C} \rightarrow ^5\text{B}$	$\beta^+ (-1 \rightarrow 0)$	9	3.35	7.57	8.6
14	$^6\text{C} \rightarrow ^7\text{N}$	$\beta^- (1 \rightarrow 0)$	$1.5 \cdot 10^{11}$	0.143	1.28	$8.7 \cdot 10^5$
18	$^9\text{F} \rightarrow ^8\text{O}$	$\beta^+ (0 \rightarrow 1)$	6720	0.7	2.37	8.42
22	$^{11}\text{Na} \rightarrow ^{10}\text{Ne}$		$> 10^8$	0.55	2.08	$> 2 \cdot 10^4$
26	$^{13}\text{Al} \rightarrow ^{12}\text{Mg}$		7	2.98	6.85	3.46
30	$^{15}\text{P} \rightarrow ^{14}\text{Si}$		153	2.99	6.87	72
34	$^{17}\text{Cl} \rightarrow ^{16}\text{S}$		1980	2.5	5.90	403
38	$^{19}\text{K} \rightarrow ^{18}\text{A}$		462	2.3	5.50	58
42	$^{21}\text{Sc} \rightarrow ^{20}\text{Ca}$		$1.2 \cdot 10^6$	1.4	3.74	$1.9 \cdot 10^4$
—	—		—	—	—	—
58	$^{29}\text{Cu} \rightarrow ^{28}\text{Ni}$		81 or 474	—	—	—

Since (11.41) the ground state of the odd isobar is normally a triplet state, all the transitions listed in this table would be forbidden on Fermi interaction (A1.132). Now, the large values of τ'_0 occurring in most cases are readily explained by supposing that the corresponding transition is a forbidden one. But the small τ'_0 values obtained for

$$^6_2\text{He}, \quad ^{10}_6\text{C}, \quad ^{18}_9\text{F}, \quad ^{26}_{13}\text{Al} \quad (5)$$

cannot be accounted for in this way; on the contrary, their occurrence constitutes one of the main arguments for the preponderance of Gamow-Teller interaction.

On the quasi-atomic model (10.24), the ground states of the first three nuclei (5) would be S states, and this would also probably be the case

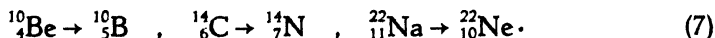
* More recent data: ^{10}Be HUGHES *et al.* [47], McMILLAN [47]; ^{14}C REID *et al.* [46], LEVY [47] ($K_\beta = 0.154$ MeV), SALOMON, GOULD and ANFINSEN [47] (same K_β as Levy), NORRIS and INGRAM [48] (half-life $1.6 \cdot 10^{11}$ sec).

for $^{26}_{13}\text{Al}$. Table A1.132 then shows that their β -decay would be simple; in the first two cases, one would have $\tau_0 = 2\tau'_0$, in the last two $\tau_0 = \frac{2}{3}\tau'_0$. The τ_0 values so obtained, viz.

$$3.34 \quad ; \quad 17.2 \quad ; \quad 5.61 \quad ; \quad 2.31 \quad (6)$$

(10^3 sec), are — except the second — of the required order of magnitude (4), determined by the evidence from class (a) nuclei. The larger τ'_0 values appearing for higher mass-numbers (from $A = 30$ on) — and perhaps also the exceptionally high value for $^{10}_6\text{C}$ — could be ascribed to the inadequacy of Wigner's approximation. In fact, for heavier nuclei, the perturbations due to the spin dependent nuclear forces would gradually tend to blur the distinction between allowed and forbidden transitions (A1.12).

Much more puzzling are the high τ'_0 values corresponding to the transitions



As compared with the allowed decay processes of the nuclei (5), these transitions are characterized by abnormally long life-times and abnormally low energy releases. It is not probable that their highly forbidden character can be ascribed to some anomalous behaviour of the initial or final state. Thus, it has been established, by the examination of the band spectrum, that the angular momentum of the nucleus ^{14}C is zero (JENKINS [48]). As regards ^{10}Be , it would be difficult to understand how the initial state of this nucleus could be so very different from that of the other unstable isobar ^{10}C , which seems to behave normally: the two isobars differ only by the exchange of two neutrons for two protons, and (on the quasi-atomic picture) lie symmetrically with respect to ^{10}B , as regards the filling of the $2p$ shell (10.32); there can thus at most be a second order perturbation due to Coulomb interaction, and this effect, estimated by COOPER and NELSON [41], although working in the right direction, seems definitely too small. This deadlock has induced OPPENHEIMER [41] to put forth suggestions with far-reaching implications on the nature of the neutrinos involved in such anomalous processes; but this interesting speculation does not seem to be promising*.

The formally similar transition (7.2)



requires some comment. As stated in 7.2, one can use the cross-section for

* Oppenheimer's suggestion that the neutrino in question would have a spin $\frac{1}{2}$ has been shown by KUSAKA [41] to lead to difficulties as regards the form of the spectrum. His assumption that there would be a difference in mass between the neutrinos taking part in "normal" and anomalous processes has been disproved by the work of Hughes and Egger described in A1.212.

the β^+ -process (8), treated as an allowed transition, to investigate the rate of energy production in a white dwarf star as a result of this reaction. All published papers on this subject conclude that it is impossible on this basis to reconcile the observed low luminosities and relatively large masses of these stars. In other words, the calculated rate of the reaction (8) would seem to be much too fast. This conclusion would suggest that the process (8) behaves in the same anomalous way as the transitions (7), which would be equally hard to understand, since it is obvious that we have here a transition of the type $^1S \rightarrow ^3S$. However, a renewed examination of the astronomical evidence has recently led Dr. SCHATZMAN to reverse the conclusion just mentioned. When due account is taken of the fact that the hydrogen in a white dwarf is localized in a thin superficial layer, there appears to be no difficulty in accounting for the rate of energy production by the reaction (8) considered as an allowed process.

APPENDIX II

ELECTROMAGNETIC PROPERTIES OF NUCLEI

A2.1. Multipole moments of a system of charged particles

A2.11. System in slowly varying external field. Let us consider a system of charged particles under the action of an external electromagnetic field. The distribution of electricity of the system is characterized by a current density $\vec{I}$ and a charge density ϱ , the external field by a vector potential $\vec{A}$ and a scalar potential B , from which the field strengths $\vec{E}$, $\vec{H}$ are derived in the usual way. The interaction energy, insofar as higher powers of the potentials may be neglected, has the form

$$\mathcal{V}_{\text{el.m.}} = - \int \vec{A} \cdot \vec{I} dv + \int B \varrho dv, \quad (1)$$

where we must take for $\vec{I}$ and ϱ their expressions in absence of any electromagnetic field. More particularly, we shall consider a *slowly varying* field, i.e. assume that the relative variations of the field components over distances of the order of the dimensions of the system and times of the order of the proper periods connected with the system are small compared with unity. In such a case, it is well-known that expression (1) for the interaction energy can be put into a more convenient form, involving the values of the electromagnetic field and its derivatives at some arbitrarily chosen point O and the successive *multipole moments* of the system with respect to this point. The properties of the system to any desired approximation are then easily derived from the expressions for such multipole moments.

We shall here carry out the transformation of the operator (1) up to the second order of approximation, on the further assumption that the velocities of the particles are small compared with the velocity of light, i.e. that the dimensions of the system are small compared with the proper wave-lengths (MØLLER and ROSENFELD [43]). This approximation involves the few first multipole moments, defined by the following formulae, in which the vector $\vec{x}$ is taken from the point O as origin:

$$\begin{aligned} \text{the total charge} & e \equiv \int \varrho dv \\ \text{the electric dipole moment} & \vec{P} \equiv \int \varrho \vec{x} dv \\ \text{the magnetic dipole moment} & \vec{M} \equiv \frac{1}{2} \int \vec{x} \wedge \vec{I} dv \\ \text{the electric quadrupole moment}^* & Q_{ik} \equiv \frac{1}{2} \int \varrho (x_i x_k - \frac{1}{3} \delta_{ik} r^2) dv. \end{aligned} \quad (2)$$

* In the expression for Q_{ik} , the x_i ($i = 1, 2, 3$) denote the Cartesian components of the vector $\vec{x}$.

We start from the expansions

$$\begin{aligned} B &= B_0 + \vec{x} \text{grad}_0 B_0 + \frac{1}{2} \vec{x}(\vec{x} \text{grad}_0) \text{grad}_0 B_0 + R \\ \vec{A} &= \vec{A}_0 + (\vec{x} \text{grad}_0) \vec{A}_0 + \vec{R}', \end{aligned} \quad (3)$$

the residuals $R, \vec{R}'$ containing higher derivatives of the potentials with respect to the point O . Remembering further that

$$\int \vec{I} dv = \dot{\vec{P}} \quad (4)$$

and putting for a moment

$$Q'_{ik} = \frac{1}{2} \int \rho x_i x_k dv,$$

we therefore get *

$$\begin{aligned} \int B \rho dv &= e B_0 + \dot{\vec{P}} \text{grad}_0 B_0 + (Q' \text{grad}_0) \text{grad}_0 B_0 + R'' \\ \int \vec{A} \vec{I} dv &= \dot{\vec{P}} \vec{A}_0 + \int \vec{I} (\vec{x} \text{grad}_0) \vec{A}_0 dv + R''', \end{aligned} \quad (5)$$

the residuals being of higher order of approximation than the second. Now,

$$\begin{aligned} \vec{I}(\vec{x} \text{grad}_0) \vec{A}_0 - \vec{x}(\vec{I} \text{grad}_0) \vec{A}_0 &= (\vec{x} \wedge \vec{I}) \vec{H}_0 \\ \vec{I}(\vec{x} \text{grad}_0) \vec{A}_0 + \vec{x}(\vec{I} \text{grad}_0) \vec{A}_0 &= \vec{I} \text{grad} [\vec{x}(\vec{x} \text{grad}_0) \vec{A}_0]; \end{aligned}$$

whence, on account of $\text{div } \vec{I} + \dot{\vec{Q}} = 0$,

$$\int \vec{I}(\vec{x} \text{grad}_0) \vec{A}_0 dv = \vec{M} \vec{H}_0 + (\dot{\vec{Q}}' \text{grad}_0) \vec{A}_0,$$

and

$$\begin{aligned} -\int \vec{A} \vec{I} dv &= \dot{\vec{P}} \vec{A}_0 - \vec{M} \vec{H}_0 + (Q' \text{grad}_0) \vec{A}_0 \\ &\quad - \frac{d}{dt} [\dot{\vec{P}} \vec{A}_0 + (Q' \text{grad}_0) \vec{A}_0] - R'''. \end{aligned} \quad (5a)$$

The time derivative of an operator, occurring in this formula, can be omitted since it does not give any contribution either to the expectation value of the interaction energy in stationary states or to the matrix-elements corresponding to transitions between states of the system with the same total energy. From (5) and (5a), we thus finally get, apart from higher order residuals,

$$\mathcal{V}_{\text{el.m.}} = e B_0 - \dot{\vec{P}} \vec{E}_0 - \vec{M} \vec{H}_0 - (Q \text{grad}_0) \vec{E}_0 - \dots \quad (6)$$

In this formula, it was possible to replace Q' by the traceless tensor Q , as defined by (2), because, for an external field, $\text{div}_0 \vec{E}_0 = 0$. The various

* If Q is a tensor and $\vec{u}$ a vector, $Q\vec{u}$ represents the vector with components $\sum_k Q_{ik} u_k$.

terms of the expansion (6) have familiar interpretations, which it is superfluous to recall. We shall here especially be concerned with the quantities known in spectroscopy under the names of *magnetic moment* and *electric quadrupole moment*: these are not the vectors $\vec{M}$ or tensors Q themselves, but certain mean values of components of these quantities, which we shall now proceed to define.

A2.12. Magnetic moment in spectroscopic sense. In the first place, we deal, in the theory of hyperfine structure, with the expectation value of the electromagnetic energy in the ground state of the nucleus investigated. As is well-known, only the projection of $\vec{M}$ on the direction of the total angular momentum contributes to the expectation value of this vector in any state in which the total angular momentum is fixed. Calling this angular momentum $\hbar\vec{J}$ and taking the nuclear magneton (1.21-2) $\mu_0 = \hbar e/2M_p$ as unit, we may thus write

$$\text{av} \vec{M} = \frac{\mu}{J} \mu_0 \text{av} \vec{J}, \quad (7)$$

μ being a constant which may depend on the quantum number J of the state considered, but not on the magnetic quantum number m defining the "orientation" of the vector $\vec{J}$. The convention usually adopted is just to call this constant μ the *magnetic moment*: it may be described as the expectation value (in units μ_0) of the projection M_z of $\vec{M}$ on some fixed direction in the substate of maximum magnetic quantum number ($m = J$).

A2.13. Electric quadrupole moment in spectroscopic sense. The definition of the electric quadrupole moment in spectroscopic sense rests on a similar basis (CASIMIR [36], § 4): the expectation value of Q can be split into two factors, one of which is independent of the magnetic quantum number, while the other is the expectation value of the tensor $J_i J_k - \frac{1}{3} \delta_{ik} \vec{J}^2$. The first factor can therefore again be determined by considering one of the components of Q , Q_{33} say, and taking its expectation value in some suitable substate; the definition of the *electric quadrupole moment* conventionally given is the expectation value of $6 \cdot \frac{1}{e} \cdot Q_{33}$ in the substate of maximum magnetic quantum number, i.e.

$$Q = \text{av}_{(m=J)} \frac{1}{e} \int \rho (3z^2 - r^2) dv, \quad (8)$$

the coordinates being referred to the fixed point O as origin. The factor $1/e$ means that the spatial distribution of the particles rather than the electric density is considered, so that Q has the dimension of the square of a length. For the deuteron, in particular, we can say, more simply, that the quadrupole moment is the average value, in the substate with $m = J$, of the operator

$$Q = \frac{1}{4} (3z^2 - r^2), \quad (9)$$

expressed in *relative* coordinates. Since $\text{av}(J_i J_k - \frac{1}{3} \delta_{ik} J^2)$ vanishes identically for $J = 0$ or $\frac{1}{2}$, an electric quadrupole interaction occurs only for nuclei with angular momentum $J \geq 1$.

A2.14. Higher multipole moments. In the application of equation (6) to problems of hyperfine structure, the ratio of successive terms of electric or magnetic origin is of the order of magnitude of the ratio of nuclear to atomic dimensions. This explains why no higher than electric quadrupole moments of nuclei have ever been detected. The possible occurrence of a magnetic octupole moment in the case of Iodine has been discussed, with negative outcome, by CASIMIR and KARREMAN [42].

A2.2. Magnetic moments of nuclei

A2.21. The empirical data. Our knowledge of magnetic moments of nuclei enables us to enunciate some general regularities, which indirectly throw light on the properties of nuclear forces:

(1) *Even nuclei* have no magnetic moment. This is, of course, in agreement with the conclusion (10.24) that the ground states of such nuclei are ^1S configurations.

(2) The magnetic moments of stable *odd nuclei* (given in the accompanying table A2.21-1 according to MILLMAN and KUSCH [41]) exhibit a regular decrease, which, as we shall see, can be brought in relation with an increasing influence of non-central nuclear couplings (15.1).

A2.21-1. Magnetic moments of stable odd nuclei	
^2_1D	0.857
^6_3Li	0.821
$^{10}_5\text{B}$	0.598
$^{14}_7\text{N}$	0.403

(3) From the list A2.21-2 of the magnetic moments of *odd mass nuclei* hitherto measured, two remarkable regularities emerge, the bearing of which on problems of nuclear structure will also presently be discussed:

- when plotted against the total angular momentum (fig. A2.21-1, 2), the magnetic moments fall into two groups, roughly distributed along two separate curves; this division obtains separately for the odd proton and the odd neutron nuclei, the curves being quite different in the two cases;
- if two isotopes differing by two neutrons have the same angular momentum, they mostly have also the same magnetic moment. This last regularity is strikingly exhibited by the figures in the fourth column (of both left and right half) of table A2.21-2, which gives the ratios μ_{A+2}/μ_A of the magnetic moments of the isotopes of mass $A+2$ and A .

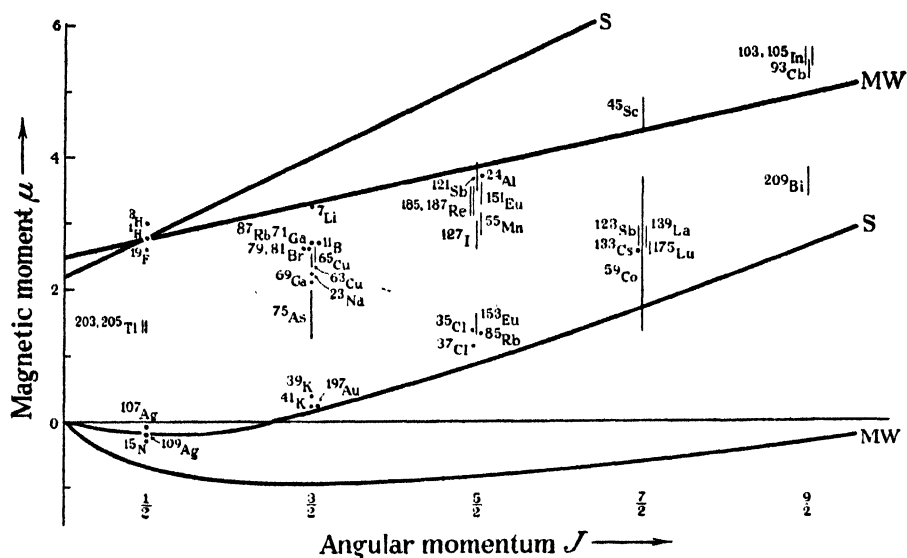


Fig. A2.21-1. Magnetic moments of odd proton nuclei. — Limiting curves (A2.23) according to Margenau and Wigner (MW) and to Schmidt (S).

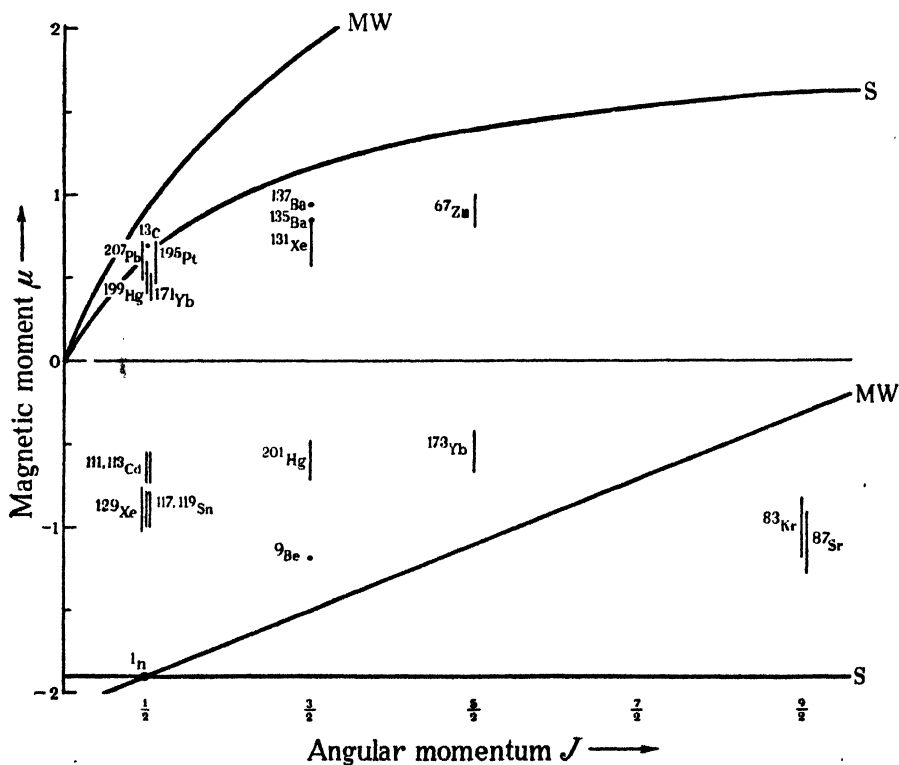


Fig. A2.21-2. Magnetic moments of odd neutron nuclei. — Limiting curves (A2.23) according to Margenau and Wigner (MW) and to Schmidt (S).

A2.21-2. Magnetic moments of odd mass nuclei							
Odd proton nuclei				Odd neutron nuclei			
Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$	Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$
^1_1H	$\frac{1}{2}$	2.7928		n	$\frac{1}{2}$	-1.9125	
^3_1H	$\frac{1}{2}$	2.9791		^3_2He	$\frac{1}{2}$	(-) 2.131	
^7_3Li	$\frac{3}{2}$	3.2532		^9_4Be	$\frac{3}{2}$	-1.176	
$^{11}_5\text{B}$	$\frac{3}{2} ?$	2.686		$^{13}_6\text{C}$	$\frac{1}{2}$	0.701	
$^{15}_7\text{N}$	$\frac{1}{2}$	(-) 0.280					
$^{19}_9\text{F}$	$\frac{1}{2}$	2.625					
$^{23}_{11}\text{Na}$	$\frac{3}{2}$	2.217					
$^{27}_{13}\text{Al}$	$\frac{5}{2}$	3.630					
$^{35}_{17}\text{Cl}$	$\frac{5}{2}$	1.368	0.83				
$^{37}_{17}\text{Cl}$	$\frac{5}{2}$	1.136					
$^{39}_{19}\text{K}$	$\frac{3}{2}$	0.391	0.56				
$^{41}_{19}\text{K}$	$\frac{3}{2}$	0.215					
$^{45}_{21}\text{Sc}$	$\frac{7}{2}$	4.4...4.8					
$^{55}_{25}\text{Mn}$	$\frac{5}{2}$	3.0					
$^{59}_{27}\text{Co}$	$\frac{7}{2}$	2...3					
$^{63}_{29}\text{Cu}$	$\frac{3}{2}$	2.2265	1.07				
$^{65}_{29}\text{Cu}$	$\frac{3}{2}$	2.3847					
$^{69}_{31}\text{Ga}$	$\frac{3}{2}$	1.994	1.27	$^{67}_{30}\text{Zn}$	$\frac{5}{2}$	0.9	
$^{71}_{31}\text{Ga}$	$\frac{3}{2}$	2.540					
$^{75}_{33}\text{As}$	$\frac{3}{2}$	1.6					
$^{79}_{35}\text{Br}$	$\frac{3}{2}$	2.1066	1.08				
$^{81}_{35}\text{Br}$	$\frac{3}{2}$	2.2706					
$^{85}_{37}\text{Rb}$	$\frac{5}{2}$	1.340		$^{83}_{36}\text{Kr}$	$\frac{3}{2} ?$	-1.0	
$^{87}_{37}\text{Rb}$	$\frac{3}{2}$	2.733		$^{87}_{38}\text{Sr}$	$\frac{5}{2}$	≈ -1.1	
$^{93}_{41}\text{Nb}$	$\frac{5}{2}$	5.3					
$^{107}_{47}\text{Ag}$	$\frac{1}{2}$	-0.10	1.9				
$^{109}_{47}\text{Ag}$	$\frac{1}{2}$	-0.19					

A2.21-2. Magnetic moments of odd mass nuclei (Continued)

Odd proton nuclei				Odd neutron nuclei			
Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$	Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$
¹¹³ ₄₉ In	$\frac{9}{2}$	5.49	1.002	¹¹¹ ₄₈ Cd	$\frac{1}{2}$	-0.65	1.0
¹¹⁵ ₄₉ In	$\frac{9}{2}$	5.49		¹¹³ ₄₈ Cd	$\frac{1}{2}$	-0.65	
¹²¹ ₅₁ Sb	$\frac{5}{2}$	3.7	0.441	¹¹⁷ ₅₀ Sn	$\frac{1}{2}$	-0.89	1.0
¹²³ ₅₁ Sb	$\frac{7}{2}$	2.8		¹¹⁹ ₅₀ Sn	$\frac{1}{2}$	-0.89	
¹²⁷ ₅₃ I	$\frac{5}{2}$	2.8					
¹³³ ₅₅ Cs	$\frac{7}{2}$	2.558	1.0108	¹²⁹ ₅₄ Xe	$\frac{1}{2}$	-0.9	1.117
				¹³¹ ₅₄ Xe	$\frac{3}{2}$	0.7	
¹³⁹ ₅₇ La	$\frac{7}{2}$	2.8		¹³⁵ ₅₆ Ba	$\frac{3}{2}$	0.837	
¹⁵¹ ₆₃ Eu	$\frac{5}{2}$	3.4		¹³⁷ ₅₆ Ba	$\frac{3}{2}$	0.936	
¹⁵³ ₆₃ Eu	$\frac{5}{2}$	1.5	1.0097				
				¹⁷¹ ₇₀ Yb	$\frac{1}{2}$	0.45	-0.65
¹⁷⁵ ₇₁ Lu	$\frac{7}{2}$	2.6		¹⁷³ ₇₀ Yb	$\frac{5}{2}$	-0.65	
¹⁸⁵ ₇₅ Re	$\frac{5}{2}$	3.3					
¹⁸⁷ ₇₅ Re	$\frac{5}{2}$	3.3	0.6	¹⁹⁵ ₇₈ Pt	$\frac{1}{2}$	0.6	0.5
¹⁹⁷ ₇₉ Au	$\frac{3}{2}$?	0.2		¹⁹⁹ ₈₀ Hg	$\frac{1}{2}$	0.5	
				²⁰¹ ₈₀ Hg	$\frac{3}{2}$	-0.6	
²⁰³ ₈₁ Tl	$\frac{1}{2}$	(1.612)	1.0097				
²⁰⁵ ₈₁ Tl	$\frac{1}{2}$	1.628		²⁰⁷ ₈₂ Pb	$\frac{1}{2}$	0.6	
²⁰⁹ ₈₃ Bi	$\frac{9}{2}$	3.6					

This table has been compiled from partial lists given by MARGENAU and WIGNER [40b], MILLMAN and KUSCH [41], and INGLIS [41]; full references to the sources will be found in these papers. Additional references: ³H H. ANDERSON and A. NOVICK, *P.R.* **71**, 372. 1947; F. BLOCH, A. GRAVES, M. PACKARD and R. SPENCE, *P.R.* **71**, 373, 551. 1947; ³He H. ANDERSON and A. NOVICK, *P.R.* **73**, 919. 1948; ¹³C (spin) F. JENKINS, *P.R.* **72**, 169. 1947; ³⁷Cl (spin) E. SCHRADER, *P.R.* **64**, 57. 1943; ⁶³, ⁶⁵Cu R. POUND, *P.R.* **73**, 523. 1948; ⁶⁹, ⁷¹Ga BECKER and KUSCH [48]; ⁷⁹, ⁸¹Br S. BRODY, W. NIERENBERG and N. RAMSEY, *P.R.* **72**, 258. 1947; R. POUND, *P.R.* **72**, 1273. 1947; ⁹³Cb W. MEEKES and R. FISHER, *P.R.* **72**, 451. 1947; ¹¹³, ¹¹⁵In T. HARDY and S. MILLMAN, *P.R.* **61**, 459. 1942; ¹²⁷I (spin) W. GORDY *et al.* [47a, b]; ²⁰⁵Tl H. POSS, *P.R.* **72**, 637. 1947.

Finally, we note that the spins and magnetic moments of two heavy odd nuclei of long life-time have been determined:

A2.21-3. Magnetic moments of long-lived odd nuclei			
Nucleus	J	μ	Reference
$^{43}_{19}\text{K}$	4	-1,290	J. ZACHARIAS, <i>P.R.</i> 61, 270. 1942 (see INGLIS [41])
$^{176}_{71}\text{Lu}$	≥ 7	3,8	

A2.22. General expression for magnetic moment. According to the terminology of 4.42, the expectation value of the magnetic moment $\vec{M}$ consists of a direct contribution $\vec{M}_{\text{nucl}}$ of the constituent nucleons and an exchange moment $\vec{M}_{\text{exch}}$; the latter, however, may be expected to be small and is generally neglected*. The quantity $\vec{M}_{\text{nucl}}$ can further be analyzed into a term $\vec{M}_{\text{spin}}$ due to the spins of the nucleons and another $\vec{M}_{\text{orb}}$ due to the "orbital" motion of the protons:

$$\vec{M}_{\text{nucl}} = \vec{M}_{\text{spin}} + \vec{M}_{\text{orb}}. \quad (1)$$

with (4.42-9)

$$\vec{M}_{\text{spin}} = \sum_i \vec{\mu}^{(i)} = \mu_0 \sum_i (\tau_+^{(i)} \mu_n + \tau_-^{(i)} \mu_p) \vec{\sigma}^{(i)} \quad (2)$$

and

$$\vec{M}_{\text{orb}} = \mu_0 \sum_i \tau_-^{(i)} \vec{l}^{(i)}, \quad \vec{l}^{(i)} = \frac{1}{\hbar} \vec{x}^{(i)} \wedge \vec{p}^{(i)}. \quad (3)$$

The most comprehensive discussion of the expectation values of expression (1) is that given by MARGENAU and WIGNER [40b]. Before we proceed with this discussion, it will be good to recall a well-known** theorem of quantum mechanics concerning average values of angular momenta, which is of constant application in calculations of magnetic moments. Consider two commuting vectors $\vec{A}$, $\vec{B}$, whose components satisfy the characteristic commutation relations of angular momenta, and let $\vec{C} = \vec{A} + \vec{B}$. Consider further a stationary state characterized by definite values of the lengths of the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, i.e. by quantum numbers A , B , C , such that $\text{av } \vec{A}^2 = A(A+1)$ etc. Then, the expectation value of the projection of $\vec{A}$ on some fixed direction Oz is given by

$$\text{av } A_z = \text{av } \frac{(\vec{A}\vec{C})}{C^2} C_z.$$

* For the exchange contribution to the magnetic moments of ^3H and ^3He , see A2.251.

** See, e.g., E. CONDON and G. SHORTLEY, *The Theory of atomic Spectra* (1935), Chapt. III, 10 (p. 64).

i.e., owing to the relation

$$2\vec{A}\vec{C} = \vec{C}^2 + \vec{A}^2 - \vec{B}^2, \quad (4)$$

by

$$\text{av } A_z = \frac{C(C+1) + A(A+1) - B(B+1)}{2C(C+1)} \text{av } C_z. \quad (5)$$

The most important contribution to the magnetic moment comes from the spins of the nucleons. With the notation (10.14-6, 7), the z -component of $\vec{M}_{\text{spin}}$ may be written

$$(M_z)_{\text{spin}} = \mu_0 [(\mu_n + \mu_p)S_z - (\mu_p - \mu_n)Y_z]. \quad (6)$$

The most convenient representation of the eigenfunction of the stationary state considered is therefore that based on the supermultiplets (P, P', P''), or more briefly P , of Wigner (10.14); each supermultiplet can further be analyzed into spin multiplets (10.13) with quantum numbers L, S , so that we have

$$\Psi = \sum c_{PLS} \Psi_{PLS}, \quad (7)$$

the sum involving all combinations compatible with the given value of the total angular momentum J ; further, the eigenfunction Ψ refers to a definite magnetic quantum number m , which will ultimately be given the maximum value $m = J$ (A2.12). In the expectation value of the operator (6) in the state (7), non-diagonal elements of the type $\int \Psi_{PLS}^* Y_z \Psi_{PLS'}$ will generally occur, but they may be neglected in comparison with the diagonal elements; if we denote the latter by the symbol av_{PLS} , we thus get

$$\text{av } (M_z)_{\text{spin}} = \sum |c_{PLS}|^2 \text{av}_{PLS} (M_z)_{\text{spin}}; \quad (8)$$

but since $\text{av}_{PLS} Y_z$ can be expressed, by application of formula (5), in terms of $\text{av}_{PLS} S_z$, we may define by

$$\text{av}_{PLS} (M_z)_{\text{spin}} = \mu_0 g_s(PS) \text{av}_{PLS} S_z \quad (9)$$

a "Landé factor" or "gyromagnetic ratio" $g_s(PS)$ of the spin magnetic moment, which depends on the quantum numbers P and S , but not on L . Transforming the factor $\text{av}_{PLS} S_z$ in (9) by another application of (5), and inserting (9) in (8), we obtain finally for the spin contribution μ_s to the magnetic moment μ , defined in A2.12,

$$\mu_s = \sum |c_{PLS}|^2 g_s(PS) \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} J. \quad (10)$$

The Landé factor g_s takes on a very simple form for the lowest supermultiplet ($PP'P''$) of the nucleus under consideration. For odd mass nuclei, this supermultiplet, according to 10.311 is $(\frac{1}{2}n, \frac{1}{2}, \pm\frac{1}{2})$, the value $Y_z = +\frac{1}{2}$ corresponding to an odd neutron, $Y_z = -\frac{1}{2}$ to an odd proton; therefore,

$$\begin{aligned} g_s &= 2\mu_n & (\text{odd neutron nuclei}) \\ g_s &= 2\mu_p & (\text{odd proton nuclei}). \end{aligned} \quad (11)$$

The lowest supermultiplet of an odd nucleus corresponds in any case to a maximum spin quantum number $S = 1$, together with $Y_z = 0$; this gives

$$g_s = \mu_n + \mu_p = \mu_d^0 \quad (\text{odd nuclei}). \quad (12)$$

These results are, of course, obvious consequences of the fact that the lowest configurations of the nuclei in question are spin-saturated, while the odd proton and the odd neutron in an odd nucleus have parallel spins. In many calculations of magnetic moments it may be assumed that only the lowest supermultiplet has to be taken into account; this simplifies formula (10):

$$\mu_s = J g_s \left[\frac{1}{2} + \sum |c_{LS}|^2 \frac{S(S+1) - L(L+1)}{2J(J+1)} \right], \quad (13)$$

and it may further be noted that in this configuration no cross-terms such as have been neglected in formula (10) occur. The expression (13) is independent of any model of nuclear structure: if the quasi-atomic model is adopted, S and L have definite values, so that there is only one term in the sum. Odd mass nuclei with neutron excess ± 1 have also been treated on the α -particle model: the role of L is then played by the quantum number K (13.22), which may have the values $J \pm \frac{1}{2}$.

The calculation of the orbital magnetic moment does not lead to such results of fairly general validity: the use of different nuclear models here gives rise to rather divergent estimates. The eigenfunctions Ψ_{PLS} , or, if we drop for a moment the index of the supermultiplet, Ψ_{LS} , can be decomposed according to the eigenvalues of the total orbital momenta $\vec{L}_p$, $\vec{L}_n$ of the protons and the neutrons:

$$\Psi_{LS} = \sum_{L_p L_n} b_{LS L_p L_n} \Psi_{LS L_p L_n}. \quad (14)$$

Again, there occur in the expectation value $\text{av}(M_z)_{\text{orb}}$, besides the diagonal elements $\text{av}_{LS}(M_z)_{\text{orb}}$, non-diagonal terms

$$\int \Psi_{LS}^*(M_z)_{\text{orb}} \Psi_{L\pm 1, S}$$

which can be shown to be negligible. We thus have

$$\text{av}(M_z)_{\text{orb}} = \sum |c_{LS}|^2 \text{av}_{LS}(M_z)_{\text{orb}}, \quad (15)$$

and, using (5),

$$\text{av}_{LS}(M_z)_{\text{orb}} = \mu_0 g_l(LS) \text{av}_{LS} L_z \quad (16)$$

with

$$g_l(LS) = \sum |b_{LS L_p L_n}|^2 \frac{L(L+1) + L_p(L_p+1) - L_n(L_n+1)}{2L(L+1)}. \quad (17)$$

Inserting (16) in (15) after a repeated application of (5), we get the orbital contribution μ_l to the magnetic moment in the form

$$\mu_l = \sum |c_{LS}|^2 g_l(LS) \frac{J(J+1) + L(L+1) - S(S+1)}{2J(J+1)} J. \quad (18)$$

Combining this with formula (10), we obtain for the magnetic moment μ

$$\mu = \mu_s + \mu_l = J \sum |c_{PLS}|^2 \left[\frac{1}{2} [g_s(PS) + g_l(PLS)] + \frac{1}{2} [g_s(PS) - g_l(PLS)] \frac{S(S+1) - L(L+1)}{J(J+1)} \right]. \quad (19)$$

The limitations of this general expression consist in the neglect of non-diagonal elements of the operators $\mathbf{Y}_z$ and $(M_z)_{\text{orb}}$.

On the *quasi-atomic model*, the ground state has definite quantum numbers L and S , and the sum on the right-hand side of (19) reduces to a single term. Moreover, if we assume that the multiplet to which the ground state belongs is either normal or inverted, this term takes a very simple form. In the case of an *inverted multiplet*, $J = L + S$ and

$$\mu = g_s S + g_l L; \quad (20)$$

for a *normal multiplet*, $J = |L - S|$ and

$$\begin{aligned} \mu &= \frac{J}{J+1} [(L+1)g_l - Sg_s] & (L > S) \\ \mu &= \frac{J}{J+1} [Lg_l - (S+1)g_s] & (L < S). \end{aligned} \quad (21)$$

On this model, the different configurations $\psi_{LSL_p L_n}$ entering into the expansion (14) can, for not too heavy nuclei, easily be determined and the expansion coefficients b explicitly calculated in terms of the assumed central interaction potential. The Landé factor $g_l(LS)$ is then given by (17). Such calculations will be discussed in the sequel (A2.252).

For heavier nuclei, the best approach (MARGENAU and WIGNER [40b]) is provided by the *liquid droplet model*, according to which all constituent nucleons contribute equally to the expectation value of the orbital momentum*. This would mean

$$g_l(LS) \approx Z/A; \quad (22)$$

but formula (18) with this value of $g_l(LS)$ can give no more than a rough estimate of the orbital contribution μ_p , from which individual nuclei will more or less deviate. The treatment of light odd mass nuclei with neutron excess ± 1 on the *α -particle model* illustrates the kind of deviations that can be expected (SACHS [39]). In this case, the orbital momentum around the figure axis can be ascribed partly to the rotation of the nucleus as a whole, partly to an orbital motion around this axis of the extra particle or hole. To the first type of rotation a gyromagnetic ratio of the form (22) may be attributed, whereas the gyromagnetic ratio g_p of orbital motion is 1 if the additional or lacking particle is a proton and 0 if it is a neutron.

* Miss WAY [39a], in her treatment of the droplet model, makes the same assumption concerning the spin momentum. This, however, as emphasized by MARGENAU and WIGNER [40b], is hardly acceptable.

Consequently, the gyromagnetic ratio g_z pertaining to the component of orbital momentum around the figure axis will be of the form

$$g_z = \theta g_p + (1-\theta) \frac{Z}{A} \quad (0 \leq \theta \leq 1), \quad (23)$$

the parameter θ denoting the ratio of the contributions from orbital motion and bulk rotation to the total orbital momentum around the figure axis. The gyromagnetic ratio g_z corresponding to any component of orbital moment perpendicular to the figure axis will be just $\approx Z/A$, except when the configuration of the next α -nucleus is spherically symmetrical (e.g. ^{15}N , ^{15}O , see 13.21), in which case it will be equal to g_z . The orbital magnetic moment arising from the bulk rotation is of the form (18), with $S = \frac{1}{2}$ and L (or rather K) $= J \pm \frac{1}{2}$, and the gyromagnetic ratio g_z . If g_z is different from g_z , it must be supplemented by a contribution μ_A from the orbital motion of angular momentum A (13.21, 13.22) around the figure axis, with gyromagnetic ratio $g_z - g_z$. In order to calculate this contribution, we expand the eigenfunction of the ground state in terms of the eigenfunctions $\Psi_{K m_k m_s}$ corresponding to a definite orbital momentum K and projection m_k of this orbital momentum on a fixed direction of space:

$$\Psi = \sum c_{K m_k m_s} \Psi_{K m_k m_s}; \quad (24)$$

since the projection m_s of the spin on the same direction is $\pm \frac{1}{2}$, while that of the total angular momentum, m , is specified (and taken $= J$), we have to take the summation over $m_k = m - m_s$ for $m_s = \pm \frac{1}{2}$, and over $K = J \pm \frac{1}{2}$. Calling Θ the angle between the fixed direction and the figure axis, we have

$$\mu_A = (g_z - g_z) A \sum_{K, K'} \sum_{m_k, m_s} c_{K m_k m_s}^* c_{K' m_k m_s} (K | \cos \Theta | K')_{m_k A}; \quad (25)$$

in this formula, $(K | \cos \Theta | K')_{m_k A}$ represents the matrix-element of $\cos \Theta$ for specified values of the quantum numbers m_k, A , with respect to which it is diagonal*. The expansion coefficients c_{KS} or $c_{K m_k m_s}$, indicating the proportions in which the two K states are mixed in the ground state, depend on the spin-orbit couplings (A2.252).

A2.221. Conjugate and self-conjugate nuclei. Owing to the symmetry of the nuclear interaction with respect to protons and neutrons (3.3), *light isobaric nuclei* (for which the asymmetry due to the Coulomb force can in first approximation be neglected) may be grouped in pairs of *conjugate nuclei*, with equal and opposite neutron excesses, having the same stationary states in that approximation: in two such conjugate nuclei, the respective

* These matrix-elements have been derived by H. CASIMIR, *Rotation of a rigid body in quantum mechanics* (thesis Leiden 1931), p. 50. In our notation,

$$(K | \cos \Theta | K) = m_k A / K(K+1)$$

$$(K | \cos \Theta | K-1) = (K-1 | \cos \Theta | K) = \sqrt{K^2 - m_k^2} \sqrt{K^2 - A^2} / K \sqrt{4K^2 - 1}.$$

roles of protons and neutrons are simply interchanged. This implies a very simple and general relation, pointed out by SACHS [46a], between the magnetic moments μ^+ and μ^- of two conjugate nuclei. In fact, we get from (2) and (3)

$$\begin{aligned}\vec{M}_{\text{spin}}^+ + \vec{M}_{\text{spin}}^- &= 2\mu_0(\mu_n + \mu_p)\vec{S} \\ \vec{M}_{\text{orb}}^+ + \vec{M}_{\text{orb}}^- &= \vec{L},\end{aligned}\quad (26)$$

whence, quite rigorously,

$$\begin{aligned}\mu^+ + \mu^- &= \sum |c_{PLS}|^2 \left[\mu_d^0 \frac{J(J+1) + S(S+1) - L(L+1)}{J(J+1)} \right. \\ &\quad \left. + \frac{J(J+1) + L(L+1) - S(S+1)}{2J(J+1)} \right] J \\ &= (\mu_d^0 + \tfrac{1}{2})J + (\mu_d^0 - \tfrac{1}{2}) \sum |c_{PLS}|^2 \frac{S(S+1) - L(L+1)}{J(J+1)} J.\end{aligned}\quad (27)$$

This formula cannot, however, be checked by experiment for the time being, since at least one of the partners in any pair of conjugate nuclei is unstable and the technique of measuring magnetic moments of unstable nuclei has not yet been developed. The only exception concerns the pair of conjugate nuclei ${}^3\text{H}$ - ${}^3\text{He}$ (A2.251).

An interesting case is that of the nuclei of zero neutron excess, which are *self-conjugate*: for those nuclei, the right-hand side of (27) represents twice the magnetic moment. In particular, the *stable odd nuclei*, for which (11.41) $J = 1$, have a magnetic moment of the form

$$\mu = \tfrac{1}{2}(\mu_d^0 + \tfrac{1}{2}) + \tfrac{1}{2}(\mu_d^0 - \tfrac{1}{2}) \sum |c_{PLS}|^2 \frac{S(S+1) - L(L+1)}{2}; \quad (28)$$

this is the basic formula for the discussion of the empirical results (A2.26).

It may be observed that the orbital contribution to (27) and (28) can be deduced directly from (17), (18) by noticing that on account of the charge symmetry of nuclear interactions we have (provided the Coulomb force be neglected)

$$|b_{LSL_p L_n}|^2 = |b_{LSL_n L_p}|^2, \quad (29)$$

whence

$$g_l^+ + g_l^- = 1 \quad (30)$$

and for zero neutron excess

$$g_l = \tfrac{1}{2}. \quad (31)$$

A2.23. The distribution of the magnetic moments of odd mass nuclei. We are now in a position to discuss the regularities enumerated in A2.21 from the theoretical point of view. We shall begin with the first collective property (3a) of odd mass nuclei, viz. the distribution of their magnetic moments along two curves, when plotted against the total angular momentum. If we assume that the ground states of the nuclei in question

belong to the lowest supermultiplet, their spin quantum number will be $S = \frac{1}{2}$ and they will in general consist of mixtures of two states with $L = J \pm \frac{1}{2}$. From this point of view, developed by MARGENAU and WIGNER [40b], we may draw two *limiting* curves, corresponding to the *pure* states with $L = +\frac{1}{2}$ and $L = J - \frac{1}{2}$: so far as our assumption as to the unperturbed lowest supermultiplet is valid, the magnetic moments should fall between these limiting curves. According to (20), (21), and (11), the limiting values of the magnetic moments are

$$\begin{aligned} L=J-\frac{1}{2}: \quad \mu &= \mu_{pn} + (J-\frac{1}{2}) g_l \\ L=J+\frac{1}{2}: \quad \mu &= -\frac{J}{J+1} \mu_{pn} + \frac{J}{J+1} (J+\frac{3}{2}) g_l, \end{aligned} \quad (32)$$

the symbol μ_{pn} denoting either μ_p or μ_n according as we are dealing with an odd proton or an odd neutron nucleus. Strictly speaking, these limiting moments are not merely functions of J , since g_l , even with its rough value (22), may be different for two nuclei with the same J ; however, for our purpose of getting a general survey, we may adopt some constant value,

$$g_l \approx 0.4, \quad (32a)$$

say. In this way, we obtain the curves marked *MW* in the fig. A2.21-1, 2.

It must be admitted that the result is not very satisfactory. Not only does the theory fail to account for the separation of the empirical values into two groups within the above limits, but these limits themselves do not follow the general trend of the empirical curves. A much better agreement in this last respect is reached if, following the original suggestion of SCHMIDT [37], one adopts for g_l , instead of the value (32a) pertaining to the droplet model, that resulting from the rather extreme assumption that the orbital momentum is entirely due to the motion of the odd particle. This would mean that in the representation (14) of the eigenfunction, only one term would remain, for which L would be equal to L_p or L_n ; the expression (17) for g_l then reduces to the standard value g_p for orbital motion (1 for a proton, 0 for a neutron):

$$g_l = g_p. \quad (33)$$

This value, inserted in (32), yields the curves marked *S* in fig. A2.21-1, 2. Curves closely representing the empirical separation into two groups would, as noticed by Schmidt himself and, more precisely, by INGLIS [38a], correspond to a sort of compromise between the extremes (32a) and (33), rather near to the latter, viz.

$$\begin{aligned} g_l &= \frac{1}{8} & (\text{odd neutron nuclei}) \\ g_l &= \frac{7}{8} & (\text{odd proton nuclei}). \end{aligned} \quad (34)$$

The outcome of this analysis is therefore the disclosure of a strong influence of the *structure* of the nucleus on its angular momentum properties: it is rather surprising that this influence should go towards

preserving Russell-Saunders coupling and, above all, concentrating almost the whole orbital motion on a single nucleon.

The implications of this situation for the choice of the model of nuclear structure best adapted to the facts have been especially discussed in two papers by INGLIS [38a, 41]. He concludes that in this respect the α -particle model is superior to the quasi-atomic one. In the crudest approximation, both models concur in predicting that all the paired nucleons would tend to form a $1S$ configuration, leaving only the orbital momentum (and spin) of the odd particle. But the question is in how far the perturbations due to the close coupling of the nuclear constituents will modify this simple picture. On the quasi-atomic model, one would expect such perturbations to be very severe, especially for heavier nuclei. On the α -particle model, on the other hand, it would seem that the $1S$ configuration formed by the paired extra neutrons would be well isolated below the excited states, so that the odd proton would presumably be only weakly coupled with these neutrons. As regards the rotation of the framework of α -particles, its coupling with the odd proton would also be small on account of the exclusion of the low rotational levels (13.15). Altogether, *a gyromagnetic ratio of the order (34) would thus seem plausible from the point of view of the α -particle model.*

A2.24. Isotopes with equal magnetic moments. We now turn to the other remarkable feature (3b) of odd mass nuclei mentioned in A2.21, viz. the existence of nuclei of this type, differing by two neutrons and having the same angular momentum and the same, or nearly the same, magnetic moment. From the consideration of this law, INGLIS [41] draws another argument in favour of the α -particle model; we shall summarize his discussion. At first sight, the approximate equality of magnetic moments seems to be simply accounted for by the droplet value $g_I \approx Z/A$ of the gyromagnetic ratio; but we have just seen that this value cannot be relied upon as describing the general behaviour of nuclear systems. We must therefore again look for an explanation in the structural properties of such systems. On the quasi-atomic model, one would expect in most cases the two neutrons to be added to an unfilled shell and to change g_I considerably. On the α -particle model, the degeneracy of the individual extra-neutron states would not be so large: in contrast to the quasi-atomic case, different orientations of the individual orbital momenta $\vec{l}^{(i)}$ relative to the body axes would involve different energies because of the lack of spherical symmetry. The two neutrons would thus often come to occupy an individual level which was previously empty; this would affect g_I only in higher order.

It may of course happen that — whatever the model of nuclear structure adopted — the addition of the two neutrons entails a large perturbation of the stationary state of the first isotope and a corresponding difference between the magnetic moments of the two nuclei. A concrete example illustrating this case is that of the pair of isotopes ^{39}K , ^{41}K , treated by

INGLIS [38a]. It will be convenient to use the description of the states concerned based on the quasi-atomic model. In conformity with the conclusions of 10.32, the ground state of ^{39}K is represented by a configuration in which the $3d$ shell is entirely filled but for one proton: owing to this "hole", the protons form a ^2D configuration, while the neutrons are in a ^1S configuration; the resulting state of the nucleus is a $^2D_{\frac{1}{2}}$. Now, the two additional neutrons will be paired in the lowest level of the $3p$ shell, giving rise to a mixture of two states of the neutron system, viz. ^1S and ^1D . The state of the ^{41}K nucleus is thus still a $^2D_{\frac{1}{2}}$, but the admixture of ^1D neutron state will tend to lower the gyromagnetic ratio g_I , in agreement with experiment.

A2.25. *Magnetic moments of light odd mass nuclei.* The following considerations will now be concerned with more individual features, chiefly pertaining to the lighter nuclei, which are susceptible of detailed study. As regards odd mass nuclei, the comparison of the theoretical results with the observed values will throw some more light on the ability of the quasi-atomic and the α -particle model to account for finer features of nuclear magnetic moments.

A2.251. *The conjugate nuclei ^3H and ^3He .* The theoretical treatment of the simplest odd mass nuclei, ^3H and ^3He , is interesting in that it can be carried out completely and affords an example of perturbation of the lowest supermultiplet by higher ones*. The comparison between theory and experiment, on the other hand, has important implications with regard to fundamental aspects of the problem of nuclear forces.

In first approximation, the ground state of the nuclei in question is a $^2S_{\frac{1}{2}}$ one, antisymmetrical in the spins of the like particles (14.111), i.e. involving the spin eigenfunctions $^2\sigma_{\pm\frac{1}{2}}(\bar{12}, 3)$ (14.111–14), in which 1, 2 refer to the like particles; this state, which belongs to the supermultiplet $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, will be more precisely denoted by $^2\tilde{S}_{\frac{1}{2}}$. The magnetic moment is, in this approximation, μ_{pn} . However, the non-central nuclear interactions bring about (17.2) a mixture of this state with others, the effect of which on the magnetic moment we have to investigate. The admixed states consist of a $^2\tilde{P}$ involving the same spin functions and belonging to the same supermultiplet, and of other states belonging to the supermultiplet $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$, viz. a $^2\tilde{S}$ and a $^2\tilde{P}$ involving the symmetric spin functions $^2\sigma_{\pm\frac{1}{2}}(\bar{12}, 3)$ (14.11–4), and two quartet states, ^4P , ^4D , with the spin functions $^4\sigma(\bar{12}, 3)$ (14.11–5), which are symmetrical in all three particles. We have to calculate the magnetic moments of all these states, and further to ascertain what combinations may give rise to non-diagonal terms in the expression for the magnetic moment of the ground state.

The spin gyromagnetic ratio g_s in the states $^2\tilde{S}$, $^2\tilde{P}$ is (as already stated) $2\mu_{pn}$. In the entirely symmetrical quartet states, we have, of course,

* The following treatment is an extension of that given by SACHS and SCHWINGER [46b].

av $\sigma_z^{(l)} = \frac{1}{3}$, and consequently $g_s = \frac{2}{3}(2\mu_n + \mu_p)$ for ${}^3\text{H}$ and $g_s = \frac{2}{3}(2\mu_p + \mu_n)$ for ${}^3\text{He}$. Finally, the spin gyromagnetic ratio in the states ${}^2\bar{S}$, ${}^2\bar{P}$ is found, from the definition (9), by direct use of the explicit expression (14.11-4) for ${}^2\sigma(\bar{1}\bar{2}, 3)$:

$$\begin{aligned} g_s &= \frac{2}{3} [\mu_p + 2(2\mu_n - \mu_p)] = \frac{2}{3}(4\mu_n - \mu_p) \quad \text{for } {}^3\text{H}, \\ \text{and} \quad g_s &= \frac{2}{3} (4\mu_p - \mu_n) \quad \text{for } {}^3\text{He}. \end{aligned}$$

In order to calculate the orbital gyromagnetic ratios, it is necessary to consider the angular dependence of the eigenfunctions. We choose as spatial coordinates the vectors $\vec{x}$ and $\vec{X}$, which represent, respectively, the radius vector joining the two like particles and that joining the third particle to the centre of gravity of the other two:

$$\vec{x} = \vec{x}^{(1)} - \vec{x}^{(2)}, \quad \vec{X} = \vec{x}^{(3)} - \frac{1}{2}(\vec{x}^{(1)} + \vec{x}^{(2)}). \quad (35)$$

If $\vec{p}$, $\vec{P}$ denote the canonically conjugate momenta, it is easily verified that, in the system of reference in which the centre of gravity of the nucleus rests at the origin,

$$\hbar(\vec{l}^{(1)} + \vec{l}^{(2)}) = \frac{1}{3} \vec{X} \wedge \vec{P} + \vec{x} \wedge \vec{p}, \quad \hbar \vec{l}^{(3)} = \frac{2}{3} \vec{X} \wedge \vec{P}. \quad (36)$$

In a state whose eigenfunction is either symmetrical or antisymmetrical with respect to $\vec{x}$ and $\vec{X}$, we have

$$\text{av } (\vec{X} \wedge \vec{P}) = \text{av } (\vec{x} \wedge \vec{p});$$

combining this with the relations (36), we readily find

$$\begin{aligned} \text{av } (l_z^{(1)} + l_z^{(2)}) &= \frac{2}{3} \text{av } L_z \\ \text{av } l_z^{(3)} &= \frac{1}{3} \text{av } L_z. \end{aligned} \quad (37)$$

Now, the eigenfunctions of the P and D states under consideration have the symmetry property just mentioned: for (GERJUOY and SCHWINGER [42]) the angular dependence of the P wave-functions can be represented by a linear combination of the components of the only vector of even parity built up from $\vec{x}$ and $\vec{X}$, viz. $\vec{x} \wedge \vec{X}$, so that these functions are antisymmetric in $\vec{x}$ and $\vec{X}$; likewise, for the ${}^4D_1$ state, we may distinguish four distinct types of angular dependence, characterized by the tensors

$$T_{ik}^{(xX)} = \frac{1}{2}(x_i X_k + x_k X_i) - \frac{1}{3} \delta_{ik} |\vec{x}| |\vec{X}|$$

and

$$T_{ik}^{(a)} = a_i a_k - \frac{1}{3} \delta_{ik} |\vec{a}|^2, \quad \text{with } \vec{a} = \vec{x}, \vec{X}, \vec{x} \wedge \vec{X},$$

from which one can form 3 symmetrical combinations $T^{(xX)}$, $T^{(\vec{x} \wedge \vec{X})}$, $T^{(x)} + T^{(X)}$, and one antisymmetrical combination $T^{(x)} - T^{(X)}$. If we accordingly introduce separately a symmetrical ${}^4D_s$ eigenfunction, comprising the three symmetrical tensors, and an antisymmetrical one, ${}^4D_a$ (so that the ${}^4D$ state

is actually described by a linear combination of ${}^4D_s$ and ${}^4D_a$), we may thus say that the orbital Landé factors in all P and D states are $g_l = \frac{1}{2}$ for ${}^3\text{H}$ and $g_l = \frac{2}{3}$ for ${}^3\text{He}$.

We can now write down the magnetic moments in the various component states, which enter additively into the general expression (19) for the magnetic moment, weighted by the absolute squares $|c_{PLS}|^2$ of the expansion coefficients of the ground state wave-function:

Component state	$\mu({}^3\text{H})$	$\mu({}^3\text{He})$
${}^2\tilde{S}$	μ_p	μ_n
${}^2\tilde{P}$	$-\frac{1}{3}\mu_p + \frac{2}{3}$	$-\frac{1}{3}\mu_n + \frac{2}{3}$
${}^2\bar{S}$	$\frac{1}{3}(4\mu_n - \mu_p)$	$\frac{1}{3}(4\mu_p - \mu_n)$
${}^2\bar{P}$	$-\frac{1}{3}(4\mu_n - \mu_p) + \frac{2}{3}$	$-\frac{1}{3}(4\mu_p - \mu_n) + \frac{2}{3}$
${}^4P$	$\frac{5}{9}(2\mu_n + \mu_p) - \frac{1}{9}$	$\frac{5}{9}(2\mu_p + \mu_n) - \frac{1}{9}$
${}^4D_{s,a}$	$-\frac{1}{3}(2\mu_n + \mu_p) + \frac{1}{3}$	$-\frac{1}{3}(2\mu_p + \mu_n) + \frac{1}{3}$

(38)

It is essential, however, also to take account of the cross-terms which have been neglected in formula (19). These cross-terms are of two types, pertaining either to the spin or to the orbital moment, and both are represented in our case: there is clearly a non-vanishing matrix-element $({}^2\tilde{P} | (M_z)_{\text{spin}} | {}^4P)$, and matrix-elements $(A | (M_z)_{\text{orb}} | B)$ for all pairs of states with the same spin multiplicity, but different symmetry with respect to $\vec{x}$ and $\vec{X}$: for it follows from $(A | l_z^{(1)} + l_z^{(2)} + l_z^{(3)} | B) = 0$ and from (36) that, if A and B have the same symmetry,

$$(A | l_z^{(1)} + l_z^{(2)} | B) = 2(A | l_z^{(3)} | B) = 0;$$

on the other hand, if A and B have different symmetry properties, one gets

$$(A | l_z^{(1)} + l_z^{(2)} | B) = -(A | l_z^{(3)} | B),$$

which means that the orbital cross-terms are equal and opposite for ${}^3\text{H}$ and ${}^3\text{He}$, so that they disappear from the sum of the magnetic moments of the two conjugate nuclei, in conformity with the general theory of A2.221. Likewise,

$$({}^2\bar{P} | \sigma_z^{(1)} + \sigma_z^{(2)} | {}^4P) = -({}^2\bar{P} | \sigma_z^{(3)} | {}^4P),$$

whence equal and opposite values, proportional to $\pm(\mu_p - \mu_n)$, follow for the matrix-element $({}^2\bar{P} | (M_z)_{\text{spin}} | {}^4P)$ for ${}^3\text{H}$ and ${}^3\text{He}$.

Using the relation $\sum |c_{PLS}|^2 = 1$ in order to eliminate the coefficient $c({}^2\tilde{S})$, presumed to be the most important one, we may finally write the magnetic moment of ${}^3\text{H}$ in the form

$$\begin{aligned} \mu = & \mu_p - \frac{2}{3}(2\mu_p - \frac{1}{3}) |c({}^2\tilde{P})|^2 - \frac{4}{3}(\mu_p - \mu_n) |c({}^2\bar{S})|^2 - \frac{4}{3}(2\mu_p + \mu_n - \frac{1}{3}) |c({}^2\bar{P})|^2 \\ & - \frac{2}{3}(2\mu_p - 5\mu_n + \frac{1}{3}) |c({}^4P)|^2 - \frac{2}{3}(2\mu_p + \mu_n - \frac{1}{3}) [|c({}^4D_s)|^2 + |c({}^4D_a)|^2] \\ & + 2 \Re [c^*({}^2\bar{P}) c({}^4P) (\mu_p - \mu_n) ({}^2\bar{P} | \sigma_z^{(3)} | {}^4P)] + 2 \Re [\sum c^*(A) c(B) (A | l_z^{(3)} | B)]; \end{aligned} \quad (39)$$

the pairs of states yielding orbital cross-terms are $^2\bar{S}^2\bar{P}$, $^4P^4D_s$ and $^4D_a^4D_s$. A similar expression could be written down for the moment of ^3He ; the two moments are connected by the general relation (27), which gives in this case

$$\mu(^3\text{H}) + \mu(^3\text{He}) = \mu_d^0 - \frac{2}{3}(\mu_d^0 - \frac{1}{2}) \{ 3[|c(^4D_s)|^2 + |c(^4D_a)|^2] + 2[|c(^2\bar{P})|^2 + |c(^2\bar{P})|^2] - |c(^4P)|^2 \}. \quad (40)$$

Moreover, as VILLARS [47] pointed out, it is necessary, since both moments have been measured with great precision, to take into account the small additional contributions due to the exchange term in the general expression for the magnetic moment (A2.22). Indeed, if the existence of such terms could be established, it would afford a very strong argument in favour of the field conception of nuclear forces. The order of magnitude of the exchange magnetic moments will depend a great deal, however, on the particular choice of the nuclear field (VILLARS [47]). Their actual calculation has been carried out by THELLUNG and VILLARS [48] on the basis of Møller and Rosenfeld's symmetrical meson theory (16.42), using the wave-function of the ground state of the three-nucleon system obtained by Fröhlich *et al.* (14.22). To this approximation, the absolute value of the effect is found to be $\mu_{\text{exch}} = 0.023$; this quantity must be added to $\mu(^3\text{H})$ and subtracted from $\mu(^3\text{He})$, so that formula (40) remains valid.

Let us now proceed to the discussion of the empirical results. Formula (40) yields information of general validity about the admixtures of P and D states to the ground state. Substituting the data, one finds (ANDERSON [48])

$$3|c(^4D_{s,a})|^2 + 2|c(^2\bar{P})|^2 - |c(^4P)|^2 = 0.13 \pm 0.08.$$

This relation is compatible with Gerjuoy and Schwinger's analysis of the ground state (17.2), according to which the admixtures of P and D_a states would be negligible, while $|c(^4D_s)|^2 \approx 0.04$. However, it would hardly be possible, on such assumptions, to explain the fact that the empirical value of $\mu(^3\text{H})$ is *larger* than the proton moment μ_p by a factor very accurately determined as 1.066636 ± 0.00001 . For the D_s admixture alone would not give rise to any non-diagonal contribution in the expression (39) and would consequently tend to lower the magnetic moment $\mu(^3\text{H})$; to counteract this effect, the exchange contribution would have to be as large as 0.27. In view of the theoretical estimate quoted above, it does not seem very likely that a field theory accounting for the nuclear interactions would yield such a considerable exchange effect.

Whether the experimental results could be accounted for on the assumption of a more complicated mixture of orbital states, giving rise to

* The matrix-element $(^2\bar{S} | l_z^{(3)} | ^2\bar{P})$ vanishes because the $^2\bar{S}$ eigenfunction, symmetrical in the spatial coordinates of the like particles, is an *even* function of the invariant $\vec{x} \cdot \vec{X}$. The space factor of the $^2\bar{S}$ eigenfunction, on the other hand, is an *odd* function of this invariant.

positive cross-terms in (39), cannot be decided without an explicit determination of the eigenfunction of the ground state, much more elaborate than that attempted by Gerjuoy and Schwinger *. But it may immediately be observed that this explanation would call for the existence of spin-orbit interactions of appreciable magnitude besides the axial dipole couplings, since the latter will not to a first approximation couple a ${}^2\tilde{S}$ state to any P state; this conclusion would be in agreement with the tentative interpretation of the fine structure of ${}^5\text{He}$ as a P doublet splitting (17.43).

A2.252. *The "2p shell" nuclei.* The next class of light nuclei to be considered comprises those which, on the simple picture of the quasi-atomic model, correspond to the gradual filling up of the $2p$ shell (10.32). The magnetic moments of the stable members of this class are all known experimentally. Since these nuclei allow of a fairly detailed treatment on the α -particle model as well as on the quasi-atomic one, they afford a very instructive comparison of these two models of nuclear structure.

The *quasi-atomic* picture offers the easier approach. In the crudest approximation, the ground state of all "2p shell" nuclei is of the ${}^2P$ type; the magnetic moment is accordingly given by (20) or (21) with (11), i.e.

$$\begin{aligned} \text{if } J = \frac{3}{2}, \quad \mu &= \mu_{pn} + g_l \\ \text{if } J = \frac{1}{2}, \quad \mu &= -\frac{1}{3}\mu_{pn} + \frac{2}{3}g_l. \end{aligned} \quad (41)$$

The calculation of the Landé factors g_l by means of formula (17) requires knowledge of the expansion coefficients b in (14). On Wigner's approximation (9.23), the latter quantities are independent of the particular way in which the interaction energy depends on the distances between the nucleons; they may thus be calculated for the case of the "long range fiction" (10.31), which reduces the work to the solution of very simple secular problems **. The results, together with the corresponding g_l values, are embodied in the following table:

A2.252-1. Ground states and orbital Landé factors of 2p shell nuclei on Wigner's approximation					
Configu- ration	Ground state	Odd proton nuclei	g_l	Odd neutron nuclei	g_l
$p^1 \quad p^{11}$	$(1S^2P)$	${}^5\text{Li} \quad {}^{15}\text{N}$	1	${}^5\text{He} \quad {}^{15}\text{O}$	0
$p^3 \quad p^9$	$\sqrt{\frac{5}{9}}(1S^2P) + \sqrt{\frac{4}{9}}(1D^2P)$	${}^7\text{Li} \quad {}^{13}\text{N}$	$\frac{1}{3}$	${}^7\text{Be} \quad {}^{13}\text{C}$	$\frac{2}{3}$
$p^5 \quad p^7$	$\sqrt{\frac{5}{9}}(1S^2P) - \sqrt{\frac{1}{9}}(1D^2P) \pm \sqrt{\frac{1}{3}}(1D^2D)$	${}^9\text{B} \quad {}^{11}\text{B}$	$\frac{2}{3}$	${}^9\text{Be} \quad {}^{11}\text{C}$	$\frac{1}{3}$
A symbol like $(1S^2P)$ represents the eigenfunction $\Psi_{LSL_p L_n}$ of a configuration in which the nucleons (protons or neutrons) in even number form by themselves a $1S$ state, while those in odd number form a ${}^2P$ state. — The stable nuclei are in <i>italics</i> .					

* See the more detailed discussion by SACHS [47b]. The later calculations of Mrs. GOEPPERT-MAYER and SACHS [48] ignore the possibility of spin-orbit couplings and use the inadequate form of nuclear potential mentioned in the footnote to 16.20.

** The necessary data for setting up the secular equations for all $2p$ configurations are contained in table IV of FEENBERG and WIGNER's paper [37a].

In constructing this table, account has been taken (as expressed by the first column) of the symmetry between particles and "holes" resulting from the symmetry of the nuclear interaction with respect to the charge of the nucleons; only in the phases of certain component wave-functions can a change occur when passing from the one to the other of equivalent configurations: if the wave-function corresponds to an odd value of $L_p + L_n - L$, it changes sign when going over from particles to "holes". It can further be verified that the inclusion of interactions explicitly depending on the spin and isotopic coordinates gives rise only to small variations of the b 's even within a wide range of variation of the interaction parameters*.

The theoretical determination of the J -values necessitates consideration of the splitting of the ground state as a result of non-central interactions. Assuming only relativistic spin-orbit coupling (ROSE and BETHE [37]), the application of Inglis' rule (15.22) would indicate an angular momentum $J = \frac{3}{2}$ for the nuclei with p^1 , p^3 and p^5 configurations, and $J = \frac{1}{2}$ for those with p^7 , p^9 , p^{11} configurations: this prediction agrees with observation except in the case of ^{11}B , which very probably has an angular momentum $\frac{3}{2}$. However, one would expect the first-order non-central couplings of mixed meson theory (16.42) or the static axial dipole couplings given by other variants of meson theory (16.3, 16.41) to be determinant in this respect, rather than the weaker relativistic couplings. In a short note, Miss PHILLIPS [40] indeed points out that the effect of the axial dipole couplings might remove the discrepancy concerning ^{11}B : the quasi-atomic model predicts above the lowest $^2P_{\frac{1}{2}}$ state two neighbouring levels with $J = \frac{3}{2}$, viz. a $^2P_{\frac{3}{2}}$ and a $^2D_{\frac{3}{2}}$ state, and the axial dipole interaction might perturb these levels to such an extent as to depress one of them below the $^2P_{\frac{1}{2}}$ state. The ground state of ^{11}B would then be a mixture of $^2P_{\frac{3}{2}}$ and $^2D_{\frac{3}{2}}$ states. On the other hand, as stated by Miss Phillips, the axial dipole coupling would in all other cases lead to the J -value expected from experiment.

If we adopt Miss Phillips' interpretation of the ground state of ^{11}B , we must get a somewhat more accurate evaluation of its magnetic moment. For the ^2D state we find the representation

$$\Psi(^2D) = \sqrt{\frac{7}{15}}(^1S^2D) + \sqrt{\frac{1}{15}}(^1D^2D) \pm \sqrt{\frac{1}{15}}(^1D^2P), \quad (42)$$

whence, by (17),

$$g_I(^2D) = \frac{8}{9}, \quad (42a)$$

and by (19),

$$\mu(^{11}\text{B}) = \mu_p + \frac{8}{9} - |c(^2D)|^2 \left[\frac{8}{9}\mu_p - \frac{1}{9} \right]. \quad (43)$$

The experimental value,

$$\mu_{\text{exp}} = \mu_p - 0.1,$$

* See ROSE and BETHE [37] and especially INGLIS [38b].

exhibits a radical departure from the theoretical one for a pure ${}^2P$ state; if it is used to determine the D -admixture, it yields

$$|c({}^2D)|^2 \approx 0.2. \quad (43a)$$

On the α -particle model, each type of odd mass nucleus requires a special investigation in order to ascertain the quantum numbers of its ground state. The assignment of the orbital momentum Λ around the figure axis follows from symmetry considerations (13.22) which do not involve any detailed knowledge of the interaction energy. But the determination of J and of the possible mixture of orbital momenta $K = J \pm \frac{1}{2}$ depends essentially on the spin-orbit coupling. With the same parameter θ as in (23), the latter interaction may be written in the general form

$$\theta \alpha \vec{\Lambda} \vec{S} + (1-\theta) \beta \vec{K} \vec{S}, \quad (44)$$

but it is very difficult (except in trivial cases) to say anything definite about the signs and orders of magnitude of the constants α , β and θ ; the only relation to be expected on general grounds between α and β is $|\beta| \ll |\alpha|$, since the former refers to the rotation of the whole nucleus, while the latter concerns the orbital motion of a single nucleon, which corresponds to larger velocities. SACHS [39], in his systematic discussion of the problem on hand, attempted to deduce the required quantities from the assumption that the spin-orbit coupling was of relativistic origin, and that the Larmor term could be neglected (15.22). However, INGLIS [39c] observed that in the case of ${}^7\text{Li}$ (17.53) and ${}^{13}\text{C}$ this neglect was not justified: in the first instance, on the contrary, the Larmor term is by far the larger, and in the case of ${}^{13}\text{C}$ the two terms are of the same order of magnitude and opposite sign, so that no definite prediction can be made on this basis regarding the angular momentum of this nucleus. Moreover, Inglis showed that the relativistic spin-orbit coupling in ${}^{11}\text{B}$ would be too small to bring about any appreciable admixture of ${}^2D_{\frac{3}{2}}$ state to the ${}^2P_{\frac{3}{2}}$ ground state, in contradiction (as we have just seen) to observation. It must again be pointed out that the spin-orbit couplings actually operative are undoubtedly of other origin, but nothing is known about their properties. We have therefore no other course at present but to take all J -values over from experiment, and further apply the coupling (44) with undetermined coefficients, except for whatever restrictions on their range of variation the choice of J might imply; in this sense we may still use the results of Sachs' paper.

The outcome of the discussion is summarized in table A2.252-2 below. The simplest cases are those for which $\Lambda = 0$: since the coupling (44)

becomes $\sim \vec{K} \vec{S}$, there is then no mixing of states with different K 's; the expression for the magnetic moment has exactly the same form as on the quasi-atomic model, only with another numerical value of the orbital Landé factor. Analogous circumstances present themselves in the case

A2.252-2. Ground states and magnetic moments of 2p shell nuclei according to α -particle model					
Nuclei	J (obs.)	A	K	μ	
${}^7\text{Li}$ ${}^7\text{Be}$	$\frac{3}{2}$	0	1	$\mu_{pn} + g_{\xi}$	
${}^9\text{B}$ ${}^9\text{Be}$	$\frac{3}{2}$	1	1	$\mu_{pn} + \frac{1}{2}(g_{\xi} + g_{\zeta})$ if $ \epsilon\rangle\rangle \alpha\rangle$	
			1,2	$\frac{3}{8}(\mu_{pn} + g_{\xi} + g_{\zeta})$ if $ \epsilon\rangle\langle\langle \alpha\rangle$	
${}^{11}\text{B}$ ${}^{11}\text{C}$	$\frac{3}{2}$	1	1	$\mu_{pn} + \frac{1}{2}(g_{\xi} + g_{\zeta})$ if $ \epsilon\rangle\rangle \alpha\rangle$	
			1,2	$0.31\mu_{pn} + 0.76g_{\xi} + 0.58g_{\zeta}$ if $ \epsilon\rangle\langle\langle \alpha\rangle$	
${}^{13}\text{N}$ ${}^{13}\text{C}$	$\frac{1}{2}$	0	1	$-\frac{1}{2}\mu_{pn} + \frac{3}{2}g_{\xi}$	
${}^{15}\text{N}$ ${}^{15}\text{O}$	$\frac{1}{2}$	—	1	$-\frac{1}{2}\mu_{pn} + \frac{3}{2}g_{\xi}$	
$g_{\xi} \approx Z/A, \quad g_{\zeta} = \theta g_p + (1-\theta)(Z/A)$					

of ${}^{15}\text{N}$ and ${}^{15}\text{O}$, in which, owing to the spherical symmetry, $\vec{A}$ in (44) becomes identical with $\vec{K}$; the value $\frac{1}{2}$ of J implies $\alpha\theta + \beta(1-\theta) > 0$. In the remaining cases of mass numbers $A=9$ and 11 , the value $J=\frac{3}{2}$ gives rise to the condition $\alpha\theta + 2\beta(1-\theta) < 0$; the mixing of states with different K -values depends on the ratio of the coefficient of spin-orbit coupling $|\alpha|$ (the β -coupling does not contribute to the mixing) to the energy separation

$$|\epsilon| = \left| \left\{ \frac{\hbar^2}{2I_c} K(K+1) + \frac{1}{2}[J(J+1) - K(K+1) - S(S+1)]\beta(1-\theta) \right\}_{K=J+\frac{1}{2}}^{K=J-\frac{1}{2}} \right| \quad (45)$$

$$= (J + \frac{1}{2}) \left| \frac{\hbar^2}{I_c} - \beta(1-\theta) \right| :$$

in the extreme case $|\epsilon| \gg |\alpha|$, there is no mixing; in the opposite extreme case, mixing ratios *

$$\begin{aligned} |c(^2D)|^2 : |c(^2P)|^2 &= 1 : 3 \quad \text{for } A=9 \\ &= 3 : 4 \quad \text{for } A=11 \end{aligned} \quad (46)$$

are found. * The magnetic moment is the sum of the three contributions μ_s, μ_l, μ_A given by (13) with (11), (18) and (25), respectively.

We are now in a position to compare the observed magnetic moments with the predictions of both nuclear models. For this purpose, the most

* The $c_{K m_K m_s}$'s take on the following values:

$$\begin{aligned} c_{1,1,\frac{1}{2}} &= \frac{1}{\sqrt{3}} & A=9 & \quad A=11 \\ c_{2,1,\frac{1}{2}} &= \frac{1}{2} \frac{1}{\sqrt{5}} & -\sqrt{\frac{3}{7}} \frac{1}{\sqrt{5}} \\ c_{2,2,-\frac{1}{2}} &= -\frac{1}{2} \frac{2}{\sqrt{5}} & +\sqrt{\frac{3}{7}} \frac{2}{\sqrt{5}} \end{aligned}$$

suitable procedure will be to analyze the measured moments into a spin and an orbital contribution in conformity with the crudest approximation of the quasi-atomic picture: this is indicated in the 3rd and 4th columns of table A2.252-3; the two following columns give the calculated values for the quantities in the 4th column according to both models. The case of ^{11}B has already been touched upon in the preceding discussion: we have seen how the quasi-atomic picture could be improved to explain the experimental result, and provided a sufficient margin is granted the spin-orbit couplings, the α -particle model also covers this result; whether the actual non-central forces are of the required magnitude, remains a matter for further inquiry. The cases of ^9Be and ^{13}C should presumably be handled in the same way as ^{11}B . The magnetic moment of ^{15}N (if the sign, as yet unknown, is taken to be negative) fits quite nicely with the unperturbed quasi-atomic picture; the α -particle model here requires the rather extreme case of purely orbital motion of the proton "hole" ($\theta = 1$) (INGLIS [39a]).

A2.252-3. Observed and calculated magnetic moments of 2p shell nuclei					
Nucleus	J (obs)	Observed moment		Calculated on quasi-atomic model	Calculated on α -particle model
^7Li	$\frac{3}{2}$	$\mu = \mu_p +$	0.4636	0.33	0.43
^9Be	$\frac{3}{2}$	$\mu = \mu_n +$	0.734	0.33	0.22...1.03
^{11}B	$(\frac{3}{2})$	$\mu = \mu_p +$	-0.1036	0.67	-1.31...0.73
^{13}C	$\frac{1}{2}$	$\mu = -\frac{1}{3}\mu_n +$	0.064	0.44	0.31
^{15}N	$\frac{1}{2}$	$\mu = -\frac{1}{3}\mu_p +$	0.65*	0.67	0.31...0.67
* Assuming the minus sign.					

All this evidence remains vague and inconclusive. The case of ^7Li , on the other hand, has been studied in much greater detail. The difference between the two models is here clearly illustrated by the approximate treatment embodied in table A2.252-3 (BETHE [38c]): the spin contribution is the same in both cases, but the orbital Landé factors are rather different. On the quasi-atomic model, the two p shell neutrons are paired and only the odd proton spin contributes; the orbital momentum is about equally shared between these three nucleons, the closed s shell does not participate in the orbital motion*, and hence $g_l \approx \frac{1}{3}$. On the α -particle model, the system consists of an α -cluster and a triton (^3H) cluster: here again, the neutrons of the triton are paired and the spin moment is that of the odd proton; but the orbital motion is now shared also by the α -cluster, so that a larger orbital gyromagnetic ratio results. Roughly, all nucleons sharing equally as in a structureless droplet, we should have $g_l = Z/A = 3/7$; somewhat more accurately, the orbital momentum would be shared between the two

* For a formal proof, taking the motion around the common centre of gravity into consideration, see ROSE and BETHE [37].

clusters in the inverse ratio of their masses, and since the specific charges of the α - and triton-clusters are $\frac{1}{2}$ and $\frac{1}{3}$, respectively, one would expect

$$g_{\xi} \approx \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2} = 0.40. \quad (47)$$

Although this value deviates less from the experimental one than that given by the quasi-atomic model, it is too small and there is no obvious way of improving it: INGLIS [39a] has verified that the polarization of the triton-cluster by the field of the α -cluster has no appreciable effect. On the other hand, various refinements of the quasi-atomic picture have not led to any substantial correction of the initial approximation. The Heisenberg and Bartlett forces would reduce the orbital Landé factor $g_l = \frac{1}{2}$ to perhaps 0.3 (ROSE and BETHE [37], INGLIS [38b]). The admixture to the ground state of the configuration ($3P^2P$) in which the spins of the $2p$ -neutrons are parallel is too small to modify the spin Landé factor significantly (INGLIS [38b]). The second order perturbations due to interference with excited configurations are negligible (INGLIS [38b]). It is not likely that the axial dipole coupling would in this case give rise to any serious perturbation of the ground state (PHILLIPS [40]). Altogether, therefore, the magnetic moment of ${}^7\text{Li}$ remains a bit of a puzzle.

A2.253. *The "2s shell" nuclei.* If we interpret the building up of a nucleus following the completion of the $2p$ shell as the filling of a $2s$ shell (10.32), we should expect the ground states of these nuclei to be of the ${}^2S$ type, with the result that the magnetic moments of the odd mass members of this class would just be μ_{pn} . The α -particle model (SACHS [39]), as shown in table A2.253, leads to the same prediction. The observed magnetic moment of ${}^{19}\text{F}$ reveals by its deviation from the predicted value μ_p a perturbation of the ground state. On the α -particle model, one might, following Sachs, think of an admixture of the state with $K = 1$ due to the energy terms arising from the centrifugal forces (which are usually neglected since they involve the square of the angular momentum). On the quasi-atomic model (INGLIS [39a]), a different interpretation suggests itself: the spin dependent interactions will bring about a small admixture of a state, belonging to a higher supermultiplet, in which the two $2s$ -neutrons have parallel spins; this would give rise to a negative contribution (since $\mu_n < 0$) to the spin magnetic moment.

A2.253. Ground states and magnetic moments of 2s shell nuclei according to α -particle model						
Nuclei	J	Λ	K	μ	μ (obs)	
${}^{17}\text{F}$ ${}^{17}\text{O}$	$\frac{1}{2}$	—	0	μ_{pn}		
${}^{19}\text{F}$ ${}^{19}\text{Ne}$	$\frac{1}{2}$	0	0	μ_{pn}	$\mu_p - 0.1646$	

A2.26. *Magnetic moments of stable odd nuclei.* Thanks to the rigorous expression obtainable (A2.221) for the magnetic moments of the stable

odd nuclei, the analysis of the empirical data in this case yields fairly precise information on the properties of the ground states of these nuclei (SACHS [46a]). Using the representation adapted to the quasi-atomic picture, we may say, since the angular momentum of the nuclei in question is $J = 1$, that the ground state is a mixture of ${}^3S$, ${}^3D$, ${}^1P$ and ${}^3P$ states. Since the P states, in the approximation of central forces, are found (10.22) to lie higher than the others, we may expect their proportions in the mixture to be small and even negligible for ${}^2\text{H}$ and ${}^{10}\text{B}$ (10.22). Combining formula (28) with the relation

$$|c({}^3S)|^2 = 1 - [|c({}^3D)|^2 + |c({}^1P)|^2 + |c({}^3P)|^2] \quad (48)$$

between the expansion coefficients, one finds

$$\mu = \mu_d^0 - \frac{3}{2}(\mu_d^0 - \frac{1}{2}) [|c({}^3D)|^2 + \frac{2}{3}|c({}^1P)|^2 + \frac{1}{3}|c({}^3P)|^2], \quad (49)$$

i.e.

$$|c({}^3D)|^2 + \frac{2}{3}|c({}^1P)|^2 + \frac{1}{3}|c({}^3P)|^2 = \beta, \quad \beta = \frac{\mu_d^0 - \mu}{\frac{3}{2}(\mu_d^0 - \frac{1}{2})}. \quad (50)$$

If the P state admixture is negligible, equation (50) directly gives the ${}^3D$ -admixture. In any case, we derive from (50) upper limits for the proportions of D and P states:

$$|c({}^3D)|^2 \leq \beta, \quad |c({}^1P)|^2 \leq \frac{3}{2}\beta, \quad |c({}^3P)|^2 \leq 3\beta. \quad (51)$$

From (49) and (50) we easily deduce

$$1 - 3\beta \leq |c({}^3S)|^2 \leq 1 - \beta. \quad (52)$$

We may repeat this argument after interchanging the roles of ${}^3S$ and ${}^3D$, which simply means interchanging also ${}^1P$ and ${}^3P$ and replacing β by $1 - \beta$; this yields the further inequalities

$$|c({}^3S)|^2 \leq 1 - \beta, \quad |c({}^1P)|^2 \leq 3(1 - \beta), \quad |c({}^3P)|^2 \leq \frac{3}{2}(1 - \beta), \quad (51a)$$

$$3\beta - 2 \leq |c({}^3D)|^2 \leq \beta, \quad (52a)$$

which are more useful when $\beta > \frac{1}{3}$. The application of these formulae to the data (table A2.21-1) yields the following results:

A2.26. Ground states of stable odd nuclei						
Nucleus	$\mu_d^0 - \mu$	β	$ c({}^3S) ^2$	$ c({}^3D) ^2$	$ c({}^1P) ^2$	$ c({}^3P) ^2$
${}^2\text{D}$	0,023	0,04	0,96	0,04	—	—
${}^6\text{Li}$	0,058	0,10	0,70...0,90	0...0,10	0...0,15	0...0,30
${}^{10}\text{B}$	0,281	0,49	0,51	0,49	—	—
${}^{14}\text{N}$	0,476	0,835	0...0,165	0,505...0,835	0...0,495	0...0,248

As regards the further question of the couplings responsible for admixtures of such orders of magnitude, little definite can be said. INGLIS [39a] has shown, in the cases of ${}^6\text{Li}$ and ${}^{14}\text{N}$, that neither the Coulomb

energy, nor an amount of spin-orbit coupling sufficient to account for the separation of the ${}^7\text{Li}$ doublet (17.5) would be able to yield more than a small fraction of the observed difference $\mu_d^0 - \mu$. The axial dipole couplings are probably much more effective.

A2.261. Unstable odd nuclei. There is a type of nucleus not covered by the foregoing discussion, viz. the odd nuclei of mass number $4a$ and neutron excess ± 2 , of which one example, the long-lived isotope ${}^{40}\text{K}$, has been investigated experimentally as regards its magnetic moment (table A2.21–3). Among the “ $2p$ shell” nuclei, the pairs of conjugate nuclei ${}^8\text{Li} - {}^8\text{B}$, ${}^{12}\text{B} - {}^{12}\text{N}$ belong to this type. They may be treated by the same method as the odd mass nuclei on the quasi-atomic model (ROSE and BETHE [37]). The characteristics of the ${}^3P$ ground states are as follows:

Configuration	Ground state		g_I		g_I
$p^4 \quad p^8$	$\pm \sqrt{\frac{7}{8}}(2P^2P) + \sqrt{\frac{1}{8}}(2D^2P)$	${}^8\text{Li} \quad {}^{12}\text{N}$	$\frac{1}{8}$	${}^8\text{B} \quad {}^{12}\text{B}$	$\frac{7}{8}$

According to (19) and (12), the magnetic moments are given by

$$\mu = \frac{1}{2} J(\mu_d^0 + g_I). \quad (53)$$

Relativistic spin-orbit coupling (limited to the Thomas effect) indicates $J = 2$ for ${}^8\text{Li} - {}^8\text{B}$ and $J = 0$ for ${}^{12}\text{N} - {}^{12}\text{B}$; but these J -values are, of course, very uncertain.

The case of ${}^{40}\text{K}$ is rather strange (INGLIS [41]). From the possible components of the ground state:

$${}^1G_4, {}^3FGH_4, {}^5DFGHI_4, \text{ etc.},$$

the singlet and triplet ones are both excluded (as predominant components), because they would most probably imply a positive magnetic moment: this is obvious for the ${}^1G_4$ state, since g_I will certainly be positive; and if we assume that in the triplet states g_s is $\approx \mu_d^0$ as for the light odd nuclei, we shall have in the most favourable case, viz. that of a ${}^3H_4$ state, $\mu = \frac{1}{6}(6g_I - g_s)$, which is positive for any plausible value of g_I . For quintet states, we expect the gyromagnetic ratio g_s to be made up of one proton and 3 neutron contributions, i.e.

$$g_s = \frac{1}{2}(\mu_p + 3\mu_n) = -1.47, \quad (54)$$

whereas g_I might be $\approx \frac{1}{4}$, corresponding to equal sharing of the orbital momentum between those four nucleons. Then, two of the possible quintet states, viz. ${}^5D_4$ and ${}^5F_4$, are found, by (19), to give negative values of acceptable order of magnitude for the magnetic moment:

$$\begin{aligned} \mu({}^5D_4) &= 2(g_s + g_I) = -2.44 \\ \mu({}^5F_4) &= \frac{7}{3}g_s + \frac{13}{5}g_I = -1.41. \end{aligned} \quad (55)$$

We are thus driven to the conclusion that the ground state of ${}^{40}\text{K}$ would probably be a mixture of the quintet states ${}^5D_4 + {}^5F_4$, although there is no obvious reason for the lack of pairing of two neutrons in this case. This difficulty does not seem to arise in the case of the heavier odd nucleus ${}^{176}\text{Lu}$, whose ground state could well be a triplet one (INGLIS [41]).

A2.27. Relativistic correction to the magnetic moment. The preceding considerations have been based on the non-relativistic expression for the magnetic moment and although there is little doubt that this approximation is quite sufficient, it may be of some interest to estimate the order of magnitude of the relativistic correction. About the exact form of the magnetic moment operator there is some uncertainty owing to the occurrence of the anomalous moments of proton and neutron. However, if we accept the principle of the explanation of these moments on field theory, we must regard them as originating from a definite current distribution (cf. 6.13) and we then simply get, for the magnetic moment of a nuclear system *,

$$\vec{M} = \sum_i e_3^{(i)} (\vec{\mu}^{(i)} + T_-^{(i)} \vec{l}^{(i)} \mu_0); \quad (56)$$

this formula differs from the non-relativistic expression (1), (2), (3) by the occurrence of the operators $e_3^{(i)}$, implying corrections of the second order in the nucleon velocities. We shall carry out the computation of this correction in two cases:

(1) one nucleon moving in a central field: this may be taken as a crude picture of the situation obtaining in odd mass nuclei (A2.23);

(2) the ground state of the deuteron.

Nucleon in central field. The well-known solution of Dirac's equation in this case ** may be written in our notation (4.21):

$$\psi = w_+(\varrho_3) \psi_+ + w_-(\varrho_3) \psi_-;$$

$$l = j - \frac{1}{2}:$$

$$\begin{aligned} \psi_+ &= \frac{1}{\sqrt{2l+1}} [\sqrt{l+m+\frac{1}{2}} Y_l^{m-\frac{1}{2}} v_+(\sigma'_z) - \sqrt{l-m+\frac{1}{2}} Y_l^{m+\frac{1}{2}} v_-(\sigma'_z)] F_+(r) \\ \psi_- &= \frac{i}{\sqrt{2l+3}} [\sqrt{l-m+\frac{3}{2}} Y_{l+1}^{m-\frac{1}{2}} v_+(\sigma'_z) + \sqrt{l+m+\frac{3}{2}} Y_{l+1}^{m+\frac{1}{2}} v_-(\sigma'_z)] F_-(r); \end{aligned} \quad (57)$$

$$l = j + \frac{1}{2}:$$

$$\begin{aligned} \psi_+ &= \frac{1}{\sqrt{2l+1}} [\sqrt{l-m+\frac{1}{2}} Y_l^{m-\frac{1}{2}} v_+(\sigma'_z) + \sqrt{l+m+\frac{1}{2}} Y_l^{m+\frac{1}{2}} v_-(\sigma'_z)] F_+(r) \\ \psi_- &= \frac{i}{\sqrt{2l-1}} [\sqrt{l+m-\frac{1}{2}} Y_{l-1}^{m-\frac{1}{2}} v_+(\sigma'_z) - \sqrt{l-m-\frac{1}{2}} Y_{l-1}^{m+\frac{1}{2}} v_-(\sigma'_z)] F_-(r). \end{aligned}$$

From the wave-equation, involving the potential energy $\mathcal{V}(r)$, one readily derives the relation

$$\int (E - \mathcal{V} + M) |\psi_-|^2 dv = \int (E - \mathcal{V} - M) |\psi_+|^2 dv \quad (58)$$

* See, e.g., MøLLER and ROSENFELD [43], p. 41 sq. In formula (37) of this paper, however, the factors $e_3^{(i)}$ are wrongly omitted from the terms containing the anomalous moments.

** See, e.g., BETHE [33], p. 311 sq.

between large and small components ψ_+ , ψ_- ; for not too high velocities, the left-hand side can be replaced by $2M \int |\psi_-|^2 dv$, so that

$$\int |\psi_-|^2 dv \approx \frac{1}{2M} \int (E - \mathcal{V} - M) (|\psi_+|^2 + |\psi_-|^2) dv;$$

the integral on the right represents the average kinetic energy of the nucleon in the stationary state ψ . Denoting by β^2 the ratio of this quantity to the rest energy M , we thus have

$$\int |\psi_-|^2 dv \approx \frac{1}{2} \beta^2, \quad (59)$$

to which we may add the normalization condition in the form

$$\int |\psi_+|^2 dv = 1 - \frac{1}{2} \beta^2. \quad (60)$$

With the help of these formulae, it is an easy matter to calculate the magnetic moment. One finds

$$l = j - \frac{1}{2}:$$

$$\text{av } \varrho_3 \sigma_z = \frac{m}{j} \left[1 - \frac{1}{2l+3} \beta^2 \right]$$

$$\text{av } \varrho_3 l_z = \frac{m}{j} \left[l - \frac{2l^2+4l+1}{2l+3} \beta^2 \right];$$

$$l = j + \frac{1}{2}: \quad (61)$$

$$\text{av } \varrho_3 \sigma_z = -\frac{m}{j+1} \left[1 + \frac{1}{2l-1} \beta^2 \right]$$

$$\text{av } \varrho_3 l_z = \frac{m}{j+1} \left[l + 1 - \frac{2l^2-1}{2l-1} \beta^2 \right].$$

Hence, the correction $\Delta\mu$ to be added to the non-relativistic expression for the magnetic moment μ is, for both values of $l = j \pm \frac{1}{2}$,

$$\Delta\mu = -\frac{1}{j+1} \beta^2 \left[\frac{1}{2} \mu_{pn} + (j^2 + j - \frac{1}{4}) g_p \right]. \quad (62)$$

For odd mass nuclei, this would mean a correction of the order of 1%.

In treating this problem, MARGENAU [40a] and CALDIROLA [46] prefer to compute directly the average value, in the state (57), of the interaction energy of the nucleon with a constant external magnetic field. Margenau's statement that the calculation of the average value of the operator (1), (2), (3) gives a different answer is, of course, formally correct, but he seems to have overlooked the fact that this operator is not the true expression for the magnetic moment. If the factors $\varrho_3^{(l)}$ are introduced, as in (56), no discrepancy arises. Margenau's treatment of the anomalous moments is thus necessarily inadequate; in this respect, Caldirola indicates the right procedure, which consists in starting from the Hamiltonian (4.221-9), in which, however, a factor ϱ_3 must be inserted in the last term.

Ground state of the deuteron *. Using the representation (4.33-18)

* The following calculation is inaccurate inasmuch as it neglects the second order correction to the Hamiltonian of the deuteron problem (15.22), and it accordingly does not

of the eigenfunction of any stationary state of the deuteron, we must in the first place fix its normalization. Multiplying it by some suitable constant C , we may write down the normalization condition, exact to the second order in the nucleon velocities,

$$|C|^2 [f^{(0)}|\psi|^2 + f^{(1)}|\psi|^2] = 1. \quad (63)$$

We may now take for $\psi^{(0)}$ and $\psi^{(1)}$ the expressions (4.34–29b), in which the radial functions are normalized to unity. Thus, $\int |\psi^{(0)}|^2 = 1$, and for $\int |\psi^{(1)}|^2$ we get for instance in the case $l = j - 1$:

$$\begin{aligned} \int |\psi^{(1)}|^2 &= \frac{\hbar^2}{2M^2} \int_0^\infty \left[\frac{d}{dr} {}^3R^{(l)} - \frac{j}{r} {}^3R^{(l)} \right]^2 dr \\ &= -\frac{\hbar^2}{2M^2} \int_0^\infty {}^3R^{(l)} \left[\frac{d^2}{dr^2} {}^3R^{(l)} - \frac{j^2}{r^2} {}^3R^{(l)} + \frac{2j}{r} \frac{d}{dr} {}^3R^{(l)} \right] dr \\ &= -\frac{\hbar^2}{2M^2} \int_0^\infty {}^3R^{(l)} \left[\frac{d^2}{dr^2} {}^3R^{(l)} - \frac{l(l+1)}{r^2} {}^3R^{(l)} \right] dr \\ &= \frac{1}{2M_0} \int_0^\infty {}^3R^{(l)} ({}^3\varepsilon^{(l)} - {}^3V^{(l)}) {}^3R^{(l)} dr \equiv \beta^2, \end{aligned} \quad (64)$$

in virtue of the wave-equation (4.331–27). The quantity β^2 represents the ratio of the mean kinetic energy *per nucleon* to the rest energy of a nucleon. The other cases yield the same result, so that we get from (63)

$$|C|^2 \approx 1 - \beta^2. \quad (65)$$

Now, for the expectation value of any operator of the form

$$A = \varrho_3^{(1)} A^{(1)} + \varrho_3^{(2)} A^{(2)},$$

in which the A 's do not contain the ϱ 's we get, again neglecting a term of higher order,

$$\begin{aligned} \text{av } A &= |C|^2 [f^3 f_1^* (A^{(1)} + A^{(2)}) {}^3f_1 + f^3 f_0^* (A^{(1)} - A^{(2)}) {}^1f_0 \\ &\quad + f^1 f_0^* (A^{(1)} - A^{(2)}) {}^3f_0]; \end{aligned}$$

calling $\overset{(0)}{A}$ the non-relativistic approximation, given by the first integral

claim to be more than an estimate of order of magnitude. Calculations taking account of this correction have been carried out, for the ${}^3S$ state, by BREIT and BLOCH [47c]. Adopting the Hamiltonian (15.22–16), they find an additional term $+\frac{1}{2}\beta^2$ to the first formula (70); with the Hamiltonian (15.22–17), the additional term is $-\varepsilon_0/M + \frac{1}{2}\beta^2$. Similar calculations by SACHS [47a] are incorrect in so far as they do not use the complete second order Hamiltonians.

SACHS [47a] and PRIMAKOFF [47] discuss the influence of axial dipole forces on the relativistic correction to the magnetic moment.

between the brackets, we may write the relativistic correction, up to the second order in the velocities, in the form

$$\Delta A = -\beta^2 \overset{(0)}{A} + 2 \Re \int {}^3f_0^* (A^{(1)} - A^{(2)}) {}^1f_0. \quad (66)$$

In all states with $l = j$, the last term of this expression vanishes. In the states with $l = j \pm 1$, this term involves a singlet-triplet transition and therefore vanishes, in particular, for all operators independent of the spin coordinates. For the magnetic moment of the deuteron, we have

$$\Delta \mu = -\beta^2 \overset{(0)}{\mu} + \Delta \mu', \quad (67)$$

the term $\Delta \mu'$ arising from the spin term only. Since

$$\text{av } T_+^{(l)} = \text{av } T_-^{(l)} = \frac{1}{2}, \quad (68)$$

we get

$$\Delta \mu' = \mu_d^0 \Re [\int {}^3f_0^* (\sigma_z^{(1)} - \sigma_z^{(2)}) {}^1f_0]_{m=j},$$

for which a straightforward calculation, using (64), yields

$$\Delta \mu' = \pm \mu_d^0 \beta^2 \frac{2j}{2j+1}; \quad (69)$$

the $\pm$ signs correspond to $l = j \mp 1$, respectively. In particular, the corrections for the components of the ground state are

$$\begin{aligned} \Delta \mu({}^3S_1) &= -\frac{1}{3} \mu_d^0 \beta^2 \\ \Delta \mu({}^3D_1) &= -\frac{3}{4} \beta^2 - \frac{1}{6} \mu_d^0 \beta^2. \end{aligned} \quad (70)$$

The order of magnitude of β^2 is 1 %.

A2.3. Quadrupole moments of nuclei

A2.31. The empirical data. Electric quadrupole moments have been determined for a large number of odd mass nuclei; in addition, the electric quadrupole moments of two odd nuclei, the stable ${}^2\text{D}$ and the long-lived ${}^{176}\text{Lu}$ are known. Recently, the analysis of the hyperfine structure of microwave molecular spectra has revealed the existence of quadrupole moments of ${}^{14}\text{N}$ (SIMMONS and GORDY [48] *, GORDY *et al.* [48a], TOWNES *et al.* [47a]) and of the ${}^{35,37}\text{Cl}$ and ${}^{79,81}\text{Br}$ isotopes (TOWNES *et al.* [47a], GORDY *et al.* [47b, c]). By this method, the absolute values of the quadrupole moments cannot be ascertained without further study, since the quantity measured is the product (A2.11-6) of the quadrupole moment by the gradient of the electric field; only the signs and the ratios of the moments of isotopes can be given directly. However, for heavier nuclei, an estimate of the field gradient, and consequently of the quadrupole moment itself, can be inferred from data of atomic spectroscopy*

* This paper contains references to previous work, experimental and theoretical. For the theory of the effect, including second order corrections, see further BARDEEN and TOWNES [48]. The authors do not all use the same convention as regards the definition of the quadrupole coupling; see a clarifying note by FELD [47].

(TOWNES [47b]); this method involves the neglect of the distortion of the electronic clouds due to molecular binding. Its application to the case

A2.31. Electric quadrupole moments of nuclei							
Nucleus	J	μ	Q 10^{-24} cm^2	Nucleus	J	μ	Q 10^{-24} cm^2
Odd mass nuclei							
Odd proton nuclei				Odd neutron nuclei			
$^{35}_{17}\text{Cl}$	$\frac{3}{2}^*$	1,368	-0.0795				
$^{37}_{17}\text{Cl}$	$\frac{3}{2}^*$	1,136	-0.062				
$^{63}_{29}\text{Cu}$	$\frac{3}{2}$	2,227	-0.1				
$^{65}_{29}\text{Cu}$	$\frac{3}{2}$	2,385	-0.1				
$^{69}_{31}\text{Ga}$	$\frac{3}{2}$	1,994	0.186				
$^{71}_{31}\text{Ga}$	$\frac{3}{2}$	2,540	0.117				
$^{75}_{33}\text{As}$	$\frac{3}{2}$	1.6	0.3				
$^{79}_{35}\text{Br}$	$\frac{3}{2}$	2,107	$Q \approx 0.2$				
$^{81}_{35}\text{Br}$	$\frac{3}{2}$	2,271	$0.77 \cdot Q$	$^{83}_{36}\text{Kr}$	$\frac{3}{2}$	-1.0	0.15
$^{115}_{49}\text{In}$	$\frac{5}{2}$	5.49	0.84				
$^{127}_{53}\text{I}$	$\frac{5}{2}$	2.8	0.8?	$^{131}_{54}\text{Xe}$	$\frac{3}{2}$	0.7	0 ± 0.1
$^{151}_{63}\text{Eu}$	$\frac{5}{2}$	3.4	1.2				
$^{153}_{63}\text{Eu}$	$\frac{5}{2}$	1.5	2.5	$^{173}_{70}\text{Yb}$	$\frac{5}{2}$	-0.65	3.9
$^{175}_{71}\text{Lu}$	$\frac{7}{2}$	2.6	5.9				
$^{185}_{75}\text{Re}$	$\frac{5}{2}$	3.3	2.8				
$^{187}_{75}\text{Re}$	$\frac{5}{2}$	3.3	2.6	$^{201}_{80}\text{Hg}$	$\frac{3}{2}$	-0.6	0.5
$^{209}_{83}\text{Bi}$	$\frac{9}{2}$	3.6	-0.4				
Odd nuclei							
Stable odd nuclei				Unstable odd nuclei			
^2_1D	1	0.857	$2.73 \cdot 10^{-3}$				
$^{14}_7\text{N}$	1	0.403	$0.01 \dots 0.1$	$^{176}_{71}\text{Lu}$	≥ 7	3.8	7
* From molecular data, however, TOWNES <i>et al.</i> [47a], derive a J -value $\frac{3}{2}$.							

of NH_3 , from which the quadrupole moment of ^{14}N should be derived, is therefore rather uncertain, and only approximate limiting values can be indicated *. On the other hand, the quadrupole moments of the $^{35}, ^{37}\text{Cl}$

For the deuteron case, see footnote to (6.12-7).

isotopes have lately been measured by an atomic beam method (DAVIS *et al.* [48]); this would enable us in this case to derive from the molecular data some direct information about the field gradients in the various molecular structures studied. All available results are collected in table A2.31; references to the sources will be found in INGLIS' oft-quoted paper [41] *. It is a most striking feature of this material that *almost all quadrupole moments are positive*, indicating an electricity distribution elongated along the axis of angular momentum of the nucleus. Moreover, *with increasing charge number, the values of the quadrupole moments pass through a maximum at about $Z = 71$.*

In addition, it has been verified (DAILEY *et al.* [46], TOWNES *et al.* [47c]) that, in accordance with the theory (A2.13), the nuclei ^{13}C , ^{15}N , ^{33}S , for which $J = \frac{1}{2}$, and ^{34}S , with $J = 0$, have no quadrupole moment.

A2.32. Quadrupole moments of odd mass nuclei and nuclear models. It is obvious that the elongated form of many nuclei revealed by the positive quadrupole moments is incompatible with the *droplet model*, according to which the only shape stable against arbitrary distortions is that of an oblate ellipsoid of revolution around the axis of angular momentum. Moreover, the quantitative discussion carried out by Miss WAY [39a] shows that the flattening due to rotation would be much too small to account for the absolute value of the negative quadrupole moment of ^{209}Bi . Just as the magnetic moment, the electric quadrupole is an essentially structural property of the nucleus. And just as in the case of the magnetic moment, it is, as emphasized by INGLIS [41], the *α -particle model* that affords the best possibilities of interpretation.

In the first place, the investigation of the stablest packings of hard spheres (WEFELMEIER [37a]) suggests that the most elongated framework of α -particles is in the neighbourhood of $Z = 71$, just where the largest quadrupole moments are observed. However, it is essential for this interpretation that the total angular momentum should orient itself along the longitudinal axis of the framework. Now, the evidence from the magnetic moments suggests, as we have seen, that the ground state of an odd mass nucleus is a doublet state, in which the direction of the total angular momentum coincides with the average direction of the orbital momentum; further, the latter seems to be largely ascribable to the motion of the odd nucleon. As pointed out by FANO [37], the exclusion principle entails a repulsion of a nucleon by an α -cluster, and consequently a concentration of the wave-function of the odd nucleon as far away as possible from the concentrations of α -clusters: for an elongated structure of the α -clusters, this means an orientation of the orbital momentum of the odd nucleon along the longitudinal axis, as required for the explanation of the positive

* All the quadrupole moments given in the table, with the exception of those of ^{79}Br , ^2H and ^{14}N , have been determined directly by methods involving only the atoms of the elements concerned. For the $^{69}, ^{71}\text{Ga}$ isotopes, the recent results of BECKER and KUSCH [48] are given. Concerning ^{127}I , see also GORDY *et al.* [48b].

quadrupole moments. The same argument applied to a flattened structure would lead to an orientation of the orbital momentum perpendicular to the figure axis; taking the zero point rotation into consideration, this situation would correspond to spherical symmetry of the average charge distribution and a vanishing quadrupole moment.

The fact that negative quadrupole moments occur only among odd proton nuclei is also in harmony with the picture derived from the interpretation of the magnetic moments. For the motion of single proton with a non-vanishing orbital momentum L_p in a central field gives rise to a negative quadrupole moment (BETHE [33], p. 557)

$$Q_{L_p} = (Q_{L_p m_p})_{m_p=L_p} = -\frac{2L_p}{2L_p+3} \text{av } r^2, \quad (1)$$

where

$$Q_{L_p m_p} = -\frac{2[3m_p^2 - L_p(L_p+1)]}{(2L_p-1)(2L_p+3)} \text{av } r^2; \quad (2)$$

and this remains true for heavier nuclei, as we shall see presently (A2.321), when the interaction with the other constituent nucleons is taken into account. If the framework of α -particles is nearly spherically symmetrical, the negative quadrupole moment of the odd proton may determine the sign of the whole effect. This simple interpretation, however, does not seem immediately applicable to the five known cases of nuclei with negative quadrupole moments, because it is not clear that their α -framework should have nearly spherical symmetry. (For the detailed discussion, see INGLIS [41].)

A2.321. Quadrupole moment due to a single proton. The preceding discussion must be completed by the proof of the property just enunciated: *in heavier nuclei, the motion of a single proton gives rise to a negative quadrupole moment.* This proof has been given by WELLES [42]. Assuming the ground state to belong to the lowest supermultiplet, we have $S = \frac{1}{2}$ and $L = J \pm \frac{1}{2}$. The substate with $m = J$ corresponds, for $L = J - \frac{1}{2}$, to definite values $m_l = L$ and $m_s = \frac{1}{2}$ of the z -components L_z and S_z ; but for $L = J + \frac{1}{2}$, we have a mixture* of two states with $m_l = L$, $m_s = -\frac{1}{2}$ and $m_l = L - 1$, $m_s = +\frac{1}{2}$:

$$\Psi_{L=J+\frac{1}{2}; S=\frac{1}{2}} = \sqrt{\frac{2J+1}{2J+2}} \Psi_{L=J+\frac{1}{2}; m_l=J+\frac{1}{2}; m_s=-\frac{1}{2}} - \frac{1}{\sqrt{2J+2}} \Psi_{L=J+\frac{1}{2}; m_l=J-\frac{1}{2}; m_s=\frac{1}{2}}. \quad (3)$$

If we therefore write the eigenfunction of the ground state (with $m = J$) in the form

$$\Psi = c_{J-\frac{1}{2}} \Psi_{L=J-\frac{1}{2}; S=\frac{1}{2}} + c_{J+\frac{1}{2}} \Psi_{L=J+\frac{1}{2}; S=\frac{1}{2}}, \quad (4)$$

* See E. CONDON and G. SHORTLEY, *The theory of atomic spectra* (1935), Chapt. V, 4, formula (8b), p. 123.

we have for the quadrupole moment

$$Q = |c_{J-\frac{1}{2}}|^2 Q_{J-\frac{1}{2}} + |c_{J+\frac{1}{2}}|^2 \left[\frac{2J+1}{2J+2} Q_{J+\frac{1}{2}}^+ + \frac{1}{2J+2} Q_{J+\frac{1}{2}}^- \right] - 2 \Re c_{J-\frac{1}{2}}^* c_{J+\frac{1}{2}} \frac{1}{\sqrt{2J+2}} q_J. \quad (5)$$

In this formula, the average values $Q_{J-\frac{1}{2}}$, $Q_{J+\frac{1}{2}}^+$, $Q_{J+\frac{1}{2}}^-$ are the diagonal elements of the operator $3z^2 - r^2$ of the odd proton with respect to the wave-functions $\Psi_{Lm_l m_s}$ with $L = m_l = J - \frac{1}{2}$, $L = m_l = J + \frac{1}{2}$ and $L = J + \frac{1}{2}$, $m_l = J - \frac{1}{2}$; the non-diagonal element

$$q_J = \int \Psi_{L=J-\frac{1}{2}; m_l=J-\frac{1}{2}; m_s=\frac{1}{2}}^* (3z^2 - r^2) \Psi_{L=J+\frac{1}{2}; m_l=J-\frac{1}{2}; m_s=\frac{1}{2}} \quad (6)$$

is generally negligible.

The computation of the Q 's necessitates a further decomposition of the eigenfunctions. In the first place, we write

$$\Psi_{Lm_l m_s} = \sum b_{Lm_l m_s; L_p L_n} \Psi_{Lm_l; L_p L_n}, \quad (7)$$

L_p being the orbital quantum number of the odd proton, L_n that of the "core" formed by the neutrons and the other protons (assumed to give rise to a spherically symmetrical charge distribution). Further,

$$\Psi_{Lm_l; L_p L_n} = \sum_{m_p + m_n = m_l} (Lm_l | m_p m_n)_{L_p L_n} \varphi_{L_p m_p} \chi_{L_n m_n}, \quad (8)$$

where φ is a function of the coordinates of the odd proton alone, and χ a function of the coordinates of all the other particles. The expansion coefficients $(Lm_l | m_p m_n)_{L_p L_n}$ can be derived by arguments from group theory*. With the expansion (7), the Q 's take the form

$$Q_{Lm_l} = \sum |b_{Lm_l m_s; L_p L_n}|^2 Q_{Lm_l; L_p L_n} + 2 \Re \sum b_{Lm_l m_s; L_p \pm 2, L_n}^* b_{Lm_l m_s; L_p L_n} q_{L_p}^{\pm}, \quad (9)$$

with diagonal elements $Q_{Lm_l; L_p L_n}$ with respect to the $\Psi_{Lm_l; L_p L_n}$ and non-diagonal elements $q_{L_p}^{\pm}$ corresponding to a change from $L_p \pm 2$ to L_p . Again, the contributions from the latter elements can be shown to be small owing to compensation of terms of both signs. The diagonal elements, according to (8), are given by

$$Q_{Lm_l; L_p L_n} = \sum_{m_p + m_n = m_l} |(Lm_l | m_p m_n)_{L_p L_n}|^2 Q_{L_p m_p}, \quad (10)$$

with $Q_{L_p m_p}$ defined by (2). The average $\text{av } r^2$ occurring in this last quantity can be estimated, e.g. by assuming the odd proton to move in some fictitious central field, as usual in the quasi-atomic model; the order of magnitude of $\text{av } r^2$ is, of course, that of the square of the nuclear radius

$$R^2 = r_0^2 A^{2/3} \approx 2A^{2/3} \cdot 10^{-26} \text{ cm}^2.$$

They are given, e.g., by E. CONDON and G. SHORTLEY, *op. cit.* Chapt. III, 14.

It will be observed that the $Q_{Lp m_p}$ occurring in the average quadrupole moment (10) of a proton of given orbital momentum in the nucleus may in general be of both signs, so that, in contrast to the case of an isolated proton, the average $Q_{L m_l; L_p L_n}$ can be positive as well as negative. However, if we make the assumption, consonant with the droplet model (A2.22-22), that all $|b_{L m_l m_s; L_p L_n}|^2$ are approximately the same, the overall averages $Q_{L; m_l=L}$ turn out, on actual calculation, to be always negative, though smaller in absolute value than the single proton moment (1). As regards the average $Q_{J+\frac{1}{2}}^-$, calculation shows that it is related to $Q_{J+\frac{1}{2}}^+$ by the simple equation

$$Q_{J+\frac{1}{2}}^- = \frac{2J-5}{2J+1} Q_{J+\frac{1}{2}}^+ \quad (11)$$

so that the total contribution corresponding to $L = J + \frac{1}{2}$ in (5) is

$$\frac{(2J-1)(J+2)}{(2J+1)(J+1)} Q_{J+\frac{1}{2}}^+; \quad (12)$$

the coefficient of $Q_{J+\frac{1}{2}}^+$ is always positive, except for $J = \frac{1}{2}$, when it vanishes, as it should (A2.13). Welles' final results are as follows:

J	$Q/\text{av } r^2$
$\frac{3}{2}$	$-0.057 (1 + 0.72 c_{J+\frac{1}{2}} ^2)$
$\frac{5}{2}$	$-0.14 (1 + 0.14 c_{J+\frac{1}{2}} ^2)$
$\frac{7}{2}$	$-0.19 (1 + 0.31 c_{J+\frac{1}{2}} ^2)$
$\frac{9}{2}$	$-0.27 (1 + 0.2 c_{J+\frac{1}{2}} ^2)$

(13)

In particular, one gets

$$\begin{aligned} \text{for Cu: } Q &\approx -0.05 (1 + 0.72 |c_{J+\frac{1}{2}}|^2) 10^{-24} \text{ cm}^2 \\ \text{for Bi: } Q &\approx -0.33 (1 + 0.2 |c_{J+\frac{1}{2}}|^2) 10^{-24} \text{ cm}^2. \end{aligned} \quad (14)$$

according to these figures, the observed negative quadrupole moments might well be due to the motion of a single proton around a spherically symmetrical core.

A2.322. Quadrupole moments of light nuclei. If we adopt quasi-atomic model representations of the ground states of light nuclei, such as those given in table A2.252-1 for the odd mass $2p$ shell nuclei and in A2.261 for the unstable odd nuclei of this class, we can compute accurately the quadrupole moments of such nuclei. This has been done by WELLES [42]. Far from being negligible, the cross-terms analogous to the $q_{Lp}^\pm$ of formula (9) above are often predominant; but this is a feature of the quasi-atomic model, connected with the small number of component wave-functions entering into the representation of the ground state. Configurations symmetrical with respect to particles and "holes", but corresponding to the same value of the angular momentum yield, of course, quadrupole moments of opposite signs and nearly equal absolute values (the difference arising only from the slightly unequal $\text{av } r^2$): these circum-

stances occur for the pairs ${}^9\text{B} - {}^{11}\text{B}$, ${}^9\text{Be} - {}^{11}\text{C}$. Table A2.322 summarizes the results obtained by Welles:

A2.322. Calculated quadrupole moments of light nuclei					
Nucleus	J	Q 10^{-26} cm^2	Nucleus	J	Q 10^{-26} cm^2
${}^7\text{Li}$	$\frac{3}{2}$	-2,7	${}^7\text{Be}$	$\frac{3}{2}$	-5,7
${}^9\text{B}$	$\frac{3}{2}$	-1,2	${}^9\text{Be}$	$\frac{3}{2}$	3,4
${}^{11}\text{B}$	$\frac{3}{2}$	1,3	${}^{11}\text{C}$	$\frac{3}{2}$	-3,9
${}^{13}\text{N}$	$\frac{1}{2}$	0	${}^{13}\text{C}$	$\frac{1}{2}$	0
${}^8\text{Li}$	2?	1,4	${}^8\text{B}$	2?	7,6
${}^{12}\text{N}$	0?	0	${}^{12}\text{B}$	0?	0

A2.33. Quadrupole moments of stable odd nuclei. Like the deuteron, the other stable odd nuclei should exhibit quadrupole moments, increasing in magnitude as the mass number increases, owing to the increasing amount of D (and perhaps P) state entering into the representation of the ground state (A2.26). The expected order of magnitude would be roughly $10^{-27} \dots 10^{-26} \text{ cm}^2$ (as for other light nuclei; cf. table A2.322). Hitherto, we have only some indication about the quadrupole moment of ${}^{14}\text{N}$ (A2.31).

APPENDIX III

STRONG COUPLING THEORY OF NUCLEAR FORCES

A3.0. For a first introduction to the subject of strong coupling theory, PAULI's booklet *Meson theories of nuclear forces* [46] can be recommended; further, section 4 of WENTZEL's article *Recent research in meson theory* [47] gives an authoritative survey of the question, with full references to the original papers. The following notes have no other purpose but to mention the main points of interest.

A3.1. Excited states of nucleons

A3.11. Theory. The Hamiltonian of a single nucleon at rest consists of a term describing the pure meson field created by it, and another expressing the interaction between this field and the nucleon. The latter contains a source density involving a spatial distribution function $D(P)$ (1.35) and some operator depending on the spin and isotopic variables $\vec{\sigma}, \tau$ of the nucleon. Likewise, the operator defining the angular momentum of the nucleon is of the form

$$\vec{s} = \frac{1}{2} \vec{\sigma} + \vec{s}_{\text{field}}, \quad (1)$$

the term $\vec{s}_{\text{field}}$, which represents the angular momentum of the meson field, being a function of the field variables alone; a similar operator in isotopic space may be introduced:

$$\mathbf{t} = \frac{1}{2} \tau + \mathbf{t}_{\text{field}}, \quad (2)$$

by means of which the charge of the nucleon (in units e) is expressed as $\frac{1}{2} - t_3$.

The fundamental problem consists in finding a canonical transformation which reduces the Hamiltonian to the diagonal form. More generally, we may assume the field originally to include also free mesons: the transformation in question then effects a separation between the proper meson field of the nucleon and that representing the free mesons. The eigenvalues of the total Hamiltonian are obtained in the form of an expansion with respect to decreasing powers of the coupling constant g (1.35); a constant term in g^2 is interpreted as the self-energy of the nucleon; terms independent of g pertain to the "free" meson field, distorted by the presence of the nucleon, and thus include a description of the scattering of mesons by the nucleon; finally, a term in g^{-2} occurs, representing the excitation energy of the nucleon due to the inertia of its spin and charge. This last term has the simple form of the proper energy of a symmetrical top, such that the components of $\vec{s}$ correspond to the projections of the angular momentum of the top on axes fixed in space, while those of $\mathbf{t}$ play the

same part as the projections of the angular momentum on axes rigidly fixed in the top. Denoting by j the quantum number of the angular momentum ($j = \frac{1}{2}, \frac{3}{2}, \dots$) and by n an eigenvalue of its projection t_3 on the "body-fixed" axis 3 ($|n| \leq j$), we obtain the energy values

$$E_{j,n} = \text{const} + \frac{1}{2} \varepsilon [2j(j+1) - n^2] \quad (3)$$

in the case of a charged meson theory (8.31), and

$$E_j = \text{const} + \frac{1}{2} \varepsilon j(j+1) \quad (4)$$

for a symmetrical or neutral theory. In these formulae, $\varepsilon \equiv \hbar^2/I$, I being defined by (1.35–16); the exact relation of the symbol g to the source constants of the meson fields (15.31) depends on the types or mixtures (15.32) of fields assumed.

A3.12. Scattering of mesons by nucleons. It was first pointed out by HEISENBERG [39] that the inertia of the spin and charge of a nucleon would have a considerable effect in reducing the scattering cross-section for mesons, especially for high meson energies, i.e. high frequencies of the meson field. While ordinary perturbation theory, when this inertia effect is neglected, leads to a cross-section increasing as the square of the meson energy, the inclusion in the calculation of the proper field of the nucleon gives rise to an extra factor which brings the cross-section down to a finite limiting value for very large meson energies. It must be stressed, however, that even without regard to spin and charge inertia, a similar modification of the scattering cross-section at high energies can be achieved by an improvement of the perturbation theory, enabling it to take account of the damping reaction of the meson field on the nucleon.

A3.13. Electromagnetic properties of nucleons. The proper meson field of a nucleon gives rise to an anomalous magnetic moment, and also to a spread of the charge distribution around the nucleon, leading to an additional electrostatic self-energy. These effects have been carefully studied by HOURIET [45]. In first approximation, one finds that the anomalous magnetic moments of proton and neutron would be equal and opposite, while the mass difference between proton and neutron as a result of the charge spread would have the wrong sign; and these defects cannot be remedied by the correction terms appearing in higher approximation.

A3.2. Properties of the deuteron

A3.21. Static interaction. The strong coupling form of static interaction between two nucleons can be derived by applying the method of canonical transformation mentioned above (A3.11) to the Hamiltonian of a pair of nucleons at rest a certain distance r apart, interacting with the meson fields: the interaction potential appears among the terms proportional to g^2 (i.e. of second degree with respect to the various source constants). On symmetrical meson theory, the result can be expressed

very simply with the help of the transformation matrix effecting the transition from body-fixed axes to axes fixed in space*; in our notation, the elements of this matrix are conveniently represented by a set of three vectors $\vec{e}$ in isotopic space, such that

$$\vec{s} = \vec{e} \vec{t} \quad , \quad \vec{t} = \vec{e} \vec{s}. \quad (1)$$

It is then found that the operators $\tau^{(1)} \tau^{(2)} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$ and $\tau^{(1)} \tau^{(2)} \mathcal{D}^{(12)}$ must be replaced (apart from a numerical factor) by $\vec{e}^{(1)} \vec{e}^{(2)}$ and

$$(\vec{e}^{(1)} \vec{x}_0) (\vec{e}^{(2)} \vec{x}_0) - \frac{1}{3} \vec{e}^{(1)} \vec{e}^{(2)},$$

respectively. Physically, this entails an intimate interdependency of the spin and charge degrees of freedom, which would result in a much more complex mixing of states of different orbital momenta and multiplicity than on the weak coupling scheme. In fact, the operators $\vec{e}^{(1)}$, $\vec{e}^{(2)}$ being given by their representation with respect to the individual quantum numbers

$$j^{(1)}, m^{(1)}, n^{(1)} \quad ; \quad j^{(2)}, m^{(2)}, n^{(2)},$$

one has to go over to the representation in terms of the quantum numbers J, K, L, I ; $j^{(1)}$, $j^{(2)}$ of the two-nucleon system, defined as follows: J, K are the quantum numbers of the *spin* angular momentum and isotopic variable, while L, I are those of the orbital and total angular momentum:

$$(\vec{s}^{(1)} + \vec{s}^{(2)})^2 = J(J+1), \quad (\vec{t}^{(1)} + \vec{t}^{(2)})^2 = K(K+1), \quad \text{etc.}; \quad (2)$$

one has

$$\begin{aligned} |j^{(1)} - j^{(2)}| &\leq J, \quad K \leq j^{(1)} + j^{(2)} \\ |J - L| &\leq I \leq J + L. \end{aligned} \quad (3)$$

Since there is now an infinite sequence of J -values corresponding to the isobaric states $j^{(1)}$, $j^{(2)}$, a state with a given value of I will involve a mixture of an infinity of states with different L and J ; e.g. the ${}^3S$ and ${}^1S$ states of the deuteron will actually be mixtures of the respective types

$${}^3S + {}^3D + {}^7D + {}^7G + {}^{11}G + \dots \quad \text{and} \quad {}^1S + {}^5D + {}^9G + \dots$$

A3.22. The S states of the deuteron. In order to discuss quantitatively the bearing of this situation on the account of the known properties of the deuteron, VILLARS [46] has transposed to the strong coupling case Rarita and Schwinger's schematic treatment of the distance dependence of the potential (16.2); i.e. he adopts for the distance dependence of the central and non-central parts of the potential, characterized by the operators in $\vec{e}$ and $\vec{x}_0$ given above, wells of different depths and equal widths. The constant ε occurring in the expression (A3.11-4) for the excitation energies

* See, e.g. H. CASIMIR, *Rotation of a rigid body in quantum mechanics*, thesis Leiden 1931, p. 7-11, 44 sq., 50 sq.

of the isobaric states is a further parameter, whose value may be varied within certain limits. In fact, we may write

$$\varepsilon = \eta M_m, \quad \eta \approx \frac{\hbar}{g^2} a \kappa, \quad (4)$$

and the condition (1.35-17) for strong coupling yields

$$\eta \ll 1. \quad (5)$$

Villars determines, for chosen values of ε and the potential width D , the potential depths required to account for the binding energy and quadrupole

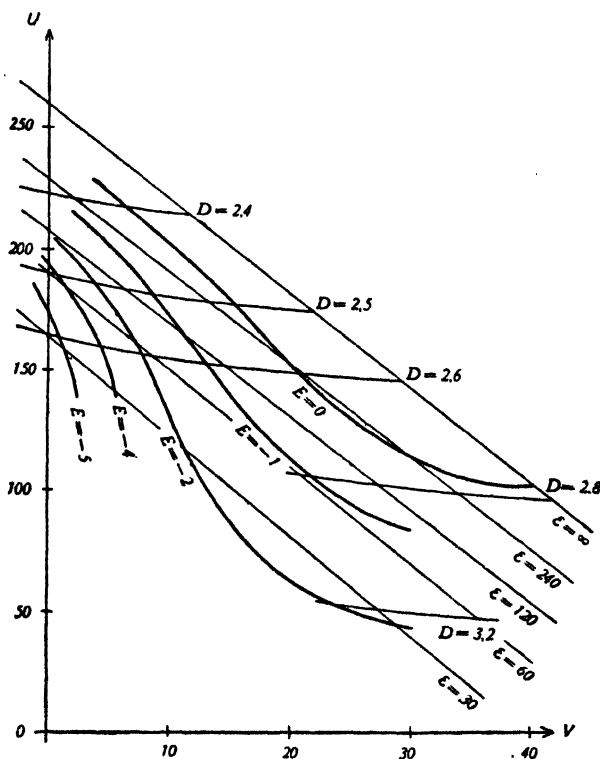


Fig. A3.22. Values of potential depths for different widths D and isobar excitation energies ε , according to VILLARS [46]. Villars' V and U correspond to $3J$ and $9aJ'$ respectively, in the notation of (16.21-1). The corresponding values of the energy E of the ${}^1S$ state of the deuteron are also indicated on the diagram. All energies are expressed in MeV, the width D in units 10^{-13} cm.

moment of the deuteron; for this purpose, he restricts himself to a mixture ${}^3S + {}^3D + {}^7D$ and verifies by control calculations that this approximation is permissible. He might have added the requirement of accounting for the magnetic moment, which would have fixed the width D for every assumed value of ε . From the figures he gives for the amounts of ${}^3D$ and ${}^7D$ admixtures, it would seem that, whatever the ε -value assumed, the

potential width should not be far from the value $D \approx d$ of Rarita and Schwinger's theory (which corresponds to $\varepsilon = \infty$); moreover, as we shall see presently (A3.3), the first saturation requirement implies that the distance-dependent factor V of the central potential operator $\vec{e}^{(1)} \vec{e}^{(2)}$ must be positive (corresponding to an attractive central potential in the ground state), and this condition puts a lower limit to the width, for which values not much below d are obtained: they vary from $2.6 \cdot 10^{-13}$ cm to $2.3 \cdot 10^{-13}$ cm for ε varying from 30 MeV to ∞ . As regards the potential depths, that pertaining to the non-central part of the interaction hardly shows any dependence on the value of ε : just a slight increase with decreasing ε ; the depth of the central potential, on the other hand, decreases markedly with decreasing ε : for $\varepsilon = 30$ MeV, e.g., it is less than half the Rarita-Schwinger value pertaining to $\varepsilon = \infty$. All these features are clearly illustrated by fig. A3.22, reproduced from Villars' paper.

Having fixed the potential depths, Villars proceeds to calculate the energy E of the $^1S + ^5D$ state. The ^5D admixture is found to depress this state to a considerable extent, so that it generally becomes a state of binding unless very high values of ε are assumed. In fact, the accompanying diagram shows that a virtual level cannot be obtained for any ε below ≈ 200 MeV. Since this figure is far larger than the extreme limit of ≈ 100 MeV above which the condition (4), (5) for strong coupling is no longer fulfilled, the outcome of this investigation is to show rather strikingly that the actual situation is much nearer the weak than the strong coupling case.

A3.23. Fast neutron scattering by protons. Even when one disregards the effect of admixture on the ^3P states, one finds that the effective potentials in the $^3P_{0,1,2}$ substates have radically different properties according to the strength of the coupling, i.e. the value of ε . In comparison with the Rarita-Schwinger case, the effect of a finite value of ε is to depress the effective potentials of the three substates, with the result that both the $^3P_0$ and $^3P_2$ potentials become attractive and the scattering ratio A (8.33–23) is accordingly decreased; for $\varepsilon = 180$ MeV and 14 MeV neutrons, Villars finds $A = 0.97$.

A3.24. Charge independence of nuclear interactions. It might be thought that a strong coupling theory could give rise to charge-independent interactions even with charged meson fields only (8.31). But an explicit calculation by JOST [46] has established that the approximate equality of like and unlike particle interactions cannot be obtained on this basis.

A3.25. Excitation of isobaric states. Processes involving a sufficient amount of energy should be expected to lead to excitation of isobaric states of the participating nucleons. Some instructive estimates of the effects to be expected have been published. JAUCH [46] discusses the *inelastic photo-dissociation of the deuteron*. Assuming the first state of excitation of the

nucleon to lie at about 45 MeV, he finds that the ratio of the photoelectric cross-sections (the only ones of importance for high energies) for the inelastic and elastic disintegrations has a maximum of about 0.07, which is reached for an initial γ -ray energy of 87 MeV. LOPES [46, 47] has calculated, upon various assumptions about the energy of the first excited state, the *inelastic scattering cross-section of very fast neutrons by protons*; he finds for the ratio of this cross-section to that for elastic scattering values of the order 10^{-3} as soon as the excitation energy of the first isobaric state is of the order of a few hundred MeV.

A3.3. Saturation requirements

The problem of the saturation of the nuclear interactions has been studied on strong coupling theory especially by COESTER [44]. His treatment is based on the approximation of the individual model; further, he replaces the non-central part of the potential by its average over all angles, so that he deals essentially with an effective central potential of the general form $J(r)\mathcal{O}$, $\mathcal{O}$ being the operator which embodies the spin and isotopic variables: in a symmetrical theory, we have (A3.21) $\mathcal{O} \equiv \vec{e}^{(1)} \vec{e}^{(2)}$; in a neutral theory, $\mathcal{O} \equiv \vec{e}_3^{(1)} \vec{e}_3^{(2)}$.

The *first saturation requirement* (11.13) is fulfilled by all theories for which $J(r) > 0$. Coester's proof of this general saturation condition may be illustrated by the simplest case of a neutral theory. In this case, the operator $\mathcal{O}$ is the cosine of the angle α between the two unit vectors $\vec{e}_3^{(1)}, \vec{e}_3^{(2)}$ symbolizing the spin states of the interacting nucleons. At small distances, i.e. $|J(r^{(12)})| \gg \epsilon$, the lowest stationary state of the system of two nucleons is obtained by minimizing the potential energy $J(r) \cos \alpha$, i.e. for $\alpha = 0$ if $J < 0$ and for $\alpha = \pi$ if $J > 0$. For a system of A nucleons, the former possibility leads to a "freezing" of all vectors $\vec{e}_3^{(i)}$ parallel to each other and, consequently, to a predominance of ordinary non-saturation forces. If, however, $J > 0$, the vectors $\vec{e}_3^{(i)}$ will tend to "repel" each other and to distribute themselves uniformly over all directions: the resulting average of the ordinary part of the total potential energy therefore vanishes, at any rate in the approximation of the Fermi gas model.

The *second saturation requirement* presents a special problem owing to the existence of isobaric nuclei of arbitrary charge: it must be examined whether this might not lead to a greater stability of nuclei in which a certain number of neutrons or protons have been replaced by isobars of negative charge, thus leading to a smaller charge number than expected for a system of neutrons and protons. The problem is discussed by Coester in the Fermi gas model approximation; he treats the limiting cases of both large and small ϵ [i.e. ϵ large or small compared with the maximum kinetic energy E_m of a nucleon (9.41-5)], but finds it possible to interpolate

between them, so as to obtain a rough idea of the behaviour of the total energy over the whole range. Neglecting Coulomb energy, one finds that no stable nucleus will exist if the values of g and ε are too small; in fact, we must have, in the first place,

$$\frac{g^2}{\hbar} \approx 5 \frac{M_m}{M}; \quad (1)$$

nuclear radii of the right order of magnitude are then obtained for

$$\varepsilon \approx 10 \text{ MeV}. \quad (2)$$

The calculation of the Coulomb energy shows that condition (1) is also compatible with the empirical dependence of the charge number Z on the mass number A . However, the approximation of the Fermi gas model, on which these conclusions are based, is far from reliable.

ADDENDA ET CORRIGENDA

ADDENDA ET CORRIGENDA

Pitié pour nous qui combattons toujours aux frontières
De l'illimité et de l'avenir

Pitié pour nos erreurs pitié pour nos péchés

GUILLAUME APOLLINAIRE, *Calligrammes*

Besides correcting the mistakes that came to my notice, I have endeavoured, in the following, to summarize the contents of the latest publications relating to the various sections of the book. This was no easy task, as it had to include, among others, an account of the new discoveries of different kinds of mesons and of the bearing of these discoveries on our conceptions of a field theory of nuclear forces. The most important additions are distinguished by a mark ✱; some of the additional matter I have found convenient to present in a *new subsection*, marked in the same way. Cross-references to any part of the *Addenda* will be preceded by the word "ad", e.g. (ad 1.332). The greatest part of the publications covered by the *Addenda* are contained in vol. 72 and 73 of the *Physical Review* and vol. 160 and 161 of *Nature*.

1.132. *Existence of the neutrino.* A very interesting survey of the whole problem has been published by H. CRANE (*Rev. mod. Phys.* 20, 278. 1948). The study, by B. WRIGHT (*P.R.* 71, 839. 1947), of the recoil nuclei from the *K*-capture process of ^{107}Cd was somewhat inconclusive. More interesting are the experiments of R. CHRISTY *et al.* (*P.R.* 72, 698. 1947) on the β -decay of ^8Li . The product nucleus ^8Be disintegrates into two α -particles, whose cloud chamber tracks may be studied: from the angle between those tracks and from their ranges the existence of a recoil momentum and an estimate of its magnitude can be inferred. The examination of 217 cases, including 28 in which the electron track was also visible, tended on the whole to favour the neutrino idea: no marked correlation was found between the directions of emission of the electron and the neutrino. Still more recently, C. SHERWIN has developed a much improved technique, based on the production of monoatomic layers of the β -emitter under investigation; this makes possible a more detailed quantitative study of the β -process. The electrons and recoil nuclei emitted in selected directions are suitably detected, and from the time of flight of the recoil nucleus its momentum is deduced. In this way, curves of distribution of recoil momenta are obtained for different values of the angle between the electron and the recoil nucleus. Experiments have been carried out with ^{32}P (*P.R.* 73, 216. 1948) and ^{90}Y (*P.R.* 73, 1173. 1948). In both cases the analysis of the results shows that there is no conservation of momentum for the electron and the product nucleus only. Sherwin further tries to derive from his data

a definite law of correlation between the directions of emission of electron and neutrino; but this part of his analysis is not very convincing.

※ 1.14. *Anomalous magnetic moment of electron.* From precise comparisons of the atomic Landé factors pertaining to certain states of the Ga and Na atoms, P. KUSCH and H. FOLEY (*P.R.* 72, 1256. 1947; 73, 412. 1948) * drew the important conclusion that *the intrinsic magnetic moment of the electron differs from the Bohr magneton by a factor 1,0012*. This anomaly has found a remarkable theoretical interpretation through the new development of quantum electrodynamics initiated by the discovery of a small anomaly in the term values of the hydrogen spectrum. J. SCHWINGER (*P.R.* 73, 416. 1948) pointed out that the electro magnetic radiation field produced by an electron in an external field should have a small finite effect on the coupling of the electron with this external field, and he showed that both the anomalies mentioned above were of this type. In particular, for the *radiative correction* $\delta\mu_e$ to the *intrinsic magnetic moment* μ_0 of the electron he derived the value

$$\delta\mu_e = \frac{\pi}{2} \cdot \frac{e^2}{\hbar} \cdot \mu_0 = 0,001162 \mu_0,$$

in excellent agreement with Kusch and Foley's results.

A consequence of the existence of the magnetic radiative anomaly is that the values of those *nuclear magnetic moments* which were determined in relation to that of the electron should be increased by the same factor. In most cases, however, this correction falls within the margin of uncertainty of the measurements. Only for the very precisely determined magnetic moments of the proton, neutron and deuteron does it lead to significant changes. The figures collected in table U.22 must be replaced (there and in the formulae quoted in the last column of the table) by the following:

$$\begin{aligned}\mu_p &= 2,7928 \\ \mu_n &= -1,9125 \\ \mu_d^0 &= 0,8803 \\ \mu_d &= 0,8575.\end{aligned}$$

In Appendix II, and especially in table A2.21-2, account has been taken of the correction so far as was thought necessary.

The small modification imposed by this correction upon the highly accurate values of μ_p and μ_d has removed a puzzling discrepancy that had arisen between the results of very precise measurements of the hyperfine doublet separations for hydrogen and deuterium and the values of these separations calculated by means of the uncorrected magnetic moments (J. NAFE and E. NELSON, *P.R.* 73, 718. 1948 **). A still more exacting test

* More detailed account: *P.R.* 74, 250. 1948. See further J. McNALLY, *P.R.* 73, 1130. 1948, who applied the same method to the Ca II spectrum, with a similar result.

** This paper contains all previous references.

is provided by the comparison of the empirical ratio of the hyperfine doublet separations with that deduced from the ratio of the magnetic moments, which has lately been determined much more accurately than either μ_p or μ_d (A. ROBERTS, *P.R.* **72**, 979. 1947; F. BLOCH, E. LEVINTHAL and M. PACKARD, *P.R.* **72**, 1125. 1947; F. BITTER *et al.*, *P.R.* **72**, 1271. 1947):

$$\mu_p : \mu_d = 3,257195 \pm 0,00002.$$

In order to obtain complete agreement, it is then necessary (A. BOHR, *P.R.* **73**, 1109. 1948) to take into consideration the asymmetry of the magnetic interaction of the electron with the proton and the neutron in the deuterium atom.

* 1.33. *New discoveries about mesons.* When this subsection was handed over to the printer, in the spring of 1947, the exhortation to "be prepared for the eventuality that other kinds of mesons ... may also play a part, perhaps even the main part, in bringing about nuclear interactions" was by no means intended as a rhetorical flourish. It was inspired by the uncertainty arising from the unexpected behaviour of negative mesons slowed down in light materials (1.332). Shortly afterwards it indeed became clear, as will be more fully explained below (ad 1.332), that the mesons observed under usual conditions had only a very weak interaction with nucleons and could not be made responsible for the forces between them. At the same time, however, a succession of spectacular discoveries revealed an unsuspected wealth of particles with different masses, intermediate between those of electron and nucleon, which offered new possibilities of interpretation of the nuclear field.

The study of the tracks produced in photographic emulsion (6.42) by cosmic radiation at high altitudes led Powell and his collaborators* to the following general conclusions: in nuclear explosions produced locally by the impinging radiation, charged particles of mass roughly 300 m can be emitted, of which some, the σ -mesons, are observed to produce nuclear explosions, showing as "stars" in the emulsion (12.30), while others, the π -mesons, decay with an extremely short life-time into a somewhat lighter charged particle, the μ -meson, and another neutral particle (we shall call it a μ^0 -meson) which must have a finite mass again somewhat smaller than that of the μ -meson. In fact, while absolute mass determinations from the emulsion tracks are difficult**, the ratio of the masses of π^- and μ -meson has been fixed with some accuracy***:

$$M_\pi : M_\mu = 1,65 \pm 0,15.$$

* C. LATTES, G. OCCHIALINI and C. POWELL, *Nature* **159**, 694. 1947 (with H. MUIRHEAD); **160**, 453, 486. 1947; G. OCCHIALINI and C. POWELL, *Nature* **162**, 168. 1948. See also F. FRANK, *Nature* **160**, 525. 1947.

** For σ -mesons, see S. LATTIMORE, *Nature* **161**, 518. 1948. For both π^- and σ -mesons, an estimate $(270 \pm 40) m$ is quoted in G. OCCHIALINI and C. POWELL, *Nature* **161**, 551. 1948 (referred to as O & P).

*** Value quoted in O & P. The method is described by C. LATTES, G. OCCHIALINI and C. POWELL, *Proc. Phys. Soc.* **61**, 173. 1948.

The evidence is thus compatible with the assignment of a mass of about $200 m$ to the μ -meson (identified with that hitherto observed in cosmic radiation) and of $(330 \pm 30) m$ to the π -meson; the mass of the μ^0 -meson is then estimated from the energy balance of the decay process at about $100 m$. It is natural to regard the π - and σ -mesons as particles of the same kind, with a positive and a negative charge respectively: before it has time to decay, the negative σ -meson is captured by a nucleus and thus induces a nuclear explosion.

A brilliant confirmation of these results was obtained when the same phenomena could be reproduced at Berkeley with much greater intensity by bombarding targets of various materials with α -particles from the large cyclotron. By using the emulsion technique, it was found that when the energy of the α -particles exceeded 300 MeV , π - and σ -mesons were produced in nuclear collisions with rapidly increasing yield*. The application of magnetic deflection, besides establishing definitely the sign of the charge of the π - and σ -mesons, made possible a much more accurate determination of their mass; a value

$$M_{\pi} = (313 \pm 16) m$$

was found, from which one derives

$$M_{\mu} = 200 m \quad , \quad M_{\mu^0} = 90 m.$$

Obviously the π -mesons** are strongly coupled to the nucleons and must essentially contribute to the nuclear interactions; their mass gives a quite acceptable value for the range of the forces. In fact, a mass value around $300 m$ corresponds more closely to the range derived from proton-proton scattering experiments (7.13) than would a mass of $200 m$. It is true, on the other hand, that the lower mass value seems to be favoured by other evidence derived from the binding energies of light nuclei (14.22), but this is of very little weight indeed; evidence from the proton-neutron system (8.33, 8.34) is inconclusive, but certainly not incompatible with a mass value $\approx 300 m$. Precisely how the nuclear forces are brought about by the π -mesons will primarily depend on their spin, about which we have no experimental indication whatever. Much the simpler course, for the time being, is to *assume that they have integral spin* (0 or 1). It is then possible to take over all the discussions bearing on "mesons" in the text of the book, except that of subsection 15.35; the only change being in the mass value. In particular, the charge independence of the nuclear interactions will then require the existence of neutral π -mesons (8.31), which would presumably be highly unstable with multiple photon emission (8.311).

The assumption that π -mesons have integral spin does not commit us to any definite value for the spin of the μ - and μ^0 -mesons. Indeed, there

* A preliminary account of the Berkeley results is given in O & P.

** Henceforth, we shall call π -mesons all those of the kind to which both π - and σ -mesons belong, whether they be positively or negatively charged, or possibly neutral.

is nothing in the known properties of these mesons to give the preference either to an integral or to a half-integral spin; cosmic ray evidence (1.333) only excludes spin 1, but is compatible with either 0 or $\frac{1}{2}$. In the following, we shall treat the case of half-integral spin; but the whole discussion would not be materially altered if we started from the alternative assumption. We shall thus consider the μ^- and μ^0 -mesons as two states (of different charges and masses) of a new kind of particle of spin $\frac{1}{2}$, for which we shall use the name " μ -meson" (just as we use "leptons" for both electrons and neutrinos). In this case, the decay process of a π -meson into a pair of μ -mesons can be treated in much the same way as its decay into a pair of leptons (15.35); we have to introduce a coupling between the π -meson and the pair consisting of a μ^- and a μ^0 -meson, with a set of coupling constants whose order of magnitude we shall represent by $\tilde{g}$. The life-time of the π -meson is given by formulae somewhat more complicated than (15.35–25) owing to the occurrence of the non-vanishing mass of the μ^0 -meson; they have been established for a pseudoscalar and a vector π -meson by C. MARTY and J. PRENTKI (C.R. 226, 787, 1948).

Now, the order of magnitude of the life-time of the π -mesons produced in the Berkeley experiments can be estimated from a consideration of the paths they describe between the moment of their emission from the target and that of their decay in the emulsion; the result is 10^{-8} sec. It is remarkable that this implies for the coupling constants $\tilde{g}$ roughly the same order of magnitude as that which had to be assumed for the coupling constants $\tilde{g}$ between mesons and leptons in order to explain the β -decay by a mechanism involving the meson field as an intermediary between nucleons and leptons (1.322). In view of this coincidence, it is tempting to imagine that the π -mesons are coupled to pairs of μ -mesons and to pairs of leptons in *exactly* the same way (in fact, as if μ -mesons and leptons were different states of a single species of elementary particles of half-integral spin). The mechanism just mentioned for the β -decay could then be maintained in the form discussed in 15.35; of course, all considerations developed in that subsection in connexion with a meson life-time of about $2 \mu\text{sec}$ (which is *that of the μ -meson*) lose any significance and should be ignored. On the other hand, it is just as likely, for all we know*, that there is no direct coupling at all between π -mesons and leptons, and that the β -decay results from a direct coupling between nucleons and leptons, as assumed in Fermi's original theory, and symbolized, as regards the order of magnitude, by a "coupling constant" $g_F \approx g\tilde{g}/\kappa^2$. As to the decay of the μ -meson, it raises a problem by itself, which will be discussed below (ad 1.332) in the light of new evidence.

According to the above scheme, there is an indirect coupling between

* The decay of a π -meson into *leptons* would not be observable with the present emulsion technique.

a nucleon and a pair of μ -mesons, through the intermediary of a π -meson; on account of $\hat{g} \approx \check{g}$, the corresponding coupling constant is of the same order of magnitude g_F as that between nucleons and leptons. Such a weak coupling is just what we should expect from the fact that a negative meson slowed down to the "K-orbit" around a light nucleus has time to decay before being captured by the nucleus; a straightforward estimate (A. LODGE, *Nature* **161**, 809, 1948) of the capture probability on the basis of the theory just outlined agrees even quantitatively with the latest data on negative meson decay (ad 1.332). The smallness of the interaction between μ -mesons and nucleons is also in qualitative agreement with the great power of penetration of cosmic ray (μ -)mesons through matter, as shown by their small scattering cross section in air and the presence of an appreciable number of mesons even deep underground. In fact, the contrast between the large cross section (of the order of nuclear dimensions) for the production of the hard component of cosmic radiation by the primary protons and the 100 times smaller scattering cross section of mesons by nucleons had led to the hypothesis of an intermediate step in the process of μ -meson production even before the discovery of the π -meson (R. MARSHAK and H. BETHE, *P.R.* **72**, 506, 1947; V. WEISSKOPF, *P.R.* **72**, 510, 1947). It is therefore rather puzzling that in the Berkeley experiments μ - and π -mesons seem to be produced in comparable numbers in the target; the interpretation of the results, however, is not yet quite definite.

If, as the above scheme implies, the μ -mesons interact with nucleons only in pairs, the capture process of a charged μ -meson should be accompanied by the emission of a (neutral) μ^0 -meson, carrying away at least 45 MeV energy. This mechanism would explain the empirical fact (cf. O. PICCIONI, *P.R.* **73**, 411, 1948; J. RETALLACK, *P.R.* **73**, 921, 1948) that the μ -meson capture does not give rise to nuclear excitation and explosion.

The π -mesons and properties of cosmic radiation. In view of the discovery of the π -meson, the usually accepted picture of the production of the (μ -)meson component of cosmic radiation has to be altered. The primary protons are expected to give rise, in nuclear collisions, to groups of charged and neutral π -mesons; and the charged component will then decay into charged and neutral μ -mesons. An interesting attempt has been made by G. CHEW (*P.R.* **73**, 1128, 1948) to analyse this mechanism in a more quantitative way. Starting from a schematized model of the multiple emission of π -mesons by excited nuclei, he fixes the parameters entering into the description of the model by comparison with suitably chosen data on the distribution of fast μ -mesons at different altitudes and latitudes; he eventually obtains a remarkably accurate check of his calculated meson distribution in energy at sea-level by the relevant empirical data. He estimates the mean free path of the primary protons for nuclear collisions at ≈ 5 cm Hg, i.e. somewhat more than would correspond to a cross-section of nuclear dimensions; the average multiplicity of π -meson production would be ≈ 5 at high altitude and would increase as the square root of

the proton energy. It is interesting that he finds that only about half the primary energy goes into the production of *charged* π -mesons.

The neutral π -mesons, supposed to be produced in the same nuclear process, would be expected to decay almost immediately into two or more photons, which would start cascade showers of the usual type (8.311). K. GREISEN (*P.R.* 73, 521, 2nd letter. 1948) enumerates a whole series of features of cosmic radiation in qualitative support of this view: the increase of burst production with altitude; the large amount of soft component in giant air showers; certain cloud chamber evidence of electrons emitted in nuclear explosions (as consecutive products of the photon decay of neutral π -mesons); the comparison of the electron with the meson spectrum, revealing the double origin of the former (decay of neutral π -mesons and of charged μ -mesons). He further suggests various experiments capable of throwing light on this aspect of the question.

A rather close estimate of the life-time of the π -mesons can be derived by considering the variation of the number of (μ -)mesons with depth underground; in fact, this distribution reflects the conditions under which these mesons are produced near the top of the atmosphere by decay of π -mesons of very high energies. It can easily be shown (K. GREISEN, *P.R.* 73, 521, 1st letter. 1948) that the occurrence of the π -meson as an intermediate step in the production of μ -mesons by nucleons has the effect of modifying the rate of production by a factor $\left[1 + \frac{R(p)}{z_0}\right]^{-1}$, z_0 being the height of the homogeneous atmosphere and $R(p)$ the range of a π -meson of momentum p , i.e. $R(p) = ct_0 p/M_\pi$, where t_0 is the life-time of the π -meson at rest. From the change of slope in the distribution curve of the μ -mesons with depth, i.e. with the momentum p , one estimates $t_0 \approx 6 \cdot 10^{-8}$ sec. The correctness of this interpretation of the change of slope of the distribution curve is confirmed by the existence of a temperature effect.

1.331. "Heavy" mesons. Some additional evidence of "heavy" mesons with a mass around $1000 m$ has been presented: (a) G. ROCHESTER and C. BUTLER (*Nature* 160, 855. 1947) interpret two events occurring in cloud chamber pictures of penetrating showers as evidence for the spontaneous decay of a neutral and a charged "heavy" meson into lighter mesons; (b) L. LEPRINCE-RINGUET *et al.* (*C.R.* 226, 1897. 1948) have observed in a photographic emulsion an event that must probably be interpreted as the production of a star by a "heavy" meson. It is, of course, not known whether these various instances refer to a single kind of particles, and any speculation as to the place occupied by "heavy" mesons in the complicated constellation of "elementary" particles would be premature.

✱ 1.332. *Behaviour of negative mesons slowed down in light materials* *. The effect discovered by Conversi, Pancini and Piccioni has now been studied for the following materials:

${}^4\text{Be}$: SIGURGEIRSSON and YAMAKAWA [47], VALLEY [47]	${}^{16}\text{S}$: SIGURGEIRSSON and YAMAKAWA [47]
${}^5\text{B}$: NERESON [48]	${}^{26}\text{Fe}$: RASETTI [41], CONVERSI <i>et al.</i> [47], VALLEY [47], NERESON [48]
water : VALLEY [47]	
${}^{12}\text{C}$: CONVERSI <i>et al.</i> [47], VALLEY [47], NERESON [48]	${}^{23}\text{Cu}$: VALLEY [47]
${}^{11}\text{NaF}$: TICHO and SCHEIN [48]	${}^{82}\text{Pb}$: NERESON and ROSSI [42, 43].
${}^{13}\text{Al}$: RASETTI [41], TICHO [47a], VALLEY and ROSSI [48], NERESON [48]	

In all elements from ${}^{16}\text{S}$ onwards no decay of negative mesons has been observed; in particular, Nereson and Rossi's precise measurement of the life-time $t_0 = 2.15 \mu\text{sec}$ refers to the *positive* mesons. By recording cloud chamber pictures of the decaying mesons, VALLEY [47] was able to establish that the mesons concerned are " μ -mesons" of mass $200 m$.

It was discovered by TICHO and SCHEIN [47a, b, 48] and independently by VALLEY and ROSSI [48], that while the life-time of the positive mesons has the constant value t_0 , that of the negative mesons exhibits a rapid decrease with increasing atomic number of the absorber: for NaF, the apparent life-time of the negative particles is $1.33 \pm 0.14 \mu\text{sec}$, for Al it is $0.74 \pm 0.17 \mu\text{sec}$. The natural interpretation of this result is that in the case of the negative mesons, there is competition between spontaneous decay with the life-time t_0 and capture by the nuclei of the absorber. The life-time t_c corresponding to the latter process will be expected to vary rapidly with the atomic number; simple considerations (J. WHEELER, *P.R.* **71**, 320, 1947) show that one would expect $1/t_c \sim Z^4$. The above figures for the apparent life-time $t_{\text{app}} = \left(\frac{1}{t_0} + \frac{1}{t_c} \right)^{-1}$ are in fact compatible with the relation

$$\frac{t_0}{t_c} = \left(\frac{Z}{Z_0} \right)^4, \quad Z_0 \approx 11.$$

The fraction of decaying negative mesons can be measured directly and compared with the calculated value t_{FP}/t_0 ; this check on the consistency of the proposed interpretation comes out satisfactorily in Ticho and Schein's

* The following recent papers will be listed and quoted here according to the system (N. 22) adopted throughout the book:

H. TICHO

1947a: — , *P.R.* **72**, 255.

1947b: — and M. SCHEIN, *P.R.* **72**, 248.

1948: — and M. SCHEIN, *P.R.* **73**, 81.

N. NERESON 1948, *P.R.* **73**, 565.

G. VALLEY

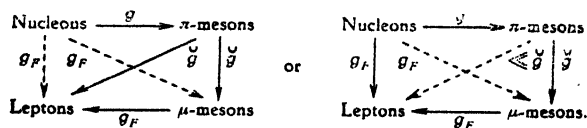
1947: — , *P.R.* **72**, 772.

1948: — and B. ROSSI, *P.R.* **73**, 177.

case, while there remains a discrepancy of uncertain origin in Valley and Rossi's measurements *.

From the theoretical side, unpublished calculations by FERRETTI and by BETHE have made it clear that the ingenious mechanism suggested by Bohr is not sufficient to prevent a negative meson from reaching the "K-orbit" in a time very short compared with the life-time for capture t_c indicated by the experiments. There is therefore no escape from the conclusion ** that the interaction of the μ -mesons with the nuclei is about 10^{12} times smaller than it would be if those mesons were responsible for the nuclear field.

Nature of μ -meson decay. New cloud chamber pictures of decaying μ -mesons, of which only one has hitherto been published (C. ANDERSON *et al.*, *P.R.* 72, 724, 1947), make it quite clear that the balance of energy and momentum not accounted for by the observed decay electron is not simply carried away by a neutrino of small or vanishing mass. In fact, in Anderson's picture, the decay positron had an energy of only 24 MeV instead of the 50 MeV expected on the usual lepton pair hypothesis. Other pictures (if their interpretation is correct) seem to indicate varying energies of the decay electron between 21 ± 6 and 55 ± 15 MeV. This feature points to a process involving more than two particles. In view of the close association of μ^- and μ^0 -mesons in the decay of the π -meson, it is tempting to assume that besides an electron and a neutrino, the decaying μ -meson produces a neutral μ^0 -meson; the so-called decay of the charged μ -meson might thus more appropriately be called a transmutation into a neutral μ^0 -meson with emission of a lepton pair. Theoretically, this process would be described by introducing a direct coupling between the μ - μ^0 -meson pair and the electron-neutrino pair; the coupling parameter, determined by the life-time t_0 of about $2 \mu\text{sec}$, would be of the order of magnitude g_F . The complete net of interactions between the various kinds of particles might then be symbolized by either of the following schemes, in which dotted arrows represent indirect couplings:



The spin of the μ -mesons is left unspecified.

* An alternative theory, according to which the μ -meson would not be captured by the nucleus, but its decay would be stimulated by electromagnetic fields (in particular by the Coulomb field of the nucleus), seems to be disproved by various observations: negative mesons stopped in a cloud chamber are not found to be associated with decay electrons (J. RETALLACK, *P.R.* 73, 921, 1948); likewise, an analysis (O. PICCIONI, *P.R.* 73, 411, 1948) of RASETTI's [41] experiments shows that not all negative mesons stopped in the absorber are accompanied by decay electrons; the meson decay with photon emission to be expected on such a view has not been observed (see below). On the theory of stimulated meson decay, see S. EPSTEIN, R. FINKELSTEIN and J. OPPENHEIMER (*P.R.* 73, 1140, 1948).

** Cf. E. FERMI, E. TELLER and V. WEISKOPF, *P.R.* 71, 314, 1947; E. FERMI and E. TELLER, *P.R.* 72, 399, 1947.

A search for γ -rays that might accompany the decay process of the μ -mesons has been carried out by E. HINCKS and B. PONTECORVO (*P.R.* 73, 257, 1122. 1948). They have established that the decay does not consist of the emission of an electron and a photon *.

2.1. Nuclear radii. From an improved theory of natural α -radioactivity, M. PRESTON (*P.R.* 71, 865. 1947) derives a new estimate of the radii of heavy nuclei, somewhat larger than that given by (2.1-1, 2).

2.21. Attention is called to E. FEENBERG's detailed semi-empirical analysis of the nuclear energy surface (*Rev. mod. Phys.* 19, 239. 1947). This should also be consulted in connexion with sections 3.22, 3.4, 10.3 and 12.1.

3.41. On page 36, the last word of line 6 should be "low".

3.42. On page 38, first line, replace "smaller" by "smaller or larger".

5.121. In formula (5.121-26), the last term on the right hand side must have the *minus* sign.

5.31. Calculation of phases. L. HULTHÉN (*Arkiv* 35A, No. 25. 1948) has further elaborated his variational method. Another method of approximation has been developed by W. RAMSEY (*Proc. Cambr.* 44, 87. 1948).

6.122. It has been pointed out by M. HAMERMESH (*P.R.* 73, 638. 1948) that there is an error in CASIMIR's [40] calculation of the effect of the quadrupole field of the deuteron in accelerating the para-ortho conversion of deuterium dissolved in water; the result given by Casimir is too large by a factor 5.

6.22. Scattering of slow neutrons by protons. The availability of intense sources of slow neutrons monochromatized by "velocity spectrometers" (cf. 6.31) has greatly facilitated scattering cross-section determinations in the range of slow neutron velocities. The variation of the cross-section with velocity can now be studied continuously; in the thermal region the increase of the cross-section with decreasing neutron energy

* An old argument of L. NORDHEIM (*P.R.* 59, 554. 1941) may be revived in this connexion. Nordheim pointed out that the ratio of the intensity of the part of the soft component of cosmic radiation which arises from the decay of the hard (μ -meson) component, to the intensity of the latter can be calculated on cascade theory on the assumption that every meson gives a definite fraction α of its energy to cascade producing particles (electrons or photons); the comparison with experimental data would thus lead to an estimate of α . Assuming that the energy of the meson is equally shared by all its decay products, one would expect $\alpha = 1$ if the decay products are an electron and a photon, $\alpha = \frac{1}{2}$ if they are an electron and a neutral particle, $\alpha = \frac{1}{3}$ if they consist of an electron and two neutral particles, etc. The empirical evidence would seem to exclude definitely the value $\alpha = 1$ in favour of a smaller value; but it is not possible to decide, e.g., between $\frac{1}{2}$ or $\frac{1}{3}$.

owing to the binding of the scattering nuclei can be followed accurately and compared with theory. The cross-section for scattering by free nuclei is actually obtained with neutrons of a few eV energy. Scattering by paraffin has been studied in this way by L. RAINWATER *et al.* (*P.R.* 73, 733, 1948).

The latest value of the proton scattering cross-section for neutrons of zero energy (6.22–32) is that obtained by E. MELKONIAN (*P.R.* 73, 1265, 1948):

$$S = (19,96 \pm 0,2) \cdot 10^{-24} \text{ cm}^2.$$

It must be pointed out that the new measurements of the oxygen cross-section (E. MELKONIAN *et al.*, *P.R.* 73, 1399, 1948) yield

$$S_O = 3,68 \cdot 10^{-24} \text{ cm}^2$$

instead of the higher value quoted in table 6.22–3. This puts Simon's value of the proton cross section, in the same table, at $(14,6 \pm 0,8) \cdot 10^{-24} \text{ cm}^2$.

* 6.31. *Scattering of slow neutrons by molecular hydrogen.* Alvarez and Pitzer's measurements have been extended, with improved technique, by R. SUTTON *et al.* (*P.R.* 72, 1147, 1947). The scattering cross-sections of gaseous ortho- and parahydrogen at 19,5° K have been determined for the whole range of neutron temperatures 10° ... 30° K ($E = 0,8673 \dots 2,386 \text{ meV}$); the results are markedly different from the provisional values (6.31–1) obtained by Alvarez and Pitzer. For 20° K neutrons, e.g., Sutton *et al.* find

$$\begin{array}{ll} \text{for } o\text{H}_2: & 124 \cdot 10^{-24} \text{ cm}^2, \\ \text{for } p\text{H}_2: & 3,97 \cdot 10^{-24} \text{ cm}^2. \end{array}$$

* 6.33. When subjected to the analysis outlined at the end of 6.33, Sutton *et al.*'s results yield a value of the proton scattering cross-section for slow neutrons $S(0) = 19,7 \cdot 10^{-24} \text{ cm}^2$, in excellent agreement with the direct measurements (6.22–32) of this quantity. Because of the smallness of the ratio $(3 \cdot {}^3a + {}^1a)^2 / ({}^3a - {}^1a)^2$, the quantity $S(0)$ depends essentially on S_{ortho} , as a comparison of (6.33–9) and (6.32–6) (in which $S_{p \rightarrow o} = 0$) will show. The agreement just mentioned therefore confers additional reliability to the determination of S_{ortho} . The measurement of S_{para} , however, is more difficult and any error in its determination would mainly affect the value of the relatively small scattering radius ${}^3a$. From the data, one finds

$${}^3a = 0,522 \cdot 10^{-12} \text{ cm}, \quad {}^1a = -2,34 \cdot 10^{-12} \text{ cm}.$$

The singlet scattering radius ${}^1a$ is compatible (cf. table 6.431–1) with a width ${}^1D \approx 2,8 \cdot 10^{-13} \text{ cm}$ for the corresponding potential well, such as one would expect, assuming charge independence (8.1), from the proton-proton scattering data. But the triplet scattering radius ${}^3a$ yields a much smaller width value for the ${}^3S$ effective potential, viz.

$${}^3D = (1,5 \pm 0,4) \cdot 10^{-13} \text{ cm}.$$

One would have to assume a one to two percent contamination of the parahydrogen sample with orthohydrogen to raise the triplet width derived from the experiments to the same value as ^1D .

However, the low ^3D value has received a striking confirmation from quite independent experiments of E. WOLLAN and C. SHULL and their collaborators (*P.R.* 73, 830, 842. 1948) on the diffraction of slow neutrons by crystalline powders. The distribution of the diffracted neutrons will depend on the *scattering amplitudes* characteristic of the nuclei of the atoms constituting the crystal; for hydrogen this scattering amplitude is clearly proportional to $(3 \cdot ^3a + ^1a)$. Now, by combining the results obtained with different suitably chosen crystals, it is possible to assign definite scattering amplitudes to the various nuclei involved. Thus, experiments with sodium containing crystals, including metallic Na and NaH, yield among others the hydrogen scattering amplitude. The resulting value of $(3 \cdot ^3a + ^1a)$ is in complete agreement with that derived from the molecular hydrogen data; combining it with the known value of $S(0)$, one finally gets $^3D = (1,6 \pm 0,2) \cdot 10^{-13}$ cm.

There is of course no *a priori* objection to having effective potential wells of different widths for the ^3S and ^1S configurations of the proton-neutron system. It must be stressed, however, that a value of ^3D as low as that just derived would be difficult to reconcile with the electric quadrupole moment of the deuteron (see ad 16.21).

6.35. J. SCHWINGER (*P.R.* 73, 407. 1948) discusses the polarization of fast neutrons in their passage through matter, owing to the action of the nuclear Coulomb field. Indeed, the electrostatic potential $\mathcal{V}_e$ gives rise to a magnetic interaction (15.22-9) on the moving neutrons

$$\frac{\hbar}{2M^2} \mu_n \vec{\sigma} (\text{grad } \mathcal{V}_e \wedge \vec{p}).$$

6.41, 6.52. The most recent data on neutron scattering and capture cross-sections have been collected by H. GOLDSMITH, H. IBSER and B. FELD (*Rev. mod. Phys.* 19, 259. 1947).

※ 6.413. *Scattering of very high energy neutrons by protons.* When the 180 MeV deuterons from the Berkeley cyclotron strike a target, they produce a narrow beam of high energy neutrons as a result of grazing collisions with the nuclei of the target, in which the proton of the deuteron is stripped off, while the neutron continues on its way. This phenomenon provides us with a powerful beam of neutrons, of well-defined direction and average energy 90 MeV. The total scattering cross-section of various nuclei for these neutrons has been measured by L. COOK *et al.* (*P.R.* 72, 1264. 1947). For hydrogen, the result is $(0,083 \pm 0,004) \cdot 10^{-24}$ cm².

※ 6.42. The angular distribution of the 90 MeV neutrons scattered by paraffin has also been determined in the angular range $\theta = 70^\circ \dots 170^\circ$

(J. HADLEY *et al.*, *P.R.* **73**, 1114, 1948). In this range, the intensity of the scattered beam is found to increase steadily with angle, with some indication of a minimum about $\vartheta = 80^\circ$. The anisotropy parameter

$$A = S(\pi)/S(\frac{1}{2}\pi)$$

is, of the order of magnitude of 3. Preliminary observations indicate that for smaller scattering angles, the variation of the scattered intensity is roughly symmetrical to that just described, with presumably another maximum at $\vartheta = 0$; it is not clear whether this maximum is greater or smaller than that in the backward direction. The consequences of this new evidence for the theory of the nuclear field will be discussed below (ad 8.33, ad 16.22).

6.431. On page 128, line 14 (not counting the table), replace *B* by *BH*.

6.52, see ad 6.41.

6.53. *Deuteron disintegration by electron impact.* The statement about the angular dependence of the effect, following formulae (23) on p. 139, needs correction. The differential cross-section of the electric effect includes an additional term containing the angular factor $Y_2^0(\vartheta)$, which arises from the longitudinal part of the field $\vec{E}_0$. This term does not give any contribution to the *total* cross-section.

The discussion of the magnetic exchange effect by Lubański and Rosenfeld, mentioned at the end, was based on the erroneous expression for K_{magn} given by Pais (8.34). Actually, the magnetic exchange effect is quite negligible.

7.12. To the "second group" of experiments (p. 148) must be added those of J. DEARNBY, C. OXLEY and J. PERRY (*P.R.* **73**, 1290, 1948) with protons of 7 MeV. The scattered protons were collected on photographic plates, the point of entry in the plate determining the scattering angle. In a note added in proof to their paper, the authors announce that preliminary results of new experiments yield larger cross-sections than those they obtain; it is therefore premature to consider the conclusions of the theoretical discussion they give. A detailed account of WILSON *et al.*'s experiments with 14.5 MeV protons has now been published in *P.R.* **72**, 1131, 1947.

As regards the reliability of the experiments of the "first group" (p. 149), it may be noted that the results obtained by Herb *et al.* at 2.1 MeV proton energy have been confirmed by new measurements at 2.08 MeV with the same apparatus (TASCHEK *et al.* [47]). On the other hand, measurements by a similar method, recently announced (W. SLEATER *et al.*, *P.R.* **73**, 1241, 1948), with protons of 2.4 and 3.0 MeV, yield values of scattering intensities at 2.4 MeV larger than Herb's by 2.9 % (while a probable error of only 2 % is claimed).

In table 7.12 (p. 150), first column, line 6 from the bottom, read 1200 instead of 1220.

7.13. An interesting attempt to establish an analytical formula giving the nuclear potential in terms of the phases has been made by C. FRÖBERG (*Arkiv* **34A**, No 28, 1948). However, since the analysis is entirely based on Born's approximation, the scope of the final formula is still too limited for practical applications. A more general iteration procedure has been outlined by E. HYLLERAAS (*P.R.* **74**, 48, 1948).

G. BREIT, A. BROYLES and M. HULL (*P.R.* **73**, 869, 1948) discuss the sensitivity of proton-proton scattering experiments with respect to the distance dependence of the nuclear potential.

The paper by J. LUBANSKI and C. DE JAGER is now published in *Physica* **14**, 8, 1948.

7.131. *Discussion of 10 MeV data.* According to a note by L. FOLDY (*P.R.* **72**, 731, 1947), the theoretical curves of fig. 7.131 correspond to a ^3P potential well of the same depth as the ^1S potential, yielding, when repulsive, a ^3P phase of -1.4° . If we take for the ^1S phase the *theoretical* value of 52.5° , the data, as pointed out by Peierls and Preston, would indicate for the ^3P phase the somewhat lower figure of about -0.8° ; this they try to account for on the assumptions stated in the text.

The 10 MeV results have also been discussed by RAMSEY* on the assumption of the meson type of potential. A ^1S phase of the order of magnitude just mentioned might be accounted for by the potential corresponding to a meson mass of $(270 \pm 30) m$. The most likely choice of the strength of the ^3P effective potential (as given by the "symmetrical" theory, 8.32) then leads to a ^3P phase of the right sign, but of too large an absolute value.

7.14, 12.33. The study of the penetration of a fast nucleon into nuclear matter has now been extended to the case of meson interaction by Dr. ZWANIKKEN. The results are not sensitive enough to the range of the interaction, however, to yield useful information concerning the value of the meson mass.

7.2. On page 157, line 3, after "For white dwarfs", insert: "it may be applied to the boundary layer; in the degenerate core,".

*** 8.33.** *Neutron-proton scattering at very high energies.* The experiments on scattering of 90 MeV neutrons have been the occasion of an unprecedented outburst of theoretical calculations, which does little credit to the system of free enterprise in the realm of physics. Each author having his own pet nuclear potential and set of constants, the result is a mass of unconnected information — *rudis indigestaque moles* — from which some general qualitative features emerge, which might have been obtained with less trouble in a more systematic way. In this subsection we shall only discuss the case of a *central nuclear potential*; the influence of non-central and non-static forces will be treated below (ad 16.22). The following table gives a survey of the scope of the various contributions:

* *Proc. R.S. A* **194**, 228, 1948.

Distance dependence	Range		Isotopic factor *	Neutron energy MeV	Angular distribution given	Reference
	^3S 10^{-13} cm	^1S 10^{-13} cm				
Well	2.8		n, c, s n, c, s n, s s	20...80 83 100 100	for 80 MeV yes $S(\pi):S(\frac{1}{2}\pi):S(0)$ for s $S(\pi):S(\frac{1}{2}\pi)$	CAMAC and BETHE [48] MASSEY <i>et al.</i> [48b] EISENSTEIN & ROHRlich [48] Wu [48a,b]
	2.8	2.8	s	80	$S(\pi):S(\frac{1}{2}\pi)$	Wu [48b]
	2.0	2.8	s	80	id.	
	1.5	2.8	s	80	id.	
		1.8	2.8	n	100	no
Exponential	2		c, s	100	yes	BARKER [48]
	1		c, s	100	yes	
Gauss	1.94	1.94	n, c, s	100	yes	Wu [48a]
	1.39	1.94	s	100	$S(\pi):S(\frac{1}{2}\pi)$	Wu [48b]
	1.04	1.94	s	100	$S(\pi):S(\frac{1}{2}\pi)$	
Meson ** ($M_m=326m$)	1.18		s	80	$S(\pi):S(\frac{1}{2}\pi)$	CHEW and GOLDBERGER [48]
* n = neutral, c = charged, s = symmetrical (15.21). ** Other (not reliable) data in EISENSTEIN and ROHRlich's [48] and Wu's [48a, b] papers.						
F. BARKER 1948, <i>Nature</i> 161, 736. M. CAMAC and H. BETHE 1948, <i>P.R.</i> 73, 191. G. CHEW and M. GOLDBERGER 1948, <i>P.R.</i> 73, 1409. J. EISENSTEIN and F. ROHRlich 1948, <i>P.R.</i> 73, 641 (Erratum <i>ibid.</i> 1411). H. MASSEY, E. BURHOP and T. HU 1948b, <i>P.R.</i> 73, 1403. T. Wu 1948a, <i>P.R.</i> 73, 934. 1948b, <i>P.R.</i> 73, 1132.						

The method used in all cases, except in that of the Gauss potential, is a straightforward numerical calculation of the phases, generally up to $l = 4$; Born's approximation for the higher phases has not been found reliable. A second approximation to Born's formula has been developed by Wu [48a] and applied by him to the case of a Gauss potential.

The total cross section is insensitive to the choice of the distance dependence of the potential and varies very little with neutron energy in the range 80 ... 100 MeV. It also depends little on the choice of the isotopic factor, being largest in a neutral theory and smallest in a symmetrical one. In any case it is of the right order of magnitude, but rather too large. The angular distribution generally* has a minimum in the neighbourhood of $\vartheta \approx \frac{1}{2}\pi$ and a maximum as well in the forward as in the backward direction. But in a neutral theory, the forward maximum is much larger than the

* The case of a neutral interaction of the Gauss type is exceptional in yielding a differential cross-section which decreases monotonically with increasing angle of scattering.

other, while the situation is reversed in a symmetrical or charged theory, in conformity with the "exchange" character of the interaction in these cases; there is no great difference between charged and symmetrical theory. It must be stressed that the presence in all cases of a "secondary" maximum in the theoretical distribution makes the interpretation of the experimental results more complicated; so long as the behaviour of the differential cross-section at small angles is not more quantitatively elucidated, it will be premature to conclude (as is sometimes hastily done) that these results establish the impossibility of a predominantly neutral form of nuclear interaction.

For a more rapid comparison of theory and experiment, the *anisotropy ratio* $A = \mathcal{S}(\pi) : \mathcal{S}(\frac{1}{2}\pi)$ is therefore not sufficient to characterize the distribution; the ratio $\mathcal{S}(\frac{1}{2}\pi) : \mathcal{S}(0)$ ought also to be taken into consideration. At present, however, we have only the empirical estimate $A \approx 3$ at our disposal. This compares with theoretical values of A of the following order of magnitude:

Distance dependence	Neutral	Symmetrical or charged
Well	≈ 10	≈ 100
Gauss	—	≈ 60
Meson ($M_m = 326m$)	—	6,8

Obviously, the assumption of a single range central potential is inadequate. In view of the recent indications (ad 6.33) of a smaller range of the ${}^3S$ effective potential, the effect of decreasing this range has been tried by WU [48b] (for the well and Gauss potentials; see above table): the total cross-section is only slightly increased, but the anisotropy ratio A falls steeply to a value of the order 5 or 6 for the smallest range value considered. However, the assumption of a small triplet range gives rise to difficulties (ad 16.21), and it is therefore gratifying that the non-central part of the ${}^3S$ interaction is found just as effective in reducing the value of A (ad 16.22).

8.34. Table 8.34-1 should be completed by the following one, corresponding to a meson mass $M_m = 296m$, nearer to that of the π -mesons (ad 1.33):

	Φ_{el} 10^{-38} cm^3	Φ_{magn} 10^{-38} cm^2	Φ_{total} 10^{-38} cm^2	H_{bar}	H_{lab}
Neutral theory	10,50	3,19	13,7	0,168	0,190
Symmetrical theory	10,05	3,21	13,3	0,176	0,198

All these results have been obtained by Dr. HULTHÉN with the collaboration of Mr. HANSSON. It will be noticed that the transformation from

the barycentric to the laboratory system of reference has an appreciable effect on the intensity ratio H .

The total photo-disintegration cross-section (table 8.34-2) has now been measured with greater accuracy:

Reference	$h\nu$ MeV	Φ_{total} 10^{-28} cm^2
R. WILSON, C. COLLIE and H. HALBAN, <i>Nature</i> 162, 185. 1948	2.62	16.2 ± 1.0
B. RUSSELL <i>et al.</i> , <i>P.R.</i> 73, 545. 1948	2.76	16.4

12.33, see ad 7.14.

13.11. According to A. HEMMENDINGER (*P.R.* 73, 806. 1948), the ground state of ^8Be has an energy larger than that of two α -particles by $(0,116 \pm 0,010) \text{ MeV}$.

13.211. More data about the excited levels of α -nuclei will be found in HORNYAK and LAURITSEN's [48] survey article.

14.12. The free deuteron scattering cross-section for slow neutrons is $(3,3 \pm 0,2) \cdot 10^{-24} \text{ cm}^2$ (value for 5 eV neutrons), according to L. RAINWATER *et al.* (*P.R.* 73, 733. 1948) (ad 6.22).

The total scattering cross-section of the deuteron for 90 MeV neutrons (L. COOK *et al.*, *P.R.* 72, 1264. 1947; see ad 6.413) is

$$(0,117 \pm 0,005) \cdot 10^{-24} \text{ cm}^2.$$

Theoretical calculations on the scattering of high energy (100 ... 200 MeV) neutrons by deuterons have been carried out by T. WU and J. ASHKIN (*P.R.* 73, 986. 1948), using Born's approximation; insofar as the Coulomb force can be neglected, the results are also applicable to proton-deuteron scattering. They assume a Gauss type of nuclear potential with neutral, charged or symmetrical form of isotopic factor and discuss the angular distribution both for elastic and inelastic scattering. The total cross-sections for 100 MeV neutrons are of the right order of magnitude, larger than the experimental figure on neutral or charged theory, and smaller on symmetrical theory. For 200 MeV neutrons, they become smaller by a factor ≈ 2 . In all cases, they are definitely smaller than the corresponding sums of neutron-neutron and neutron-proton cross-sections (ad 16.22). The formulae for the case of a non-central potential are also given in the paper quoted, but without numerical discussion.

15.34. In table 15.34, the expression given for the scalar field must be replaced by 0. The simplification of the expressions originally given in Rosenfeld's paper for the scalar and pseudovector cases is effected by a canonical transformation of the nucleon variables; see a forthcoming note by LE COUTEUR and ROSENFELD.

15.35. In contradiction to the statement following table 15.35, Sakata's

value of the β -decay constant on pseudoscalar meson theory, given in that table, does not correspond to the most natural choice of the Hamiltonian. As pointed out by E. NELSON (*P.R.* 60, 830. 1941), such a choice would lead to a formula involving the product

$$\left(f_1 - f_2 \frac{M_p + M_m}{M_m}\right)^2 \cdot \left(\check{f}_1 - \check{f}_2 \frac{m}{M_m}\right)^2.$$

Since this contains the same combination of constants as the meson life-time, the exceptional position of the pseudoscalar meson field, discussed at the end of 15.35, would not normally arise. This correction, of course, has now only academic interest.

✱ 16.20, 16.21. The question of how far the assumptions of the Rarita-Schwinger theory, especially that of equal ranges for the central and non-central forces, involve an arbitrary restriction, has been investigated at my request by Mr. DEMEUR. For this purpose, he introduced different ranges for the two kinds of potentials just mentioned and tried to adjust all the parameters of the theory thus extended so as to account for the properties of the ground state of the deuteron. His conclusion is that Rarita and Schwinger's original choice is essentially unique; of course, some slight variations (including unequal ranges) are allowed by the margin of uncertainty of the empirical data, the most sensitive one being the electric quadrupole moment. In particular, such a low value of the range of the central part of the ground state potential as seems to be indicated by the slow neutron interference effects (ad 6.33) would be entirely excluded by the accepted value of the quadrupole moment*.

✱ 16.22. *Scattering of high energy neutrons by protons on Rarita and Schwinger's theory.* We have seen (ad 8.33) that (as might be expected) the assumption of a central nuclear potential is unsuited to account for the observed angular distribution of the scattered neutrons of 90 MeV energy: it yields for the anisotropy ratio a value which is in any case too high and for most types of distance dependence much too high. It is therefore interesting that Rarita and Schwinger's theory, which seems to give a fairly accurate model of the static nuclear forces, including their axial dipole component, shows in this respect a definite trend in the right direction. Considering that a central meson potential gives a value of 1 only about twice too large, it may be conjectured that the inclusion of non-central terms in such a meson theory might achieve agreement with experiment. Leaving this, however, for future examination, I shall only collect here the data obtained on the simple Rarita-Schwinger theory. In this case, they

* This conclusion was actually drawn already by Schwinger in 1941 by a more qualitative argument; see formula (6.12-13).

An investigation similar to Demeur's has just been published by W. GUINDON (*P.R.* 74, 145. 1948).

Reference	Neutron energy MeV	Total cross section			Anisotropy ratio		
		n 10^{-24} cm ²	c 10^{-24} cm ²	s 10^{-24} cm ²	n	c	s
MASSEY <i>et al.</i> [48b]	83	0,234*	0,158*	0,141*	0,58*	3,80*	9,56*
EISENSTEIN and ROHRLICH [48]	100	0,205	—	0,113	—	—	†
J. ASHKIN and T. WU, <i>P.R.</i> 73, 973. 1948	100	0,205	0,205	0,129*	0,31	3,69	10*
	150	0,143	0,143	0,114	0,011	5,43	17,6
	200	0,108	0,108	0,090*	0,030	12,1	35,5
† $\mathcal{S}(\pi) : \mathcal{S}(\frac{1}{2}\pi) : \mathcal{S}(0) = 4,03 : 0,51 : 1,22$							

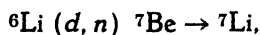
are fortunately coherent, since there was hardly anything else to do but to adopt the parameter values given in Rarita and Schwinger's paper. In the table above the references by authors' names are to the papers quoted in the table ad 8.33; the asterisks distinguish the figures derived by exact calculation, the others having been obtained by Born's approximation.

Eisenstein and Rohrlich point out that decreasing the range of the *non-central* part of the potential has the effect of increasing the total cross-section and decreasing the anisotropy ratio. Both Massey *et al.* and Ashkin and Wu give complete angular distributions; the latter authors give them also for proton-proton scattering (neglecting the Coulomb force).

H. SNYDER and R. MARSHAK (*P.R.* 72, 1253. 1947) have drawn attention to the fact that at such high energies as are dealt with here, the velocity dependent nuclear interactions might begin to become significant; at any rate, no precise comparison of theoretical and experimental results should be attempted without taking them into consideration. An estimate of the scattering cross-section including the effect of the non-static forces may be obtained by applying Møller's relativistic treatment to the meson field description of the interaction between nucleons: this corresponds to a relativistic extension of Born's method. In a special example, the authors quoted show that the result, for the energies considered, may differ by as much as a factor 2 from that of Born's approximation applied to the static interaction.

16.33. A detailed treatment of the ground state of the deuteron on Hulthén's unsymmetrical theory is given by C. FRÖBERG (*Arkiv* 35A, No 17. 1948).

17.51. Addition to table 17.51: D. ZAFFARANS, B. KERN and A. MITCHELL (*P.R.* 74, 105. 1948) have measured the γ -ray from the reaction



using 11,5 MeV deuterons. They find for the excitation energy of ${}^7\text{Li}$ the value $0,474 \pm 0,004$ MeV.

A1.2. The large table of nuclei contains some additional data and references pertaining to the class (b) nuclei (table A1.22) ${}^6\text{He}$, ${}^{14}\text{C}$, ${}^{22}\text{Na}$, ${}^{34}\text{Cl}$, ${}^{38}\text{K}$, ${}^{58}\text{Cu}$. The fact that the decay of ${}^{22}\text{Na}$ leads to an excited state of the product nucleus might perhaps provide an explanation of the anomalous character of the process in this case.

A2.21. *New determinations of magnetic moments.* R. POUND (*P.R.* 73, 112, 1948) gives the following values:

Nucleus	J	μ
${}^{31}_{15}\text{P}$	$\frac{1}{2}$	(+) $1,1314$
${}^{69}_{31}\text{Ga}$	$\frac{3}{2}$	$2,0165$
${}^{71}_{31}\text{Ga}$	$\frac{3}{2}$	$2,5611$
${}^{127}_{53}\text{I}$	$\frac{5}{2}$	$2,8122$

A2.31. Correction to table A2.31: according to new measurements of W. GORDY *et al.* (*P.R.* 74, 243, 1948), the ratio of the quadrupole moments of ${}^{81}\text{Br}$ and ${}^{79}\text{Br}$ is 0,835.

Trivial misprints

- 2.21. P. 24, last line before the table, instead of "accuary" read "accuracy".
- 5.12. P. 70, line 3, on the right-hand side of the second formula (20), the argument of the cosine should be $z - \frac{1}{2} l\pi$.
- 6.21. P. 99, the left-hand side of formula (6.21-4) must be e^{ikz} (not e^{ikr}).
- 6.412. P. 121, table 6.412, in the heading of the last column, instead of "cm" read "cm²".
- 7.12. In the caption of fig. 7.12-1, instead of "HEYDENBERG" read "HEYDENBURG". In the caption of fig. 7.12-2, lines 6 and 7, instead of "according" read "according to".
- 7.131. P. 153, line 8 from bottom, the last word should read "exemplified".
- 8.1. P. 159, last line but one, instead of "on an arbitrary way" read "in an arbitrary way".
- 14.21. P. 302, formula (14.21-3), the binding energy of ${}^5\text{He}$ should be $-27,4$ MeV.

INDEXES

AUTHOR INDEX

Abbreviations

<i>Arkiv</i>	Arkiv för Matematik, Astronomi och Fysik
<i>Cambr. Conf.</i>	Cambridge International Conference on Elementary Particles (Volume I of the Report published by the Physical Society, London, 1947)
<i>C. R.</i>	Comptes Rendus des séances de l'Académie des Sciences de Paris
<i>P. R.</i>	Physical Review
<i>Proc. Cambr.</i>	Proceedings of the Cambridge Philosophical Society
<i>Proc. Cop.</i>	Det Kgl. Danske Videnskabernes Selskabs mat.-fysiske Meddelelser
<i>Proc. Dublin</i>	Proceedings of the Royal Irish Academy
<i>Proc. Jap.</i>	Proceedings of the Physico-Mathematical Society of Japan
<i>Proc. Lund</i>	Det Kgl. Fysiografiska Sällskapet i Lunds Förhandlingar
<i>Proc. R. S.</i>	Proceedings of the Royal Society, London
<i>Sow. Phys.</i>	Physikalische Zeitschrift der Sowjet-Union

A

M. Ageno, see E. Amaldi

J. Allen

1941: — and N. Smith, *P.R.* 59, 618. 8.34

1942: — , *P.R.* 61, 692. 1.14

W. Allen and C. Hurst 1940, *Proc. Phys. Soc.* 52, 501. 6.412

L. Alvarez

1938: — , *P.R.* 54, 609. 6.31

1940a: — and F. Bloch, *P.R.* 57, 111. 1.22

1940b: — and K. Pitzer, *P.R.* 58, 1003. 6.13

E. Amaldi

1936: — and E. Fermi, *P.R.* 50, 899. 6.52

1940: — , D. Bocciarelli and G. Trabacchi, *Ric. sci.* 11, 121. 6.441

1942a: — , D. Bocciarelli, B. Ferretti and G. Trabacchi, *Naturwiss.* 30, 582. 6.42

1942b: — , D. Bocciarelli, B. Ferretti and G. Trabacchi, *Ric. sci.* 13, 502. 6.42

1943: M. Ageno, — , D. Bocciarelli and G. Trabacchi, *Nuovo Cim.* 1, 253 (also *P.R.* 71, 20. 1947). 6.412, 6.413, 14.12, 14.15

H. Anderson 1948, *P.R.* 73, 919. A2.251

C. Anfinson, see A. Solomon

H. Aoki

1939: — , *Proc. Jap.* 21, 232. 6.412, 6.42, 14.12

see S. Kikuchi

N. Arley

1938: — , *Proc. Cop.* 16, No. 1. 6.212, 6.22

1945: — , *Proc. Cop.* 23, No. 7. 1.23

1946a: — , *Physica* 12, 177. 1.23

1946b: — , *P.R.* 70, 975. 1.23

W. Arnold and A. Roberts 1946, *P.R.* 70, 766 (see further *P.R.* 71, 878. 1947). 1.22, 6.12.

L. Van Atta, see F. Myers

B

- E. Bagge**
 1939: — , *Ann. Physik* **35**, 118. 12.32, 12.33, 12.35
 1942: — , *Physik. Z.* **43**, 226. 14.15
 1943: — , *Physik. Z.* **44**, 461. 7.14
- C. Bailey**
 1946: — *et al.*, *P.R.* **70**, 583. 6.411, 6.412
 see R. Nuckolls, R. Taschek
- J. Bardeen**
 1937: — , *P.R.* **51**, 799. 9.42
 1948: — and C. Townes, *P.R.* **73**, 97, 627 (Erratum 1204). A2.31
- W. Barkas**
 1939a: — , *P.R.* **55**, 691. 10.32
 1939b: — and M. White, *P.R.* **56**, 288. 14.15
 see M. White
- H. Barschall**
 1940: — and M. Kanner, *P.R.* **58**, 590. 6.42, 14.12, 17.42
 see J. Coon, J. Wheeler
- G. Beck and S. Tsien** 1942, *Cahiers Physique* **8**, 19. 17.41, 17.42
- G. Becker and P. Kusch** 1948, *P.R.* **73**, 584. A2.21, A2.31
- F. Belinfante**
 1939: — , *Physica* **6**, 887. 4.41
 1940: — , *Physica* **7**, 449. 4.41
- S. de Benedetti**, see R. Shutt
- H. Bethe**
 1933: — , *Hdb. d. Physik* (2nd Ed., vol. XXIV/1). N. 14, 12.23, A2.27, A2.32
 1937: — , *Rev. mod. Phys.* **9**, 69. 6.212, 6.31
 1938a: — , *P.R.* **54**, 436. 3.3
 1938b: — and C. Critchfield, *P.R.* **54**, 248, 862. 7.2
 1938c: — , *P.R.* **53**, 842. A2.252
 1940: — , *P.R.* **57**, 260, 390. 15.32, 16.11, 16.32, 17.1
 1946: — , *P.R.* **70**, 821. 1.331
 see M. Rose
- H. Beyer**
 1940: — and M. Whitaker, *P.R.* **57**, 976. 6.22, 14.12
 see P. Powers
- K. Birus**, see Y. Nishina
- T. Bjørdal**, see B. Trumpy
- J. Blair**, see R. Taschek
- G. Blass** 1944, *Z. Physik* **122**, 163. 1.321
- F. Bloch**
 1940: — , *P.R.* **58**, 829. 17.41
 see L. Alvarez
- I. Bloch**, see G. Breit
- D. Bocciarelli**, see E. Amaldi
- J. Bøggild** 1945, *Proc. Cop.* **23**, No. 4. 17.51
- D. Bohm and C. Richman** 1947, *P.R.* **71**, 567. 6.431
- E. Bohr**, see K. Siegbahn
- N. Bohr**
 1934: — , *Structure et propriétés des noyaux atomiques* (Reports of the 7th Solvay Conference), p. 216 sqq., especially p. 227. 1.21
 1936: — , *Nature* **137**, 344, 351. 9.11
 1937: — and F. Kalckar, *Proc. Cop.* **14**, No. 10. 9.11

- T. Bonner 1937, *P.R.* 52, 685. 6.42
 E. Booth and C. Hurst 1937, *Proc. R. S. A* 161, 248. 6.412
 L. Borst and W. Harkins 1940, *P.R.* 57, 659. 14.14
 M. Bound, see B. Rossi
 G. Breit
 1936a: — , E. Condon and R. Present, *P.R.* 50, 825. 5.21, 7.11, 7.12, 7.13
 1936b: — and E. Feenberg, *P.R.* 50, 850. 11.21
 1937a: — , *P.R.* 51, 248, 778. 15.22
 1937b: S. Share and — , *P.R.* 52, 546. 15.22
 1938a: — , *P.R.* 53, 153. 15.22
 1938b: — and E. Wigner, *P.R.* 53, 998. 11.22, 11.4
 1938c: — and J. Stehn, *P.R.* 53, 459. 17.53
 1939a: — , H. Thaxton and L. Eisenbud, *P.R.* 55, 1018. 5.21, 6.432, 7.11, 7.12, 7.13
 1939b: — , L. Hoisington, S. Share and H. Thaxton, *P.R.* 55, 1103. 6.432, 8.1
 1939c: C. Kittel and — , *P.R.* 56, 744. 5.321, 16.32
 1939d: L. Hoisington, S. Share and — , *P.R.* 56, 884. 6.432, 7.13
 1940: — , C. Kittel and H. Thaxton, *P.R.* 57, 255. 16.32
 1947a: — , *P.R.* 71, 215. 6.213
 1947b: — and P. Zilsel, *P.R.* 71, 232. (See further P. Zilsel, B. Darling and — , *P.R.* 72, 576. 1947.) 6.213
 1947c: — and I. Bloch, *P.R.* 72, 135. A2.27
 E. Bretscher, see J. Chadwick
 F. Brickwedde, J. Dunning, H. Hoge and J. Manley 1938, *P.R.* 54, 266. 6.22, 6.31
 G. Briggs, see M. Goldhaber
 F. Brown 1939, *P.R.* 56, 1107. 14.111, 14.22
 H. Brown and D. Inglis 1939, *P.R.* 55, 1182. 13.22
 R. Buckingham
 1941: — and H. Massey, *Proc. R.S. A* 179, 123. 6.432, 14.11, 14.12
 see H. Massey
 E. Burhop and H. Massey 1948, *Proc. R.S. A* 192, 156. 14.14

C

- P. Caldirola 1946, *P.R.* 69, 608. A2.27
 H. Carroll
 1938: — and J. Dunning, *P.R.* 54, 541. 6.22, 17.42
 1941: — , *P.R.* 60, 702. 6.22, 14.12
 H. Casimir
 1936a: — , *Physica* 3, 936. 6.13
 1936b: — , *On the interaction between atomic nuclei and electrons* (Verhand. Teyler's Tweede Genootschap 11). A2.13
 1940: — , *Physica* 7, 169. 6.122
 1942: — and G. Karreman, *Physica* 9, 494. A2.14
 J. Chadwick
 1937: — , N. Feather and E. Bretscher, *Proc. R. S. A* 163, 366. 8.34
 see E. Rutherford
 R. Chaminade, see R. Maze
 F. Champion, see C. Powell
 G. Chertock 1944, *P.R.* 65, 51. 7.2
 L. Chilton, see C. Powell

- R. Christy
 1939: — and S. Kusaka, *P.R.* 55, 665. 16.21
 1941: — and S. Kusaka, *P.R.* 59, 414. 1.333
- J. Clay and C. Levert 1946, *Physica* 12, 321. 8.311
- G. Cocconi
 1944: — and V. Tongiorgi, *Nuovo Cim.* 2, 93. 1.332
 1946: — and V. Tongiorgi, *P.R.* 70, 855. 1.332
- F. Coester 1944, *Helv. phys. Acta* 17, 35. A3.3
- V. Cohen
 1939: — , H. Goldsmith and J. Schwinger, *P.R.* 55, 106. 6.22
 1940: — , H. Goldsmith and M. Hamermesh, *P.R.* 57, 352. 6.22
 see W. Zinn
- G. Collins, see E. Guth
- J. Combes, see S. Gorodetzky
- E. Condon, see G. Breit
- M. Conversi
 1944: — and O. Piccioni, *Nuovo Cim.* 2, 40, 71. 1.332
 1946a: — and O. Piccioni, *P.R.* 70, 859. 1.332
 1946b: — and O. Piccioni, *P.R.* 70, 874. 1.332
 1947: — , E. Pancini and O. Piccioni, *P.R.* 71, 209, 557. 1.332
- C. Cook
 1947: L. Langer, — and M. Sampson, *P.R.* 71, 906. A1.21
 1948: — , L. Langer and H. Price, *P.R.* 73, 1395. A1.212
- J. Coon
 1946a: — , R. Davis and H. Barschall, *P.R.* 70, 104. 6.42, 14.12
 1946b: — and H. Barschall, *P.R.* 70, 592. 14.12
- E. Cooper and E. Nelson 1941, *P.R.* 59, 216. A1.22
- H. Crane
 1938: — and J. Halpern, *P.R.* 53, 789. 1.14
 1939: — and J. Halpern, *P.R.* 56, 232. 1.14
- E. Creutz
 1939: — , *P.R.* 56, 893. 7.12
 see M. White, R. Wilson
- C. Critchfield
 1948: — , *P.R.* 73, 1. 14.13
 see H. Bethe
- D**
- B. Dailey et al. 1946, *P.R.* 70, 984. A2.31
- S. Dancoff
 1936: — and D. Inglis, *P.R.* 50, 784. 15.22
 1940: — , *P.R.* 58, 326. 17.43
- J. Darby and J. Swan 1948, *Nature* 161, 22. 14.12
- B. Darling, see G. Breit
- L. Davis et al. 1948, *P.R.* 73, 525. A2.31
- R. Davis, see J. Coon
- S. de Benedetti, see R. Shutt
- P. Dee and C. Gilbert 1937, *Proc. R.S. A* 163, 265. 6.42
- L. Delsasso, see W. Fowler, M. White
- D. Dennison 1940, *P.R.* 57, 454. 13.21, 13.211
- K. Diebner, W. Herrmann and E. Grassmann 1942, *Physik. Z.* 43, 440. 6.411, 6.52
- P. Dirac 1935, *Principles of quantum mechanics* (2nd Ed.) 4.21

- J. Dunning, see F. Brickwedde, H. Carroll, H. Hanstein, P. Powers,
A. Reid

E

- C. Eggler, see D. Hughes
L. Eisenbud
1941: — and E. Wigner, *Proc. Nat. Acad. Wash.* **27**, 281. 15.21
see G. Breit, E. Wigner
M. Eisner and N. Rosen 1947, *P.R.* **71**, 835. 6.432
D. Elliott and L. King 1941, *P.R.* **60**, 489. 3.3
C. Ellis, see E. Rutherford
W. Elsasser 1934, *J. Physique* **5**, 71. 13.12
L. Emo, see J. Richardson
I. Estermann, see J. Halpern
H. Euler 1937, *Z. Physik* **105**, 553. 12.21
G. Evans, see E. Williams

F

- U. Fano 1937, *Naturwiss.* **25**, 602. A2.32
A. and L. Farkas 1935, *Proc. R.S. A* **152**, 152. 6.122
L. Farkas
1938: — and U. Garbatski, *J. chem. Physics* **6**, 260. 6.122
1939: — and L. Sandler, *Trans. Far. Soc.* **35**, 337. 6.122
see A. Farkas
N. Feather, see J. Chadwick
E. Fedorov and N. Perfilieva 1937, *Sov. Phys.* **11**, 660. 6.441
E. Feenberg
1937a: — and E. Wigner, *P.R.* **51**, 95. 10.20, A2.252
1937b: — and M. Phillips, *P.R.* **51**, 597. 10.20
1937c: — , *P.R.* **52**, 667. 11.21, 11.4
1941a: — , *P.R.* **59**, 149. 12.13
1941b: — , *P.R.* **59**, 593. 12.14
1941c: — , *P.R.* **60**, 204. 12.12
1946a: — and G. Goertzel, *P.R.* **70**, 597. 10.34
1946b: — and H. Primakoff, *P.R.* **70**, 980. 17.1
see G. Breit, M. Phillips
B. Feld 1947, *P.R.* **72**, 1116. A2.31
E. Fermi
1935: — , *P.R.* **48**, 570. 6.52
1936: — , *Ric. sci.* **7**, 13. 6.212, 6.213, 6.52
see E. Amaldi
B. Ferretti
1941a: — , *Ric. sci.* **12**, 843. 16.32
1941b: — , *Ric. sci.* **12**, 993. 16.31, 16.311, 16.32
1943a: — , *Nuovo. Cim.* **1**, 25. 8.33
1943b: — , *Nuovo. Cim.* **1**, 229. 15.34
see E. Amaldi
M. Fierz 1937, *Z. Physik* **104**, 553. A1.11
R. Finkelstein 1947, *P.R.* **72**, 415. 8.311, 15.35
J. Fisk, see P. Morse
S. Flügge
1939: — , *Z. Physik* **113**, 587. 6.12, 16.20
1941: — , *Physik. Z.* **42**, 221. 13.15

- W. Fowler
 1936: — , L. Delsasso and C. Lauritsen, *P.R.* 49, 561. 3.3
 1939: — and C. Lauritsen, *P.R.* 56, 840. 17.51
 J. Fox, see M. White
 F. Frank 1947, *Proc. Phys. Soc.* 59, 408. A1.212
 J. Fremlin, see C. Gilbert
 J. Frenkel 1936, *Sov. Phys.* 9, 553. 9.11
 A. Fréon
 1945: — , *Le Méson* (Réunions d'études), p. 99. 1.332
 see R. Maze
 W. Fretter 1946, *P.R.* 70, 625. 1.331
 D. Frisch 1946, *P.R.* 70, 589. 6.411
 O. Frisch, H. v. Halban and J. Koch 1937, *Proc. Cop.* 15, No. 10. 6.22, 6.52
 H. Fröhlich
 1946: — , W. Ramsey and I. Sneddon, *Cambr. Conf.*, p. 166. 5.231, 8.33, 14.22
 1947: — , K. Huang and I. Sneddon, *Proc. R.S. A* 191, 61. 14.22
 W. Frye, see F. Hoyt
 K. Fuchs 1939, *Proc. Cambr.* 35, 242. 3.4
 W. Furry 1936, *P.R.* 50, 784. 15.22

G

- G. Gamertsfelder and M. Goldhaber 1946, *P.R.* 62, 556. 6.52
 G. Gamow 1934, *Z. Physik* 89, 592. 3.41
 U. Garbatski, see L. Farkas
 E. George 1946, *Cambr. Conf.* (not published) 1.331
 E. Gerjuoy and J. Schwinger 1942, *P.R.* 61, 138. 17.2, A2.251
 A. Gibert and J. Rossel 1946, *Helv. phys. Acta* 19, 285. 6.212, 6.22
 W. Gibson, L. Green and D. Livesy 1947, *Nature* 160, 534. 8.34
 C. Gilbert
 1937: — , C. Smith and J. Fremlin, *Nature* 139, 796. 1.22
 see P. Dee
 M. Goeppert-Mayer and R. Sachs 1948, *P.R.* 73, 185. A2.251
 G. Goertzel, see E. Feenberg
 M. Goldblatt *et al.* 1947, *P.R.* 72, 973. A1.212
 M. Goldhaber
 1937: — and G. Briggs, *Proc. R.S. A* 162, 127. 6.22, 14.12
 see G. Gamertsfelder, L. Schulz
 H. Goldsmith, see V. Cohen, P. Powers
 T. Goloborodko and A. Leipunski 1939, *P.R.* 56, 891. 6.411
 P. Gombás 1943, *Z. Physik* 121, 523; 122, 497. 12.23
 W. Good
 1940: — and G. Scharff-Goldhaber *P.R.* 58, 89. 6.411
 W. Gordy
 1947a: — *et al.*, *P.R.* 72, 249. A2.21
 1947b: — *et al.*, *P.R.* 72, 259. A2.21, A2.31
 1947c: — *et al.*, *P.R.* 72, 344. A2.31
 1948a: — *et al.*, *P.R.* 73, 633. A2.31
 1948b: — *et al.*, *P.R.* 73, 635. A2.31
 see J. Simmons
 S. Gorodetzky
 1942: — , Thèse Paris. 1.331
 1945: — , *Le Méson* (Réunions d'études), p. 115. 1.33

- 1946: — and J. Combes, *Cambr. Conf.*, p. 71. 1.331
- R. Gould, see A. Solomon
- G. Graham and H. Halban 1945, *Rev. mod. Phys.* **17**, 297. 8.34
- E. Grassmann, see K. Diebner
- E. Graves 1940, *P.R.* **57**, 855. 17.51
- L. Green, see W. Gibson
- B. Grönblom and R. Marshak 1939, *P.R.* **55**, 229. 13.13
- A. Gross, see A. Reid
- E. Guth
- 1939a: — , *P.R.* **55**, 411. 6.53
- 1939b: G. Collins, B. Waldmann and — , *P.R.* **56**, 876. 6.53
- A. Guthrie and R. Sachs 1942, *P.R.* **62**, 8. 13.211
- H**
- L. Hafstad
- 1938: — and E. Teller, *P.R.* **54**, 681. 13.13, 13.22, 17.52
see L. Rumbaugh, M. Tuve
- H. Halban
- 1938: — , *Nature* **141**, 644. 8.34
see O. Frisch, G. Graham
- T. Hall and P. Koontz 1947, *P.R.* **72**, 196. 17.42
- J. Halpern
- 1937: — , I. Estermann, O. Simpson and O. Stern, *P.R.* **52**, 142. 6.31
see H. Crane
- M. Hamermesh
- 1946: — and J. Schwinger, *P.R.* **69**, 145. 6.32
- 1947: — and J. Schwinger, *P.R.* **71**, 678. 6.33
see V. Cohen
- H. Hanstein
- 1940a: — and J. Dunning, *P.R.* **57**, 565. 6.22
- 1940b: — , *P.R.* **57**, 1045. 6.22
- 1941: — , *P.R.* **59**, 489. 6.22
- W. Harkins, see L. Borst
- L. Haworth, see J. Manley
- R. Haxby, see J. Williams
- O. Haxel and H. Volz 1943, *Z. Physik* **120**, 493. 6.52
- W. Heisenberg
- 1937: — , *Leipziger Ber.* **89**, 369. 7.14, 12.30
- 1939: — , *Z. Physik* **113**, 61. 15.33, A3.12
- H. Heitler, see C. Powell
- W. Hepner and R. Peierls 1942, *Proc. R. S. A* **181**, 43. 16.13
- R. Herb
- 1939: — , D. Kerst, D. Parkinson and G. Plain, *P.R.* **55**, 998. 7.12, 7.13
see C. Hudson
- W. Herrmann, see K. Diebner
- N. Heydenburg
- 1941: — and N. Ramsey, *P.R.* **60**, 42. 17.42
see M. Tuve
- E. Hill 1937, *P.R.* **51**, 370. 12.24
- H. Höcker 1942, *Physik. Z.* **43**, 236. 14.11, 14.12, 14.14, 14.15
- H. Hoge, see F. Brickwedde
- L. Hoisington
- 1940: — , *P.R.* **57**, 1077. 15.22
see G. Breit, H. Thaxton

- B. Holmberg 1944, *Proc. Lund* 14, No. 22. 16.42
 T. Holstein, see H. Primakoff
 W. Hornyak and T. Lauritsen 1948, *Rev. mod. Phys.* 20, 191. 17.3
 A. Houriet 1945, *Helv. phys. Acta* 18, 473. A3.13
 F. Hoyt and W. Frye 1940, *P.R.* 58, 784. 7.13
 N. Hu
 1945: — , *P.R.* 67, 339. 15.34
 see J. Jauch
 T. Hu, see H. Massey
 K. Huang, see H. Fröhlich
 C. Huddleston, see D. Hughes
 C. Hudson, R. Herb and G. Plain 1940, *P.R.* 57, 587. 17.51
 D. Hughes
 1946: — , *P.R.* 69, 371. (See also *P.R.* 71, 387. 1947.) 1.331
 1947: — , C. Eggleter and C. Huddleston, *P.R.* 71, 269. A1.22
 1948: — and C. Eggleter, *P.R.* 73, 809. A1.212
 L. Hulthén
 1942a: — , *Arkiv* 28 A, No. 5. 5.11, 5.22, 5.231
 1942b: — , *Arkiv* 29 B, No. 1. 5.22, 5.231, 5.32, 5.321, 5.323
 1943a: — , *Arkiv* 29 A, No. 33. 8.33, 16.42
 1943b: — , *Arkiv* 30 A, No. 9. 8.1, 16.42
 1944a: — , *Arkiv* 31 A, No. 15. 16.33, 16.5, 17.1
 1944b: — , *Proc. Lund* 14, No. 2. 16.33
 1944c: — , *Proc. Lund* 14, No. 8. 5.121
 1944d: — , *Proc. Lund* 14, No. 21. 5.31, 8.33
 1945a: — , *Proc. Lund* 15, No. 22. 5.121, 5.231
 1945b: — , *Rev. mod. Phys.* 17, 263. 16.33
 1946a: — , *C.R. 10e Congrès des math. scandin.*, p. 201. 5.31
 1946b: — and A. Pais, *Cambr. Conf.*, p. 177. 8.33
 F. Hund
 1927: — , *Z. Physik* 43, 788. 10.14
 1937: — , *Z. Physik* 105, 202. 10.10, 10.20, 10.31, 11.41
 1939: — , *Ann. Physik* 36, 319. 1.321
 1940: — , *Z. Physik* 117, 1. 1.321
 C. Hurst, see W. Allen, E. Booth
 E. Hylleraas
 1937: — , *Z. Physik* 107, 258. 5.11
 1941: — and V. Risberg, *Avhandl. Norske Vid. Akad. (mat.-naturv. Kl.)*
 No. 3. 5.22, 5.231, 14.22
 1945: — , *Fra Fysikkens Verden* 7, 175. 5.31

I

- M. Inghram, see L. Norris
 D. Inglis
 1936: — , *P.R.* 50, 783. 15.22
 1937: — and L. Young, *P.R.* 51, 525. 10.31, 11.4
 1938a: — , *P.R.* 53, 470. A2.23, A2.24
 1938b: — , *P.R.* 53, 880. A2.252
 1939a: — , *P.R.* 55, 329. 17.53, A2.252, A2.253, A2.26
 1939b: — , *P.R.* 55, 988. 6.12
 1939c: — , *P.R.* 56, 1175. 17.52, 17.53, A2.252
 1941: — , *P.R.* 60, 837. 13.12, A2.21, A2.23, A2.24, A2.261, A2.31, A2.32
 see H. Brown, S. Dancoff

J

- J. Jacobsen and O. Kofoed-Hansen 1945, *Proc. Cop.* **23**, No. 12. 1.14
 L. Jánosy
 1939: — , *Proc. Cambr.* **35**, 616. 1.34
 see J. McConnell, B. Rossi
 J. Jauch
 1944: — and N. Hu, *P.R.* **65**, 289. 16.41
 1945: — , *P.R.* **67**, 125. 16.41
 1946: — , *P.R.* **69**, 275. A3.25
 F. Jenkins 1948, *P.R.* **73**, 639. A1.22
 H. Jensen 1939, *Naturwiss.* **27**, 793, 841. 3.221
 T. Johnson, see R. Shutt
 H. Johnston, see G. Murphy
 R. Jost 1946, *Helv. phys. Acta* **19**, 113. A3.24

K

- F. Kalckar
 1935: — and E. Teller, *Proc. R.S. A* **150**, 520. 6.122
 see N. Bohr
 W. Kanne, see G. Ragan
 M. Kanner, see H. Barschall, R. Ladenburg
 G. Karreman, see H. Casimir
 J. Kellogg
 1946: — and S. Millman, *Rev. mod. Phys.* **18**, 323. 1.21
 see I. Rabi
 N. Kemmer
 1937a: — , *Helv. phys. Acta* **10**, 47. 4.33, 5.132
 1937b: — , *Nature* **140**, 192. 11.21, 11.32
 1938a: — , *Proc. Cambr.* **34**, 354. 8.31
 D. Kerst, see R. Herb
 S. Kikuchi
 1939a: — and H. Aoki, *Proc. Jap.* **21**, 75. 6.412, 14.12
 1939b: — , H. Aoki and T. Wakatsuki, *Proc. Jap.* **21**, 410. 6.42
 L. King, see D. Elliott
 C. Kittel
 1942: — , *P.R.* **62**, 109. 13.3, 17.43, 17.53
 see G. Breit
 O. Klein
 1938: — , *J. Physique* **9**, 1. 4.141
 1945: — , *Arkiv* **31A**, No. 14. 1.23
 F. Kligman 1940, *Journal exp. and theor. Phys.* (Russian) **10**, 15. 16.20
 J. Koch, see O. Frisch
 O. Kofoed-Hansen, see J. Jacobsen
 J. Konopinski
 1943: — , *Rev. mod. Phys.* **15**, 209. 13.15, A1.0, A1.21, A1.22
 1947: — , *P.R.* **72**, 518. A1.212
 H. Kopfermann 1939, *Kernmomente.* 1.21, 1.22
 H. A. Kramers 1938, *Hd.- u. Jahrb. d. chem. Phys.*, Bd. I. 4.14, 5.23
 H. Kratz, see R. Taschek
 P. Kruger
 1937: — , W. Shoupp and F. Stallmann, *P.R.* **52**, 678. 6.42, 14.12
 1938: — , W. Shoupp, R. Watson and F. Stallmann, *P.R.* **53**, 1014.
 14.12
 see J. Laughlin

- P. Koontz, see T. Hall
 P. Kusaka
 1941: — , *P.R.* 60, 61. *A1.22*
 1943: — , *P.R.* 64, 256. *1.333*
 see R. Christy
 P. Kusch, see G. Becker, S. Millman

L

- R. Ladenburg
 1937: — and M. Kanner, *P.R.* 52, 911. *6.412*
 see J. Wheeler
 W. Lamb and L. Schiff 1938, *P.R.* 53, 651. *4.43*
 L. Landau and J. Smorodinsky 1944, *J. Physics U.S.S.R.* 8, 154. *7.13, 17.42*
 L. Langer, see C. Cook
 R. Lapp
 1943: — , *P.R.* 64, 254, 255. *1.333*
 1946: — , *P.R.* 69, 321. *1.333*
 J. Laughlin and P. Kruger 1947, *P.R.* 71, 736. (See further *P.R.* 73, 197. 1948.)
 6.42
 C. Lauritsen, see W. Fowler
 T. Lauritsen, see W. Hornyak
 D. Leatherberry, see H. Thaxton
 A. Leipunski
 1936a: — , *Proc. Cambr.* 32, 301. *1.14*
 1936b: — , L. Rosenkewitsch and D. Timoshuk, *Sow. Phys.* 10, 625.
 6.441
 see T. Goloborodko
 L. Leprince-Ringuet
 1946a: — and M. Lh  ritier, *J. Physique* 7, 65. *1.331*
 1946b: — , M. Lh  ritier and R. Richard-Foy, *J. Physique* 7, 69. *1.331*
 C. Levert, see J. Clay
 P. Levy 1947, *P.R.* 72, 248. *A1.22*
 M. Lh  ritier, see L. Leprince-Ringuet
 W. Libby and E. Long 1939, *P.R.* 55, 339. *6.22, 6.31*
 C. van Lieer and G. Uhlenbeck 1937, *Physica* 4, 531. *9.11*
 D. Livesey, see W. Gibson, C. Powell
 E. Long, see W. Libby
 J. Lopes
 1946: — , *P.R.* 70, 5. *A3.25*
 1947: — , *P.R.* 72, 355. *A3.25*
 J. Lubanski and L. Rosenfeld 1945, *Experientia* 1, 198. *6.53*
 E. Luebke, see J. Manley

M

- H. Maier-Leibnitz 1938, *Naturwiss.* 26, 614. *17.51*
 J. Manley
 1942: — , L. Haworth and E. Luebke, *P.R.* 61, 152. *6.52*
 see F. Brickwedde
 H. Margenau
 1937: — and D. Warren, *P.R.* 52, 790. *14.21, 14.22*
 1938a: — and W. Tyrrell, *P.R.* 54, 422. *14.22*
 1938b: — and H. Carroll, *P.R.* 54, 705. *14.24*
 1939: — , *P.R.* 55, 1173. *14.24*

- 1940a: — , *P.R.* 57, 383. A2.27
 1940b: — and E. Wigner, *P.R.* 58, 103. A2.21, A2.22, A2.23
 1941: — , *P.R.* 59, 37. 13.12
 see D. Warren
- C. Marhoefer, see M. Wiedenbeck
- R. Marshak, see B. Grönblom
- J. Marshall 1946, *P.R.* 70, 107. 6.22, 14.12
- H. Massey
 1947a: — and R. Buckingham, *P.R.* 71, 558 (fig. 2 is printed by error on p. 563). 14.12
 1947b: — and T. Hu, *Nature* 160, 794. 16.20
 1948: — and R. Buckingham, *P.R.* 73, 260. 14.13
 see R. Buckingham, E. Burhop
- J. Mattauch
 1937: — , *Naturwiss.* 25, 738. 3.221
 1938: — , *Physik Z.* 39, 892. 1.22
 1939: — , *Naturwiss.* 27, 185. 3.221
- A. May, see C. Powell
- R. Maze, R. Chaminade and A. Fréon 1945, *J. Physique* 6, 202. 1.332
- J. McConnell
 1945: — , *Proc. Dublin* 50 A, 189. 1.23
 1947: — and L. Jánossy, *Nature* 159, 335. 1.23
- E. McMillan 1947, *P.R.* 72, 591. A1.22
- S. Millman
 1941: — and P. Kusch, *P.R.* 60, 91. 1.21, A2.21
 see J. Kellogg
- Y. Miyazaki, see Y. Nishina
- C. Møller
 1940: — and L. Rosenfeld, *Proc. Cop.* 17, No. 8. 6.12, 15.31, 15.32, 15.33, 15.34, 16.42
 1943: — and L. Rosenfeld, *Proc. Cop.* 20, No. 12. 4.41, 4.43, 8.31, A2.11, A2.27
- A. Monroe, see H. Thaxton
- P. Morse, J. Fisk and L. Schiff 1936, *P.R.* 50, 748. 5.11, 6.432
- H. Moseley and N. Rosen 1947, *P.R.* 71, 835. 6.432
- S. Moszkowski and R. Sachs 1948, *P.R.* 73, 184. 16.20
- G. Murphy and H. Johnston 1934, *P.R.* 46, 95. 1.22
- F. Myers and L. Van Atta 1942, *P.R.* 61, 19. 6.11, 8.34
- N**
- E. Nelson, see E. Cooper
- N. Nereson
 1942: — and B. Rossi, *P.R.* 62, 417. 1.332
 1943: — and B. Rossi, *P.R.* 64, 199. 1.332
- Y. Nishina
 1941a: — , K. Birus, Y. Sekido and Y. Miyazaki, *Sci. papers Inst. phys. chem. Res.* 38, 353. 8.311
 1941b: — and K. Birus, *Sci. papers Inst. phys. chem. Res.* 38, 360. 8.311
- L. Nordheim 1939, *P.R.* 55, 506. 15.35
- A. Nordsieck 1940, *P.R.* 58, 310. 6.12
- L. Norris and M. Inghram 1948, *P.R.* 73, 350. A1.22
- A. Novick 1947, *P.R.* 72, 972. A1.212
- R. Nuckolls, C. Bailey *et al.* 1946, *P.R.* 70, 805. 14.12
- H. Nye, see W. Rarita

O

- G. Occhialini, see C. Powell
 K. Ochiai 1937, *P.R.* 52, 1221. 14.13
 W. Opechowski
 1943a: — , *Physica* 10, 473. 1.332
 1943b: — , *Ned. T. Natuurk.* 10, 281. 1.332
 J. Oppenheimer
 1939: — and J. Schwinger, *P.R.* 56, 1066. 13.211
 1941: — , *P.R.* 59, 908. A1.22
 G. Ortner 1940, *Wiener Ber.* 149, 259, 7.14, 12.35, 12.36

P

- A. Pais
 1943: — , *Proc. Cop.* 20, No. 17. 6.51, 8.34
 1945: — , *Nature* 156, 715. 8.311
 1946: — , *Proc. Cambr.* 42, 45. 5.331, 8.33
 see L. Hulthén
 E. Pancini, see M. Conversi
 H. Parker 1937, *P.R.* 51, 683. (M.A. thesis, Univ. of Carolina.) 13.3
 D. Parkinson, see R. Herb
 W. Pauli
 1940: — , *P.R.* 58, 716. 1.11
 1946: — , *Meson theories of nuclear forces.* A3.0
 R. Peierls
 1947: — and M. Preston, *P.R.* 72, 250. 7.131
 see W. Hepner
 N. Perfilieva, see E. Fedorov
 D. Perkins 1947, *Nature* 160, 299. 7.14
 B. Peters and C. Richman 1941, *P.R.* 59, 804. 6.53
 S. Petersson, see K. Siegbahn
 M. Phillips
 1940: — , *P.R.* 57, 160. A2.252
 1941: — and E. Feenberg, *P.R.* 59, 400. 3.3
 see E. Feenberg
 O. Piccioni, see M. Conversi
 K. Pitzer, see L. Alvarez
 G. Plain
 see R. Herb, C. Hudson
 C. Powell
 1940: — , H. Heitler and F. Champion, *Nature* 146, 716. 6.42
 1944: F. Champion and —, *Proc. R. S. A* 183, 64. 6.42
 1946a: — , G. Occhialini, D. Livesey and L. Chilton, *J. sci. Instr.* 23, 102. 6.42
 1946b: — and G. Occhialini, *Cambr. Conf.*, p. 150. 6.42
 1947a: A. May and —, *Proc. R.S. A* 190, 170. 7.12
 1947b: H. Heitler, A. May and —, *Proc. R.S. A* 190, 180. 14.13, 17.42
 P. Powers, H. Goldsmith, H. Beyer and J. Dunning 1938, *P.R.* 53, 947. 6.22, 14.12
 R. Present
 1941: — , *P.R.* 60, 28. 2.1, 12.13
 see G. Breit, W. Rarita
 M. Preston, see R. Peierls
 H. Price, see C. Cook

H. Primakoff1937: — , *P.R.* 52, 1000. 14.131939: — and T. Holstein, *P.R.* 55, 1218. 1.341947: — , *P.R.* 72, 118. A2.27

see E. Feenberg

W. Privette, see H. Thaxton

J. Pruett 1948, *P.R.* 73, 1219. A1.212**R****I. Rabi**1936: J. Kellogg, — and J. Zacharias, *P.R.* 50, 472. 1.211939: J. Kellogg, — , N. Ramsay and J. Zacharias, *P.R.* 56, 728.
1.21, 6.121940: J. Kellogg, — , N. Ramsay and J. Zacharias, *P.R.* 57, 677.
6.12

see J. Schwinger

G. Ragan, W. Kanne and R. Taschek 1941, *P.R.* 60, 628. 7.12, 7.13

N. Ramsay, see I. Rabi

N. Ramsey, see N. Heydenburg, E. Salant

W. Ramsey

1947: — , *Proc. R.S. A* 191, 195. 8.33

see H. Fröhlich

W. Rarita

1937: — and R. Present, *P.R.* 51, 788. 5.3211, 6.432, 14.21, 14.221941a: — , J. Schwinger and H. Nye, *P.R.* 59, 209. 16.20, 16.23, 16.321941b: — and J. Schwinger, *P.R.* 59, 436. 6.121, 16.20, 16.21, 16.22, 16.231941c: — and J. Schwinger, *P.R.* 59, 556. 16.20, 16.22, 16.23F. Rasetti 1941, *P.R.* 60, 198. 1.332A. Reid, J. Dunning, S. Weinhouse and A. Gross 1946, *P.R.* 70, 431.
A1.22

R. Richard-Foy

1945: — , *Le Mésun* (Réunions d'études), p. 83. 1.331

see L. Leprince-Ringuet

J. Richardson and L. Emo 1938, *P.R.* 53, 234. 8.34

C. Richman, see D. Bohm, B. Peters

V. Risberg, see E. Hylleraas

A. Roberts, see W. Arnold

G. Roberts, see E. Williams

R. Roberts, see L. Rumbaugh

G. Rochester, see B. Rossi

M. Rose and H. Bethe 1937, *P.R.* 51, 205; a correction *P.R.* 51, 993. 17.3,
A2.252, A2.261

N. Rosen, see M. Eisner, H. Moseley

L. Rosenfeld

1940: — , *Mém. Acad. Roy. Belg.* 18, No. 6. 4.411945a: — , *Nature* 156, 141. 12.331945b: — , *Proc. Cop.* 23, No. 13. 15.31, 15.34, 16.42

see J. Lubański, C. Møller

L. Rosenkewitsch, see A. Leipunski

J. Rossel

1947: — , *Helv. phys. Acta* 20, 105. 6.212, 6.22

see A. Gibert

- B. Rossi
 1940: — , L. Jánossy, G. Rochester and M. Bound, *P.R.* 58, 761.
 8.311
 see N. Nereson
- S. Rozental
 1941a: — , *Proc. Cop.* 18, No. 7. 15.35
 1941b: — , *Proc. Cop.* 60, 612. 15.35
 1945: — , *Proc. Cop.* 23, No. 11. 15.35
- S. Rubin 1941, *P.R.* 59, 216. 17.51
- L. Rumbaugh, R. Roberts and L. Hafstad 1938, *P.R.* 54, 657. 17.51
- E. Rutherford, J. Chadwick and C. Ellis 1930, *Radiations from radioactive substances.* 2.1

S

- R. Sachs
 1939: — , *P.R.* 55, 825. 13.22, A2.22, A2.252, A2.253
 1941: — and E. Teller, *P.R.* 60, 18. 6.212, 6.22
 1946a: — , *P.R.* 69, 611. A2.221, A2.26
 1946b: — and J. Schwinger, *P.R.* 70, 41. A2.251
 1947a: — , *P.R.* 72, 91. A2.27
 1947b: — , *P.R.* 72, 312. A2.251
 see M. Goepfert-Mayer, A. Guthrie, S. Moszkowski
- S. Sakata
 1940: — and Y. Tanikawa, *P.R.* 57, 548. 8.311
 1941: — , *Proc. Jap.* 23, 291. 15.35
- E. Salant and N. Ramsey 1940, *P.R.* 57, 1075. 6.413
- M. Sampson, see C. Cook
- L. Sandler, see L. Farkas
- G. Scharff-Goldhaber, see W. Good
- E. Schatzman 1946, *Thèse Paris.* 7.2
- L. Schiff, see W. Lamb, P. Morse
- T. Schmidt 1937, *Z. Physik* 106, 358. A2.23
- L. Schulz and M. Goldhaber 1945, *P.R.* 67, 202. 6.52
- J. Schwinger
 1937a: — and E. Teller, *P.R.* 52, 286. 6.30, 6.32
 1937b: — , *P.R.* 52, 1250. 6.34
 1937c: — and I. Rabi, *P.R.* 51, 1003. 6.35
 1940: — , *P.R.* 58, 1004. 6.32, 6.33
 1941: — , *P.R.* 60, 164. 6.12
 1942: — , *P.R.* 61, 387. 15.32, 16.41
 1946: — , *P.R.* 69, 681. 6.35
 see V. Cohen, E. Gerjuoy, M. Hamermesh, J. Oppenheimer,
 W. Rarita, R. Sachs
- S. Seely, see W. Zinn
- F. Seitz, see E. Wigner
- Y. Sekido, see Y. Nishina
- R. Shankland, see R. Wilson
- S. Share, see G. Breit
- W. Shepherd, see J. Williams
- R. Sherr
 1945: — , *P.R.* 68, 240. 6.413
 see R. Taschek
- W. Shoupp, see P. Kruger

- R. Shutt, S. de Benedetti and T. Johnson 1942, *P.R.* 62, 552. 1.332
- K. Siegbahn
 1944: — and E. Bohr, *Arkiv* 30B, No. 3. A1.21
 1945a: — and S. Petersson, *Arkiv* 32B, No. 5. A1.21
 1945b: — and H. Slätis, *Arkiv* 32A, No. 9. A1.21
 1946: — , *Arkiv* 34B, No. 6. 17.51
 1947: — and H. Slätis, *Nature* 159, 471. 17.51
- A. Siegert 1937, *P.R.* 52, 787. 4.43
- T. Sigurgeirsson and A. Yamakawa 1947, *P.R.* 71, 319. 1.332
- J. Simmons and W. Gordy 1948, *P.R.* 73, 713. A2.31
- L. Simons 1940, *Proc. Cop.* 17, No. 7. 6.22
- O. Simpson, see J. Halpern
- H. Slätis, see K. Siegbahn
- W. Sleator jr. 1947, *P.R.* 72, 207. 6.413
- C. Smith, see C. Gilbert
- N. Smith, see J. Allen
- J. Smorodinsky, see L. Landau
- I. Sneddon, see H. Fröhlich
- A. Solomon, R. Gould and C. Anfinson 1947, *P.R.* 72, 1097. A1.22
- A. Sommerfeld 1939, *Atombau u. Spektrallinien*, Bd. II. 6.51, 7.11
- F. Stallmann, see P. Kruger
- H. Staub and H. Tatel 1940, *P.R.* 58, 820. 17.42
- J. Stehn, see G. Breit
- W. Stephens
 1940: — , *P.R.* 57, 938. 17.40
 1947: — , *Rev. mod. Phys.* 19, 19. 1.22
- O. Stern, see J. Halpern
- N. Svartholm 1945, *The binding energies of the lightest atomic nuclei*. Thesis
 Lund. 2.22, 5.24, 14.22
- J. Swan, see J. Darby

T

- Y. Tanikawa, see S. Sakata
- R. Taschek
 1942: — , *P.R.* 61, 13. 14.13
 1947: R. Sherr, J. Blair, H. Kratz, C. Bailey and — , *P.R.* 72, 622.
 14.13
 see G. Ragan
- H. Tatel
 1942: — , *P.R.* 61, 450. 6.42
 see H. Staub
- E. Teller
 1936: — , *P.R.* 49, 420. 6.30
 1938: — and J. Wheeler, *P.R.* 53, 778. 13.15, 13.21
 see L. Hafstad, F. Kalckar, R. Sachs, J. Schwinger
- H. Thaxton
 1939a: — and A. Monroe, *P.R.* 56, 616. 7.13
 1939b: — and L. Hoisington, *P.R.* 56, 1194. 7.13
 1940a: — and A. Monroe, *P.R.* 57, 246. 7.13, 8.1
 1940b: — , W. Privette and D. Leatherberry, *P.R.* 58, 200. 16.32
 1941: — , *P.R.* 60, 173. 7.13
 see G. Breit
- A. Thellung and F. Villars 1948, *P.R.* 73, 924. A2.251

- D. Timoshuk, see A. Leipunski
 S. Tomonaga 1938, *Z. Physik* 110, 573. 9.43
 V. Tongiorgi, see G. Cocconi
 C. Townes
 1947a: — *et al.*, *P.R.* 71, 644. A2.31
 1947b: — , *P.R.* 71, 909. A2.31
 1947c: — *et al.*, *P.R.* 72, 513. A2.31
 see J. Bardeen
 G. Trabacchi, see E. Amaldi
 B. Trumphy and T. Bjørdal 1942, *Bergens Museums Arbok*, No. 5. 8.311
 S. Tsien
 1940: — , *J. Physique* 1, 1, 103. 17.42
 1944: — , *C.R.* 218, 996. 17.42
 see G. Beck
 M. Tuve
 1936: — , N. Heydenburg and L. Hafstad, *P.R.* 50, 806. 7.12, 14.13
 1938: — , N. Heydenburg and L. Hafstad, *P.R.* 53, 239. 7.12
 1939: — , N. Heydenburg and L. Hafstad, *P.R.* 56, 1078. 7.12, 7.13
 W. Tyrrell
 1939: — , *P.R.* 56, 250. 14.21
 see H. Margenau

U

- G. Uhlenbeck, see C. van Lier

V

- L. Van Atta, see F. Myers
 C. van Lier and G. Uhlenbeck 1937, *Physica* 4, 531. 9.11
 F. Villars
 1946: — , *Helv. phys. Acta* 19, 323. A3.22
 1947: — , *Helv. phys. Acta* 20, 476. A2.251
 see A. Thellung
 G. Volkoff
 1942a: — , *P.R.* 62, 126. 12.12, 17.1
 1942b: — , *P.R.* 62, 134. 17.1
 H. Volz
 1937: — , *Z. Physik* 105, 537. 11.30
 1943: — , *Z. Physik* 121, 201. 6.52
 see O. Haxel

W

- T. Wakatsuki, see S. Kikuchi
 B. Waldmann, see E. Guth
 H. Wambacher 1940, *Wiener Ber.* 149, 157. 12.30
 D. Warren
 1937: — and H. Margenau, *P.R.* 52, 1027. 14.21
 see H. Margenau
 S. Watanabe 1939, *Z. Physik* 113, 482. 9.41, 12.21, 12.22, 12.24
 R. Watson, see P. Kruger
 R. Watts and D. Williams 1946, *P.R.* 70, 640. 14.21, A1.212
 K. Way
 1936: — and J. Wheeler, *P.R.* 50, 675. 11.121

- 1937: — , *P.R.* 51, 552. 11.121
 1939a: — , *P.R.* 55, 963. A2.22, A2.32
 1939b: — , *P.R.* 56, 556. 14.21
- W. Wefelmeier
 1937a: — , *Naturwiss.* 25, 525. 13.13, A2.32
 1937b: — , *Z. Physik* 107, 332. 13.13
- S. Weinhouse, see A. Reid
- V. Weisskopf 1937, *P.R.* 52, 295. 9.11
- C. v. Weizsäcker 1938, *Naturwiss.* 26, 209, 225. 12.23, 13.14
- S. Welles 1942, *P.R.* 62, 197. A2.321, A2.322
- G. Wentzel 1947, *Rev. mod. Phys.* 19, 1. A3.0
- P. Wenzel 1934, *Z. Physik* 90, 754. 17.41
- H. Wergeland 1941, *Norske Vidensk. Selsk. Skrifter* No. 1. 13.12, 13.14
- J. Wheeler
 1936: — , *P.R.* 50, 643. 11.121
 1937a: — , *P.R.* 52, 1083. 13.12, 13.15, 13.21, 13.3, 14.11
 1937b: — , *P.R.* 52, 1107. 13.3
 1940: — and H. Barschall, *P.R.* 58, 682. 17.42
 1941a: — , *P.R.* 59, 16. 13.11
 1941b: — , *P.R.* 59, 27. 13.11
 1941c: — and R. Ladenburg, *P.R.* 60, 754. 1.331
 see E. Teller, K. Way
- M. Whitaker, see H. Beyer
- M. White
 1939: — , L. Delsasso, J. Fox and E. Creutz, *P.R.* 56, 512. 3.3
 1940: L. Delsasso, — , W. Barkas and E. Creutz, *P.R.* 58, 586. 3.3
 1941: — , E. Creutz, L. Delsasso and R. Wilson, *P.R.* 59, 63. 3.3, A1.21
 see W. Barkas
- G. Wick 1938, *Nature* 142, 993. 1.321
- M. Wiedenbeck and C. Marhofer 1945, *P.R.* 67, 54. 6.11
- E. Wigner
 1933a: — , *P.R.* 43, 252. 2.3, 14.21
 1933b: — , *Z. physik. Chemie* B 23, 28. 6.122
 1934a: — and F. Seitz, *P.R.* 46, 509. 12.23
 1934b: — , *P.R.* 46, 1002. 12.23
 1936: — , *Proc. Nat. Acad. Wash.* 22, 662. 11.22
 1937a: — , *P.R.* 51, 106. 10.10, 10.14, 10.20, 15.21
 1937b: — , *P.R.* 51, 947. 10.10, 10.14, 10.31, 10.32
 1939a: — and L. Eisenbud, *P.R.* 56, 214. 17.1
 1939b: — , *P.R.* 56, 519. A1.12, A1.21
 see G. Breit, L. Eisenbud, E. Feenberg, H. Margenau
- W. Wildhack 1940, *P.R.* 57, 81. 7.2
- D. Williams, see R. Watts
- E. Williams
 1938: — , *Nature* 142, 431. 12.33
 1940a: — and G. Roberts, *Nature* 145, 102. 1.332
 1940b: — and G. Evans, *Nature* 145, 818. 1.332
- J. Williams
 1937: — , W. Shepherd and R. Haxby, *P.R.* 52, 390. 17.51
 see I. Zlotowski
- A. Wilson 1938, *Proc. Cambr.* 34, 365. 5.231
- R. Wilson
 1947a: — and E. Creutz, *P.R.* 71, 339. 7.12

1947b: — , *P.R.* 71, 384. 7.12

1947c: — , B. Wright and R. Shankland, *P.R.* 71, 560. 7.12
see M. White

B. Wright, see R. Wilson

Y

A. Yamakawa, see T. Sigurgeirsson

L. Young, see D. Inglis

H. Yukawa 1935, *Proc. Jap.* 17, 48. 1.32

Z

J. Zacharias, see I. Rabi

P. Zilsel, see G. Breit

W. Zinn, S. Seely and V. Cohen 1939, *P.R.* 56, 260. 6.412, 14.12

I. Zlotowski and J. Williams 1942, *P.R.* 62, 29. 17.51

SUBJECT INDEX

Actual level

(see Virtual level)

α -cluster

α -clustering 189, 191, 256 sq.
and mass defect 257
lability 269, 276 sq.
degree of dissociation 277
life-time compared with periods of rotation and vibration of α -nucleus 277 sq.

α -nuclei

(see α -particle model)
definition 191
and second saturation requirement 231
properties of ${}^8\text{Be}$ 271

α -particle

(see α -cluster, α -nuclei, α -particle model)
and nuclear radius 22
binding energy compared to deuteron's 28
binding energy of ${}^4\text{He}$ 304–306, 355 sq.
scattering in He 270 sq.
interaction of two —s 271–275
 first approximation 272
 second approximation 274
 range 271, 274 sq.
 comparison with intermolecular forces [272–274]
additivity 274

α -particle model

definition 189, 191
geometrical model of α -nuclei 275
binding energies of α -nuclei 275 sq.
 maximum 33
 additivity of α -particle interaction 275
 higher approximations 276
rotations and vibrations of α -nuclei [277–281]
 surface vibrations 277
 vibrational quanta 278
 rotational quanta 278
 reduction of low lying rotation levels [278–281]
stationary states of lightest α -nuclei [281–285]
 theory 281–283
 data 283–285

stationary states of nuclei of mass [number $4a \pm 1$: 285–287
doublet of ${}^7\text{Li}$ 370, of ${}^{13}\text{C}$ 411
magnetic moments 400 sq., 409–414
quadrupole moments 422 sq.
comparison with quasi-atomic model:
 doublet of ${}^7\text{Li}$ 372 sq.
 magnetic moments of odd mass nuclei [404
magnetic moments of light odd mass [nuclei 409–414
equal magnetic moments of isotopes 404

Angular distribution

proton-neutron scattering 123–126
 on meson theory 171–175
 on various theories 340, 344, 347
proton-proton scattering 148, 344 sq.
neutron-deuteron scattering 296 sq.
proton-deuteron scattering 299
nucleon-helium scattering 358–362
photodisintegration of deuteron 133, [175–179

Angular momentum

connexion with statistics 7
general invariance 199
conservation of — in β -decay 379
of odd nuclei 31, 205, 207, 237
of even nuclei 30, 205, 207
of nuclei of mass number $4a \pm 1$: [285–287
of deuteron 7, 90
rotation of α -nuclei 278–281
of lightest α -nuclei 281–285
and pairing of like nucleons 30
and fine structure of nuclear levels [357, 410–414, 416
and magnetic moment 392
and quadrupole moment 392 sq.

Anisotropy ratio

definition 174
on meson theory 175 sq.
on various theories 340, 344, 347
on strong coupling 431

Anti-particle

- definition 3
- anti-lepton 3 sq.
- anti-proton 8-10
- anti-neutron 10

Asymptotic form of eigenfunction

- (see Phase of eigenfunction)
- two-nucleon system 69 sq.

Axial dipole coupling

- operator of — 94, 314
- on meson theory
 - as static interaction 322
 - as non-static interaction 329, 349
- as a perturbation 335 sq., 349
- on Rarita-Schwinger theory 337-342
- on meson theories with cut-off 342-345
- and deuteron ground state 333-335, 338, [342-345, 350 sq.
- and ground state of ^{11}B 410 sq.
- sign of — and sign of deuteron quadrupole moment 335
- and mixtures of states of different orbital [momentum and spin 333 sq., 355, [410 sq.
- and saturation properties 353-355
- and P -doublet splitting 365 sq.

Barycentric system of reference

- reduction of wave-equation to — 54
- transformation to laboratory system 102, [296

 β -decay

- mechanism XVII, 3
- on meson field theory 14
- energy balance 4
- stability conditions 4, 37 sq.
- stability of isobars 30
- relation between life-time and decay-[energy 377, 386
- Fermi interaction 378

Gamow-Teller interaction 378

- simple and complex spectra 378, 380
- selection rules for allowed transitions [378-380
- calculation of matrix-elements 380-382, [384
- of neutron 8, 385
- of ^3H and neutrino mass 385-387
- positron emitters of neutron excess —:
 - transition allowed 379
 - matrix-elements 381
 - data 382-385
- Coulomb energy difference 34 sq., 383
- class (b) nuclei of mass $4a + 2$:
 - definition 379
 - matrix-elements 381 sq.
 - data 387
 - anomalies 388
- β -decay parameter
 - on meson theory 331
 - from data 385
- deuteron formation by proton collision [155-157, 388 sq.

Binding energy

- (see Mass-defect)
- and mass-defect XVII
- energy surface 35-37
- semi empirical formula 24
- method of "equivalent two-body problems" 303 sq.
- of "last" nucleon 33
- of deuteron 7, 90
- of α -nuclei 275 sq.
- of ^3H and ^5He by different methods 302
- of ^2H , ^3H , ^4He 304-306
- of ^6Li and ^6He 306 sq.
- of ^3H and ^4He on Rarita-Schwinger [theory 355 sq.

Born's approximation

- phases of states of higher orbital momentum 89
- collision of fast nucleons 258-261

Central interactions

two-nucleon system 59-61

types of distance dependence 66-68

	bound <i>S</i> states	<i>S</i> phases	phases higher <i>l</i>	<i>n-p</i> scattering	<i>p-p</i> scattering	fast nucleon penetration	radiative processes	<i>n-d</i> scattering	ground state of		
									³ H, ³ He	⁴ He	⁶ He
Well	75	86		101, 127-130	152 sq.	265	135-138				
Exponential	76	87		131	153	265 sq.		296-298	305	305	
Morse				131	153						
Gauss *				131	152 sq.	154 sq. 265-268		299	302, 305	305	302
Meson	79	88	89, 172	130 sq. 171-176	152 sq.	265	175-180		305 sq.	305 sq.	
Hulthén's	76 sq.	87		131							
Hylleraas'				131							

* (see Gauss potential)

general form of central potential 160-162

on meson theory 170 sq.

summary of argument 180 sq.

determination of central potential from

[saturation requirements 233 sq.

representation of short range potential

[229

and saturation properties 233 sq.

argument for force between like nucleons

[306

Charge

(see Isotopic factor, Isotopic variable)

positive and negative —

of electron XVII

of proton 6, 8

of meson 163 sq.

coupling between — and spin 21

— -symmetry of nuclear forces 33-35,

[167

— -multiplicity: two nucleon system 58

— -multiplet 199

— -conservation 164

— -transfer effect 167-169

— -exchange

and saturation properties 226 sq.

and non-static interactions 165,

[325 sq.

— -inertia 19-21

— -number and Coulomb energy of light

[nuclei 215 sq.

Charge independence

definition and limitations 158-160

neutral and symmetrical meson theories

[165-167

on strong coupling theory 163, 431

Compound nucleus

definition 185

in reactions involving ⁷Li 369-371**Congruent nucleons**

definition 254

spatial correlations 254-257

Contact interaction

relativistic invariance 53

no finite binding energy 73

on meson theory 322 sq.

Coulomb interaction

approximate expression 23

of light nuclei 215 sq.

of ³He 34, 294, 301

for pairs of light isobars 34 sq.

difference

between ⁶Li and ⁶He 307between ⁵Li and ⁵He 357

positron emitters of neutron excess —1:

[34 sq., 215 sq., 383

between α -particles 272, 274

and density variation within nucleus

[242 sq.

influence of Coulomb field on β -decay[of ³H 386

Cut-off meson theories

- definition 323
- non-static interactions on — 325 sq.
- symmetrical pseudoscalar theory 342 sq.
- Ferretti's theory (neutral pseudoscalar [and charged scalar] 343
- Bethe's neutral vector theory 344 sq.
- saturation properties 354
- Hulthén's unsymmetrical theory (neutral [scalar and small charged pseudoscalar] 345
- saturation properties 354

Density

- constant (to a first approximation) 22, [195
- variation in surface layer 240
- variation due to proton repulsion 242 sq.
- mixed — 219 sq., 222

Deuteron

- ground state:
 - parity 90
 - binding energy 7, 90
 - admixture of *D* state 91–93, 334, 344
 - angular momentum 7, 90
 - magnetic moment 93, 95, 419 sq.
 - electric quadrupole moment 91–95, 335, [344, 347
 - magnetic interaction 95–98
 - importance of axial dipole coupling [338, 342–345, 350 sq.
 - on Rarita-Schwinger theory 338
- ¹S virtual state:
 - from slow neutron scattering 110 sq.
 - from radiative capture of slow [neutrons 111
 - from scattering of slow neutrons by [hydrogen molecules 116
- relativistic corrections 319
- states of — on strong coupling 428–431
- disintegration
 - by photons 132–135, 175–180
 - by electron impact 138–141
- deuteron formation by proton collision [155–157, 388 sq.

Dichotomic variable

- main properties 43–50
- linearisation of wave-equation 50 sq.

Dipole moment

- (see Electric dipole moment, Magnetic [moment])

Disintegration star

- (see Penetration of fast nucleon into [nucleus])

Effective central potential

- (see Central interactions)
- definition 59
- four types in two-nucleon system 59, [161 sq.
- on meson theory 171
- on Rarita-Schwinger theory 339 sq.
- on Bethe's neutral vector meson theory [344
- ¹S — on Schwinger's mixed meson [theory 347 sq.

Electric dipole moment

- definition 390
- exchange — 64
- Siebert's theorem 64
- photodisintegration of deuteron 132–135, [176–179
- disintegration of deuteron by electron [impact 138–141

Electric quadrupole moment

- definition 390
- in spectroscopic sense 392
- of deuteron 91–95, 335, 344
 - on Schwinger's mixed meson theory [347
- sign of deuteron — and axial dipole [interaction 335
- effect on photodisintegration of deuteron [132
- of odd mass nuclei:
 - on liquid model 422
 - on α -particle model 422 sq.
 - on quasi-atomic model 425 sq.
 - due to single proton 423–425
- of odd nuclei 425 sq.
- empirical data 420–422

Electromagnetic interactions

- transmission by field 10
- nucleon in external electromagnetic field [52, 418
- and neutron-proton exchange 62 sq.
- Siebert's theorem 64
- slowly varying external electromagnetic [field 63, 390–392
- ground state of deuteron 90–93, 95–98, [335, 344, 347, 350 sq., 419 sq.

radiative processes 132-141, 175-180,
[341 sq., 345
gauge invariance and charged meson
[field 163
multipole moments 390, 393

Electron

positive and negative XVII, 3
spin 3
statistics 3
disintegration of deuteron by electron
[impact 138-141

Even nuclei

(see α -nuclei)
definition 29
stationary states 206 sq.
ground state 207, 212
angular momentum 30
magnetic moment 393

Exchange interactions

definition 26
types of central exchange potentials 61,
[160 sq.
electromagnetic — 62 sq.
spin dependent — 214
saturation property 222, 227
on Wigner's approximation 208-211
on individual nuclear model 218-221
on Fermi gas model 246 sq.
and stability of isobars 236
and interaction of α -particles 274

Exchange moments

definition 63
effect on radiative processes 132-141,
[176-180
exchange magnetic moment 397
of ^3H and ^3He 408

Exchange operator

definition 47
and exclusion principle 61
and exchange potential 61
neutron-proton exchange 62, 165
and general form of central potential
[160-162
and sources of meson field 164
average value in supermultiplet states
[210 sq., 214-216

Exclusion principle

(see Statistics)
formulation 3
and dichotomic variables 47-50
and exchange operators 61
symmetry character of eigenfunction 201

Exponential potential

(see Central interactions)

Fermi gas model

definition 188 sq.
kinetic energy of ground state 192-194
of excited states 194
kinetic properties 195-198
second saturation requirement 231-233
mass-defects 238-240
nuclear radius 241 sq.
variation of nuclear density 242 sq.
second approximation 243-258
virial theorem 257 sq.
penetration of fast nucleon into heavy
[nucleus 258-268

Fine structure of nuclear levels

general remarks 356 sq.
and β -spectra 356
and angular momentum 357
of ^5He and ^5Li :
correlation between them 357
scattering of nucleons by ^4He 358-365
axial dipole coupling 365 sq.
spin-orbit coupling 366-368
of ^7Li :
empirical data 369-371
interpretation 370
relativistic doublet splitting 372 sq.
of ^{13}C 411

Gauss potential

(see Central interactions)
binding energy
of light α -nuclei 231
of heavy isobars 232
of standard heavy nucleus 247-250
nuclear radius of heavy nucleus 242
interaction of two α -particles 273, 276

Heavy nuclei

(see Nuclear matter, Penetration of fast
[nucleon etc., Standard heavy nucleus)
Bohr's droplet model 185-188
Fermi gas model 188 sq., 192-198,
[238-268

second saturation requirement 231-233
magnetic moments 400

Hulthén's potential
(see Central interactions)

Hylleraas' potential
(see Central interactions)

Individual model
definition 190
Slater-determinants 217
exchange interactions 218-221
saturation conditions 224-227
doublet states of 3-nucleon system
[293-295]

Individual nuclei
(see Deuteron, α -particle)
light α -nuclei: (see following column)

stationary states 207, 281-285
binding energies 275
light odd nuclei of mass number 4a:
stationary states 207
magnetic moments 416
quadrupole moments 426
stable odd nuclei (${}^3_1\text{H}$, ${}^6_3\text{Li}$, ${}^{10}_5\text{B}$, ${}^{14}_7\text{N}$):
stationary states 206
angular momentum 31, 207, 237
magnetic moment 393, 402, 415 sq.
quadrupole moment 426
light nuclei of mass number $4a + 2$:
stationary states 205-207
 β -decay 387
odd mass nuclei:
stationary states 204 sq., 207,
[285-287]
 β -decay 34 sq., 215 sq., 383
magnetic moments 394-396, 409-414
quadrupole moments 420-422
additional references:

${}^3_1\text{H}$, ${}^3_2\text{He}$	Coulomb energy difference 34, 294, [301]	${}^{11}_5\text{B}$	ground state and magnetic momen [410 sq]
${}^3_1\text{H}$	binding energies 304-306, 355 sq. β -spectrum and mass of neutrino [385-387]	${}^{11}_6\text{C}$	β -decay and existence of neutrino
${}^5_2\text{He}$, ${}^5_3\text{Li}$	Coulomb energy difference 357 fine structure 363-365	${}^{12}_6\text{C}$	slow neutron scattering cross-sectio [10]
${}^5_2\text{He}$	stationary states 302	${}^{13}_6\text{C}$	fine structure 411
${}^6_2\text{He}$, ${}^6_3\text{Li}$	difference in spin dependent energy [214]	${}^{13}_7\text{N}$	β -spectrum 383
	Coulomb energy difference 306 sq. binding energies 306 sq.	${}^{14}_6\text{C}$	anomalous β -decay 388
${}^7_4\text{Be}$, ${}^7_3\text{Li}$	Coulomb energy difference 34 K -capture 369-371, 383 sq. existence of neutrino 5	${}^{16}_8\text{O}$	slow neutron scattering cross-sectio [10]
${}^7_3\text{Li}$	fine structure 369-373 magnetic moment 413 sq.	${}^{22}_{11}\text{Na}$	anomalous β -decay 388
${}^8_3\text{Li}$	orbital shell configuration 357	${}^{30}_{14}\text{Si}$, ${}^{30}_{15}\text{P}$	difference in spin dependent energ [21]
${}^8_4\text{Be}$	stationary states 271, 276	${}^{35}_{16}\text{S}$	β -spectrum and neutrino mass 387
${}^9_4\text{Be}$, ${}^9_5\text{B}$	Coulomb energy difference 34	${}^{38}_{17}\text{Cl}$	β -decay and existence of neutrino
${}^{10}_4\text{Be}$, ${}^{10}_6\text{C}$	" " " 35	${}^{39,41}_{19}\text{K}$	magnetic moment 405
${}^{10}_4\text{Be}$	anomalous β -decay 388	${}^{40}_{19}\text{K}$	" " 397, 416
		${}^{56}_{26}\text{Fe}$	binding energy 275
		${}^{88}_{36}\text{Kr}$	β -decay and existence of neutrino
		${}^{176}_{71}\text{Lu}$	magnetic moment 397, 416 quadrupole moment 421

Isobars

of anomalous charge and spin 19-21
stability 30, 234-237
atomic weights 31-33
unstable — and charge symmetry 33-35
pairs of stable — 40

Isotopes

mass-defects 33
number of stable — of given charge 40
with same magnetic moment 393-396,
[404 sq.]

Isotopic factor

- definition 165
- charged theory 165
- neutral theory 165, 167
- symmetrical theory 166 sq.
- on strong coupling 432
- in various types of interactions 313 sq.
- and saturation properties 225-227, 233 sq.
- evidence
 - from scattering of fast neutrons by
 - [protons 171-176, 340 sq.
 - from neutron-deuteron scattering
 - [298 sq.
 - from proton-deuteron scattering 300
 - from photodisintegration of deuteron
 - [175-179, 341 sq.
 - from saturation properties 233 sq.

Isotopic variable

- definition 51
- and exclusion principle 47-50
- and sources of meson fields 164-167
- and types of nuclear interactions 313 sq.

K-capture

- mechanism 4
- energy balance 4
- stability condition 4
- relation between life-time and energy 377
- of ${}^7\text{Be}$ 369-371, 383 sq.

Kinetic energy

- order of magnitude 27
- on Fermi gas model 192-194
- and surface energy 240
- and virial theorem 257 sq.

Lepton

- (see Electron, Neutrino)
- definition XVII
- interaction with mesons 14

Life-time

- of neutron 8, 385
- of meson 16-18, 330-332
- of β -unstable nucleus 377, 386

Light nuclei

- quasi-atomic model 189 sq.
- resonating group model 190

 α -particle model 191

- Wigner approximation 191
- mass-defects
 - empirical 33
 - on Wigner's approximation 212-214
- binding energies of lightest nuclei
 - [301-307, 355 sq.
- Coulomb energy 215 sq.
- fine structure
 - of light nuclei 356-373
 - of ${}^6\text{He}$ and ${}^6\text{Li}$ 357-368
 - of ${}^7\text{Li}$ 369-373
- stationary states 203-207
- magnetic moments 400, 405-416
- quadrupole moments 425 sq.
- β -decay 382-389
- positron emitters of neutron excess —1 :
 - [34 sq., 215 sq.
- light α -nuclei and second saturation
 - [requirement 231
- estimate of spin dependent interactions
 - [214

Like nucleons

- (see Pairs of nucleons)
- definition XIX
- system of two —: effective potentials 59
- argument for force between — 306
- pairing 29
- no pairing in ground state of ${}^{40}\text{K}$ 416

Liquid model

- saturation properties 25
- kinetic effects 27
- Bohr's droplet — 185-188
- statistical treatment 186 sq.
- evaporation 187 sq.
- level density 187
- magnetic moments 400
- quadrupole moments 422

Long range fiction

- definition 210
- estimate of spin dependent interactions
 - [214
- sufficient saturation conditions 227-229
- stability of isobars 234-237
- ground states of light odd mass nuclei
 - [409

Magnetic moment

- definition 390
- in spectroscopic sense 392

and nuclear models 404, 409-414
 and meson field 6, 8
 relativistic correction 417-420
 on strong coupling 21
 exchange — 63
 Landé factor or gyromagnetic ratio [398-401
 general expression for — 397-401
 spin — 397
 orbital — 397
 exchange — 397, 408
 Wigner's approximation 397-400, 403
 quasi-atomic model 400
 liquid model 400
 α -particle model 400 sq.
 effect on
 photodisintegration of deuteron 134, [176-179
 radiative capture of slow neutrons by [protons 137, 180
 disintegration of deuteron by electron [impact 138-141
 of proton 6, 428
 of neutron 8, 428
 of deuteron 93, 95
 of even nuclei 393
 of stable odd nuclei 393, 402, 415 sq.
 of unstable odd nuclei 397, 416
 of odd mass nuclei 393-396, 402-404
 of light odd mass nuclei 409-414
 of conjugate nuclei 401 sq., 408
 of self conjugate nuclei 402, 415 sq.
 of ^3H and ^3He 405-409

Many-body interactions

mechanism on field theory 18
 saturation properties 25 sq.
 evidence from binding energies of [lightest nuclei 304-306

Mass

(see Meson mass)
 of neutron 7
 of meson 15
 of neutrino 386 sq.

Mass-defect

(see Binding energy)
 main properties 23 sq.
 of light nuclei 33
 of light nuclei on Wigner's approxi-
 [ation 212-214

on Fermi gas model 238-240
 influence of spatial correlations on — [257

Meson

(see Meson field, Meson mass)
 mass determination 15
 spin 12, 18
 decay 14, 16-18
 of heavier — into lighter one 332
 life-time (theory) 330-332
 scattering of — by nucleons 428
 neutral meson:
 existence 165
 instability 167-170
 decay into photons 169 sq.

Meson field

definition 11
 covariance properties 12
 gauge invariance 163
 mass-range relation 12-14
 range (see Meson mass, Range of [nuclear force)
 Fourier amplitude 163 sq.
 components: Hermitian and non-
 [Hermitian 163, 165
 charged 163
 neutral 165
 neutral and symmetrical theories 165-167
 [(see Isotopic factor)
 argument for charged fields from
 [saturation properties 226 sq.
 mixture of —s (see Mixed meson [theories)
 cut-off of radial singularity (see Cut-off [meson theories)
 sources 164-167
 source constants 322, 349
 interaction with leptons 14, 330-332
 lepton source constants 330-332
 tables of data for the four types of [meson fields:
 static interactions 322
 velocity dependent interactions 329
 life-time of free meson 330
 β -decay parameter 331
 and static nuclear interactions 13
 charged field 163-165
 neutral and symmetrical theories [165-167
 central interaction 170 sq.
 derivation of general types 320-323
 mechanism of β -decay 14, 330-332
 and anomalous magnetic moments 6, 8

scattering of fast neutrons by protons
[171-176
photodisintegration of deuteron 175-179
radiative capture of slow neutrons by
[protons 180

Meson mass

direct determination 15
from proton-proton scattering 152 sq.
from scattering of fast neutrons by
[protons 173, 176
from radiative capture of neutrons by
[protons 180
from binding energies of lightest nuclei
[305 sq.

Meson potential

(see Central interactions)

Mixed density

definition 219
and exchange interactions 220
and saturation property 222

Mixed meson theories

definition 324
non-static interactions on — 326
"correspondence" point of view 326 sq.,
[329
velocity dependent interactions on —
[327-329
Schwinger's — (symmetrical pseudoscalar
[and vector of unequal masses) 346-348
Møller and Rosenfeld's — (same with
[equal masses) 348-350
spin-orbit coupling and ${}^5\text{He}$ doublet 368

Models of nucleus

(see α -particle model, Fermi gas model,
[Individual model, Liquid model,
[Quasi-atomic model, Resonating
[group structure)
table of — 192
collective 190, 269-289
and magnetic moments 404, 409-414

Morse potential

(see Central interactions)

Neutrino

definition XVI, 3
existence 5
spin 3

statistics 3
mass 386 sq.

Neutron

spin 7, 116 sq.
statistics 7
mass 7
 β -decay 8, 385
magnetic moment 8, 428
polarized — s 118
— cluster 218, 225 sq., 354

Neutron excess

definition 23
critical values 38-40
estimate for large mass numbers 193
quantum number of charge multiplet
[199 sq.
dependence of binding energy on — 24,
[239

Non-central interactions

(see Axial dipole coupling, Spin-orbit
[coupling)

general evidence 311
types of — 314
on meson theory:
static — 322
non-static — 325-329

Non-static interactions

(see Velocity dependent interactions)
on meson theory 165, 325-329
spin precession 165, 325 sq.
charge exchange 165, 325 sq.
types of — 314

Nuclear matter

(see Penetration of fast nucleon into
[nucleus, Standard heavy nucleus)
constant density 22, 195
variation of density 240, 242 sq.
temperature 186
entropy 187
kinetic energy 192-194
specific heat 194
viscosity and heat conductivity 195-198
velocity of sound 242, 250

Nuclear radius

main properties 22
positron emitters of neutron excess —1:
[34 sq.

pairs of light isobars 34 sq.
on Fermi gas model 241 sq.

Nuclear reactions

(see Penetration of fast nucleon into
[nucleus, Photodisintegration of the
[deuteron, Radiative capture of neu-
[trons by protons, Scattering)
according to Bohr's droplet model 185
as sources of neutrons 108 sq., 119 sq.
in neutron detection 110, 121
deuteron formation by proton collision
[155-157, 388 sq.
radiative capture of neutron by deuteron
[300
disintegration of deuteron by nucleon
[impact 300
evaporation of nucleons 187 sq.
and ${}^7\text{Li}$ fine structure 369-371
and mass of neutrino 386

Nucleon

(see Congruent nucleons, Like nucleons,
[Neutron, Nucleon radius, Pairs of
[nucleons, Penetration of fast nucleon
[into nucleus, Proton)
definition XVI
excited states 19-21, 427 sq., 431 sq.
scattering of mesons by — 428
electromagnetic properties 428
magnetic moment due to single — 403

Nucleon radius

definition 23
on strong coupling theory 19-21
on Fermi gas model 241 sq.

Odd mass nuclei

number of — 29
stationary states 204
ground state 207, 211
order of doublets 316 sq.
positron emitters of neutron excess —1:
[34 sq., 215 sq., 382-385
magnetic moments
of — 393-396, 402-404
of light — 405-414
quadrupole moments 420-426

Odd nuclei

definition 29

stationary states 206 sq.
ground state 207, 212, 235-237
nuclei of mass $4a+2$ with odd isobar of
[zero neutron excess 387 sq.
angular momentum 31
magnetic moments
of stable — 393, 402, 415 sq.
of unstable — 397, 416
quadrupole moments 421, 425 sq.

Orbital momentum

(see Orbital shell)
of two-nucleon system 55
S states 72-80, 83-86
states of higher — 74, 88 sq., 172
mixture of states of different — 91,
[333 sq., 355, 410 sq., 416
"mixed" resonance 360-362
selection rules in β -decay 380
orbital magnetic moment 397

Orbital shell

definition 200
— structure
of light nuclei 203-208
and mass-defects of light nuclei 213
order of multiplets 317
configuration of ${}^8\text{Li}$ 357

Ortho-deuterium

(see Para-deuterium)

Ortho-hydrogen

(see Para-hydrogen)

Pairs of nucleons

pairing of like nucleons 29
pairs of protons 215
symmetrical and antisymmetrical coupling
[of nucleons 209
enumeration
of pairs of like nucleons with parallel
[spins 210
of bonds (pairs of nucleons with same
[space-function) 210
spatial correlations of congruent and
[non-congruent nucleons 254-257

Para-deuterium

para-ortho conversion 95
scattering of slow neutrons 115

Para-hydrogen

- para-ortho conversion 95
- scattering of slow neutrons 111-116

Parity

- two-nucleon system 58
- general invariance 199
- of ground state of deuteron 90
- of states of light nuclei 203-208
- of states of light α -nuclei 281-285
- conservation in β -decay 379

Penetrability

- of potential barrier 22

Penetration of fast nucleon into nucleus

- determination of range of nuclear force [154 sq., 268
- differential collision cross-section 258-261
- total collision cross-section 261-263
- energy loss 263-265
- Williams' theorem 264
- range of nucleon in nuclear matter 266
- disintegration stars 266-268

Phase of eigenfunction

- definition 70
- calculation by variational method 81 sq.
- S states of low energy 83-86
 - potential well 86
 - exponential potential 87
 - Hulthén's potential 87
 - meson potential 88
- states of higher orbital momentum 88 sq.
- S phase from proton-proton scattering [149 sq.
- P phase from proton-proton scattering [153, 345
- phases in scattering
 - of neutron by deuteron 297
 - of proton by deuteron 299
 - of helium by helium 270 sq.
 - of nucleon by helium 359 sq., 363 sq.

Photodisintegration of the deuteron

- threshold frequency
 - and neutron mass 7
 - and deuteron binding energy 90
- theory 132-135
- on meson theory 175-180
- on Rarita-Schwinger theory 341 sq.
- on Bethe's neutral vector meson theory [345

Potential energy

- order of magnitude 28
- invariance requirements 53
- decomposition in ordinary and exchange [terms 217-221
- in supermultiplet state 208-211
- on Fermi gas model 238 sq.
- second approximation 243-250
- relativistic corrections 314-319
- and saturation properties 224
- and virial theorem 257 sq.

Proton

- spin 5
- statistics 5
- magnetic moment 6, 428
- quadrupole moment due to single — [423-425
- slow neutron scattering cross-section 110
- proton-proton scattering 142-154, 344 sq.
- deuteron formation by proton collision [155-157, 388 sq.
- scattering in helium 364 sq.
- scattering of α -particles in hydrogen 365

Quadrupole moment

- (see Electric quadrupole moment)

Quasi-atomic model

- (see Orbital shell)
- definition 189 sq.
- stationary states of light nuclei 203-207
 - compared with α -particle model 286
- second saturation requirement 231
- comparison with α -particle model
 - doublet of ${}^7\text{Li}$ 372 sq.
- magnetic moments of odd mass nuclei [404
- magnetic moments of light odd mass [nuclei 409-414
- equal magnetic moments of isotopes [404
- stationary states of light nuclei 203-207
- calculation of matrix-element of β -decay [380-382
- magnetic moments 400
 - of light odd mass nuclei 409-414
 - of light odd nuclei 416
 - of ${}^{39}, {}^{41}\text{K}$ 405
 - of ${}^{40}\text{K}$ 416
- quadrupole moments 425 sq.

Radiative capture of neutrons by protons

theory 135-138

on meson theory 180

experimental data 136 sq.

evidence about ^1S level of deuteron 111*Range of nuclear force*

(see Meson mass)

mass-range relation on meson field

limited range [theory 12-14

and uncertainty relations 12-14

and nuclear radius 22

representation of short range potential

[229

lower limit from deuteron properties 93

from disintegration "stars" 154 sq., 268

and neutron-deuteron scattering 299

range of interaction between two

[α -particles 271*Rarita-Schwinger theory*

definition 337

deuteron ground state 338

proton-neutron scattering 338-341

photodisintegration of deuteron 341 sq.

binding energies of ^3H and ^4He 355 sq. ^5He doublet splitting 366

saturation properties 354

strong coupling 429-431

Relativistic effects

order of magnitude 27

two-nucleon system, "large" and "small"

contact interaction 53 [components 56

second order corrections to Hamiltonian

[314-319

relativistic spin-orbit couplings 316-319,

[328, 410 sq., 416

and ^5He doublet 366-368and ^7Li doublet 372 sq.and ^{13}C doublet 411

relativistic correction

in β -decay 386

to magnetic moment 417-420

Resonating group structure

definition 190 sq.

general method 287-289

stationary states of 3-nucleon system

[290-293

neutron-deuteron scattering 296-298

application to ^3H and ^5He 302*Saturation properties*

definition 24

connexion with range 25

types of interactions with — 25

of exchange interactions 222, 247 sq.

of velocity dependent interactions 223

first saturation requirement 224

necessary conditions 225-227

sufficient conditions 227-230

second saturation requirement 224

condition from light α -nuclei 231

condition from heavy nuclei 231-233

determination of central potential from
[— 233 sq.saturation conditions and stability of
[isobars 234-237Watanabe's modification of saturation
[conditions 247 sq.

and axial dipole couplings 353-355

"conditional" saturation 354

on strong coupling 432 sq.

Scattering

proton-neutron:

theory 98-101, 338-341

transformation to laboratory system
[102

slow neutrons by bound protons

[103-105

experimental data 107-127

angular distribution 123-126, 171-175,
[340, 344, 347

proton-proton:

theory 142-146, 344 sq.

experimental data 146-150

neutron-deuteron:

experimental data 295

angular distribution 296

theory 296-299

proton-deuteron 299 sq.

experimental data 299

theory 300

collision of fast nucleons 258-261

inelastic — of protons and α -particles by
[^7Li 369, 371

of slow neutrons

by bound nucleus 105 sq.

by bound deuteron 106

by hydrogen molecule 111-116

of polarized neutrons by protons 118

of fast neutrons by protons 119-122,

[175 sq.

- of α -particles
 - by helium 270 sq.
 - by hydrogen 365
- of nucleons by helium 358–365
 - theory 358–362
 - experiments 363–365
- of mesons by nucleons 428

Slater-determinant

- and exclusion principle 48
- and quasi-atomic model 190
- and individual model 217
- and Fermi gas model 244
- of ${}^8\text{Be}$ system 272, 276

Spatial correlations

- correlation operator 245
- correlation energy 257
- symmetrically and antisymmetrically
 - [coupled nucleons 209]
- α -clustering 256 sq.
- in standard heavy nucleus 254–257
- in electron systems 257

Spin

- definition XVI
- as dichotomic variable 44 sq.
- of lepton 3
- of proton 5
- of neutron 7
- of meson 18
 - inertia 19–21
- coupling between — and charge 21
- influence on scattering of slow neutrons
 - [106]
- multiplet 200
 - order of nuclear multiplets 316 sq.
 - order of ${}^6\text{He}$ doublet 363, 366 sq.
- saturation of —s 30, 218, 225, 228, 355
- parallel —s 218, 225, 228, 354
- exchange and saturation of binding
 - [226]
- types of spin dependence of nuclear
 - [interactions 312–314]
- precession and non-static interactions
 - [165, 325 sq.]
- mixture of states of different — 355
- transitions in β -decay 379, 381 sq.
- magnetic moment 397

Spin dependence of nuclear interaction

- evidence
 - from deuteron 95–98

- from proton-neutron scattering 110
- from radiative capture of slow
 - [neutrons 111]
- from scattering of slow neutrons by
 - [hydrogen molecules 116]
- excluded on Wigner's approximation 191
- estimate of — 213–215
- and saturation conditions 225 sq.
- singlet-triplet separation for 6-nucleon
 - [systems 307]
- singlet-triplet separation for deuteron
 - [and axial dipole interaction 338,
 - [342–345, 350 sq.]

Spin-orbit coupling

- types of — 314
- relativistic 316–319, 328, 410 sq., 416
 - Thomas precession 316
 - Larmor precession 316
- as non-static interaction on meson theory
 - [329]
- and ${}^6\text{He}$ doublet:
 - relativistic 366–368
 - mixed meson theory 368
- and ${}^7\text{Li}$ doublet (relativistic) 372 sq.
- and magnetic moments of ${}^3\text{H}$ and ${}^3\text{He}$ 409

Stable nuclei

- classification 29
- Fuchs' theory 35–40
- stability conditions 4, 37 sq., 234–237
- stable odd nuclei 29, 31, 237, 393, 402,
 - [415 sq., 426]

Standard heavy nucleus

- (see Nuclear matter, Penetration of fast
 - [nucleon into nucleus])
- definition 193
- binding energy on Fermi gas model
 - [243–250]
- momentum distribution 250–254
- spatial correlations 254–257

"Star"

- (see Penetration of fast nucleon into
 - [nucleus])

Static interactions

- transmission by meson field 11, 162–167
- types of — 314
- on meson theory 320–324
 - derivation 320 sq.

- types of — 322 sq.
- cut-off of singularity 323
- mixture of fields 324
- limitation of scope 327
- on strong coupling 428 sq.

Stationary states

- (see Angular momentum, Fine structure
- [of nuclear levels, Supermultiplet])
- classification 199–203
 - charge multiplet 199
 - spin multiplet 200
 - supermultiplet 200–203
- of two-nucleon system:
 - multiplicity 55
 - angular momentum 55
 - orbital momentum 55
 - "large" and "small" components 56
 - parity 58
 - charge multiplicity 58
 - radial eigenfunctions 68–75, 91, 334
 - states of binding 75–81
 - phases 81–89
 - on strong coupling 428–431
- of three-nucleon system 290–295
 - multiplicity 291
 - on resonating group structure 290–293
 - on individual model 293–295
- of lightest nuclei 301–307
- of light nuclei 203–207, 409–414, 416
- of deuteron:
 - ground state 90–98, 338, 344, 350 sq.
 - ^1S virtual state 110–116
- of ^8Be 271 sq., 276, 282, 284
- of ^{11}B 410 sq.
- distribution of —
 - in heavy nuclei 186 sq.
 - on Fermi gas model 184
- rotations and vibrations
 - of α -nuclei 277–281
 - of lightest α -nuclei 281–285
 - of nuclei of mass number $4a \pm 1$:
 - [285–287]

Statistics

- definition XVI
- connexion with angular momentum 7
- Bose — 7
- Fermi — 3
- of lepton 3
- of proton 5
- of neutron 7

Strength of nuclear potential

definition 67

Strong coupling 19–21, 427–433

Supermultiplet

- definition 200–203
- Wigner approximation 191
- quantum numbers of ground state
 - [211 sq.]
- nuclear potential energy in — state
 - [208–211]
- mass-defects of light nuclei 212–214
- estimate of spin dependent interactions
 - [214]
- sufficient saturation conditions 227–229
- allowed β -transitions:
 - selection rules 378–380
 - classes of β -unstable nuclei 379
 - calculation of matrix-elements 380–382
 - magnetic moments 397–400, 403,
 - [409–414, 416]

Surface energy

- general expression 24
- on Fermi gas model 239 sq.

Three-nucleon systems

- stationary states on resonating group
 - [structure 290–293]
- multiplicity 291
- fundamental wave-equation 292
- doublet states on individual model
 - [293–295]
- scattering of neutrons by deuterons
 - [295–299]
- scattering of protons by deuterons
 - [290 sq.]
- radiative capture of slow neutron by
 - [deuteron 300]
- disintegration of deuteron by nucleon
 - [impact 300]
- binding energy of ^3H 302, 304–306
 - on Rarita-Schwinger theory 355 sq.
- binding energy of ^3He 294, 301
- β -decay of ^3H 385–387
- magnetic moments of ^3H and ^3He
 - [405–409]

Two-nucleon systems

- first order Hamiltonian 53
- stationary states:
 - multiplicity 55

parity 58
 charge multiplicity 58
 angular momentum 55
 orbital momentum 55
 "large" and "small" components 56
 radial wave-equations 59
 central potential 60 sq., 68-75
 non-central potential 91, 334
 states of binding 76-81
 phases 81-89
 comparison of proton-neutron and
 [proton-proton potential 158-160
 general form of central potential 160-162,
 [233 sq.
 central potential on meson theory 170 sq.
 effect of non-central force at small
 [energies 336 sq.
 Møller and Rosenfeld's mixed meson
 [theory 348-350
 deuteron 81-89
 and axial dipole interaction 338,
 [342-345, 350 sq.
 on Rarita-Schwinger theory 338
 on Bethe's neutral vector meson
 [theory 344
 on Schwinger's mixed meson theory
 [347
 on strong coupling 428-431
 proton-neutron scattering 98-131,
 [171-176
 on Rarita-Schwinger theory 338-341
 on Bethe's neutral vector meson
 [theory 344
 on Schwinger's mixed meson theory
 [347
 on strong coupling 431
 photodisintegration of deuteron 132-135,
 [175-179
 on Rarita-Schwinger theory 341 sq.
 on Bethe's neutral vector meson
 [theory 345
 disintegration of deuteron by electron
 [impact 138-141
 deuteron formation by proton collision
 [155-157, 388 sq.
 proton-proton scattering 142-154
 on Bethe's neutral vector meson
 [theory 344 sq.
 on Schwinger's mixed meson theory
 [347 sq.
 radiative capture of slow neutrons
 [135-138, 180

Uncertainty relations

connexion with limited range 12-14

Variational method

variation-iteration method 80, 302,
 [304-306
 Hartree-Fock method and quasi-atomic
 [model 189 sq., 231
 on Fermi gas model 232, 241, 243,
 [248-250
 combined with resonating group
 [method 302
 and two-nucleon system:
 states of binding 78
 phases 81 sq.
 and three-nucleon system, energy of
 [doublet state 293-295
 and interaction of two α -particles 272 sq.,
 [276 sq.
 applied to
 ^3H and ^5He 302
 ^2H , ^3H and ^4He 304-306
 ^6Li and ^6He 306 sq.
 ^3H and ^4He on Rarita-Schwinger
 [theory 355 sq.

Velocity dependent interactions

representation by integral operator 222
 saturation property 223
 first order — 314
 second order — 314-319
 on meson theory 327-329, 349
 derivation 328
 types of — 329

Virtual level

definition 84 sq.
 of deuteron 110 sq., 116
 of ^8Be 271
 of ^5He 302, 357, 359 sq., 367 sq.

Virtual processes

definition 12
 photon decay of neutral meson 169 sq.
 virtual polarization of α -particle 273
 decay of heavier meson into lighter one
 [332

Well potential

(see Central interactions, Rarita-
 [Schwinger theory)

LIST OF TABLES

Units and Constants

U.21	Fundamental constants	XV
U.22	Nuclear constants	XV, 438

Physical properties of the proton-neutron system

6.22-1	Scattering of thermal neutrons by protons	107
-2	Scattering of cooled neutrons by paraffin	108
-2a	Scattering of thermal neutrons by cooled hydrogen gas	109
-3	Scattering of resonance neutrons by protons	109, 447
6.31	Scattering of neutrons by molecular hydrogen	113
6.34	Scattering radii and excited deuteron level for neutron spin $\frac{1}{2}$	117
6.411	Scattering of neutrons of 0.1 ... 1 MeV energy by protons	120
6.412	Scattering of D(d,n) neutrons by protons	121
6.413	Scattering of very fast neutrons by protons	122
6.42-1	Investigation of recoil proton tracks	123
-2	Scattering anisotropy of fast neutrons according to the Italian physicists	125
6.431-1	Scattering parameters in terms of width of potential well	127
-2	Finite range corrections	128
-3	Widths and depths of potential wells	130
6.432	Data on proton-neutron potentials	131

Central interaction between protons

7.12	S-phases derived from first group of proton-proton scattering experiments	150, 449
7.13	Parameters of proton-proton potentials from scattering data	153

The charge independence of nuclear interaction

8.1	Comparison of proton-proton and proton-neutron potential	158
8.33	Scattering of fast neutrons by protons; theoretical results	175
8.34-1.	Theoretical photodisintegration cross-sections for $h\nu = 2.62$ MeV	177, 452
-2	Measurements of total photodisintegration cross-section	179, 453

Survey of nuclear models

9.3	Nuclear models	192
-----	----------------	-----

Nuclear multiplets

10.21	Nuclei of odd mass number	204
10.211	Light nuclei of odd mass number	205
10.22	Nuclei of mass number $A = 4a + 2$	206
10.221	Light nuclei of mass number $A = 4a + 2$	205
10.23	Nuclei of mass number $A = 4a$	207
10.231	Light nuclei of mass number $A = 4a$	207
10.24	Ground states of light nuclei	207
10.311	Sign of P'' for odd A	211

Saturation properties of nuclear forces

11.11-1	Decomposition of mean nuclear energy	220
-2	Decomposition of mean nuclear energy in states with saturated spins	221

Properties of heavy nuclei

12.33	The quantities $F(u)$, $G(x)$ and $\hbar$ for some types of nuclear potentials	265
-------	---	-----

Collective nuclear models

13.13	Binding energies of α -nuclei	275
13.21	Parity of levels of ^{12}C	282
13.211	Rotation parameters of light α -nuclei	285

Properties of light nuclei

14.12	Scattering cross-sections of deuteron for neutrons	295, 453
-------	--	----------

General types of non-central and non-static nuclear forces

15.21	Types of static and first order nuclear interactions	314
15.31	Static interactions according to meson field theory	322
15.34	Velocity dependent interactions on meson field theory	329, 453
15.35	Value of β -decay parameter $1/\tau_0$ on meson theory	331

Non-central couplings and properties of heavier nuclei

17.51	Excited state of ^7Li	369, 455
-------	--------------------------------	----------

Allowed β -transitions

A1.132	Allowed β -transitions of class (b) nuclei, on Gamow-Teller interaction	382
A1.21	Positron emitters of neutron excess —1	383
A1.22	Decay processes of nuclei of mass $4a + 2$, involving an odd isobar without neutron excess	387, 456

Electromagnetic properties of nuclei

A2.21-1	Magnetic moments of stable odd nuclei	393
-2	Magnetic moments of odd mass nuclei	395, 396, 456
-3	Magnetic moments of long-lived odd nuclei	397
A2.252-1	Ground states and orbital Landé factors of $2p$ shell nuclei on Wigner's approximation	409
-2	Ground states and magnetic moments of $2p$ shell nuclei according to α -particle model	412
-3	Observed and calculated magnetic moments of $2p$ shell nuclei	413
A2.253	Ground states and magnetic moments of $2s$ shell nuclei according to α -particle model	414
A2.26	Ground states of stable odd nuclei	415
A2.31	Electric quadrupole moments of nuclei	421, 456
A2.322	Calculated quadrupole moments of light nuclei	426

TABLE OF ATOMIC NUCLEI

INTRODUCTION

Description of table

In the following table, the nuclei are classified by *mass number*, which is, of course, the rational arrangement from a theoretical point of view. Only those properties which are directly related to nuclear forces are listed in the table; fig. 3.42 at the end of the book provides some information about the abundances of the stable isotopes, while the electromagnetic properties are fully dealt with in Appendix II.

Columns 1 to 5 give the characteristic numbers concerning the constituent nucleons and the chemical symbols. Use is made of the following types:

clarendon	for stable nuclei,
antique	for α -unstable nuclei *,
<i>italics</i>	for β -unstable nuclei.

Isomers are distinguished by an asterisk affecting the chemical symbol. Only those nuclei have been listed whose charge number is certain and mass number either certain or probable. Remaining uncertainties in this respect, or in the position of isomers, are indicated by brackets.

Columns 6, 7, 8 contain the characteristics of the decay of unstable nuclei. *Column 6* gives the life-time: s = second, m = minute, h = hour, d = day, a = year. *Column 7* describes the mode of decay: K = K -capture, $I.T.$ = isomeric transition. Whether or not the emission of γ -rays accompanies the decay process is indicated between brackets after the symbol of the mode of decay; when known, the different γ -rays emitted are enumerated and the relative probabilities of the various competing modes of decay given. The symbols for the γ -rays are provided with indices only when the decay scheme has been investigated, so that it is established that these γ -rays are emitted after the decay process; the order in which they appear between the brackets is that of their successive emission, in so far as this order can be fixed unambiguously **. *Column 8* contains the energies of the particles or γ -rays emitted. For α -rays it gives the total energy liberated (i.e. the energy of the α -particles plus that of the recoil nucleus); only the α -rays corresponding to transitions between ground levels are considered. For β -particles, the maximum energy of the spectrum is given. In each case (except for isomeric transitions), the γ -ray energies are listed under the product nucleus, since they correspond to transitions between levels of that nucleus.

* For nuclei which are both α - and β -unstable, italics are used.

** E.g. in a case like $^{198}_{79}\text{Au}$, the order of succession of $\gamma_2 \gamma_3$ cannot be fixed by means of the available evidence.

Columns 9 to 12 are concerned with binding energies; first the difference between atomic weight and mass number is given, then the total binding energy in two different units, finally the average binding energy per nucleon. In these data the last decimal has been retained only because it may become significant in the computation of *differences* of binding energies, which are often known with greater accuracy than the binding energies themselves.

List of references

The table is entirely the work of Mr. A. H. WAPSTRA * and presents, it is hoped, various improvements on the similar one of Mattauch and Flügge, on which it is primarily based. It embodies all "class A" and "class B" nuclei of G. SEABORG's table (*Rev. mod. Phys.* 16, 1, 1944), those contained in the Plutonium project table (*Rev. mod. Phys.* 18, 513, 1946) ** and later discovered or confirmed nuclei. Older references to the original publications will be found appended to the tables just quoted, and a similar list of more recent publications accompanies the present one.

This list, however, offers only a *selection* of the latest articles; it is hoped that, by consulting those, the reader will be able to trace any other reference he may be interested in. Usually, the articles quoted are those from which the data adopted in the table have been taken; whenever the quotation refers only to the data of some particular columns of the table, these columns are indicated by their ordinal numbers. In some cases, however, it has also been thought advisable to quote papers whose results show some discrepancy from those adopted; such quotations are preceded by the indication "disc". Finally, a number of items in the list are just explanatory remarks, indicating e.g. how a particular mass-defect given in the table has been computed. It must be noted that new data have been incorporated in the table as late as September 1948; this entails, unfortunately, some discrepancies of minor importance between the information given by the table and that contained either in the text or in the diagrams at the end of the book.

Additional remarks

Mass-defects of light nuclei ($A < 65$). The mass-defect determinations in this part of the table follow the general lines indicated by Mattauch and Flügge; the only modifications of M & F's figures are those rendered necessary by the results of recent investigations, as indicated in the list of references.

An independent study of the available evidence for the range $A = 20 \dots 38$ has also been carried out by H. R. ALLAN, who has very kindly put the results of his discussion at our disposal and carefully

* Now at the *Instituut voor Kernfysisch Onderzoek*, Ooster Ringdijk 18, Amsterdam.

** An article by J. SURUGUE (*J. Physique* 8, 145, 1946) has also been consulted.

compared them with the corresponding portion of the present table. This comparison reveals a satisfactory agreement up to $A = 28$, and again in the interval $A = 35 \dots 38$; in almost all cases, the differences between the two sets of mass-defects fall within the limits of the estimated accuracy of the determination. In the interval $A = 29 \dots 34$, however, there is a large unexplained discrepancy (of the order of one mMU) between the values derived from nuclear reaction data (adopted by Allan) and those based on the mass spectrograph measurement of ^{32}S (adopted by M & F and in the present table).

The mass-defects of the nuclei in the interval $A = 39 \dots 43$ can only be fixed in relation to each other, but cannot yet be connected to the rest of the series; this is indicated, as in M & F, by the undetermined correction $-\epsilon$.

Mass-defects of heavier nuclei ($A = 65 \dots 205$). For this group, M & F give only the mean packing fraction for all stable isotopes of the same element, which is of little use. The corrected table M & F* gives the individual mass-defects only of $^{203}, ^{205}\text{Tl}$, and of a few other nuclei which have a single stable isotope. In other cases, however, when the mean packing fraction is obtained for pairs of isotopes of nearly equal masses, this value can be attributed to each of them and the individual mass-defects estimated in this way. This has been done, using M & F*'s data, for such pairs as $^{69}, ^{71}\text{Ga}$, $^{99}, ^{101}\text{Ru}$, etc. Moreover, mass spectrographic data (quoted in M & F and M & F*) for other groups of isotopes, such as $^{78}, ^{82}, ^{84}, ^{86}\text{Kr}$, were used to estimate the corresponding mass-defects.

Pairs of neighbouring "stable" isobars. The three pairs of isobars $^{113}_{48}\text{Cd} - ^{113}_{49}\text{In}$, $^{115}_{49}\text{In} - ^{115}_{50}\text{Sn}$ and $^{123}_{51}\text{Sb} - ^{123}_{52}\text{Te}$ are given as stable in violation of the general stability conditions (3.42). The newly discovered isotope $^{138}_{57}\text{La}$, if stable, would give rise to a double anomaly of the same kind, $^{138}_{56}\text{Ba} - ^{138}_{57}\text{La} - ^{138}_{58}\text{Ce}$. It is probable, however, that in those cases one of the partners of each pair is actually unstable, with a very long life-time, and transforms into the other one, by β -decay if the isobar of lower charge number is the unstable one, and by K -capture in the opposite case. Although all attempts to establish the occurrence of either transmutation have hitherto failed in the cases mentioned, the instance of the isobaric pair $^{187}_{75}\text{Re} - ^{187}_{76}\text{Os}$ would tend to strengthen the belief in the correctness of the above conjecture. While this pair was until quite recently considered as another example of the anomaly under discussion (and is still indicated as such in fig. 3.42 and in the diagram of stable and unstable nuclei), it has now been found that $^{187}_{75}\text{Re}$ is β -active, with a life-time of $(4 \pm 1) \cdot 10^{12}$ years*.

Mass-defects of the heavy radioactive nuclei. For this class of nuclei, the data given by M & F* are erroneous (A. WAPSTRA, *Nature* 161, 529. 1948). In order to obtain a more consistent set of data, Wapstra

* S. NALDRETT and W. LIBBY, *P.R.* 73, 487 (Erratum 929). 1948; N. SUGARMAN and H. RICHTER, *P.R.* 73, 1411. 1948.

first determines the differences between the mass-defects of the members of each radioactive family and that of its last member (viz. ^{206}Pb , ^{207}Pb , ^{208}Pb and ^{209}Bi); then, he fixes the relative positions of the different families on the energy surface by requiring that all isotopes of any element should lie on a set of parallel parabolas (one for each sheet of the energy surface); or, at any rate, that they should lie as closely as possible to such a set. The parameters of these parabolas can be fixed by using isotopes within the same family. The distance between the sheets of the energy surface corresponding to even and odd nuclei is found by considering the isobars of mass 210; the sheet corresponding to odd mass nuclei is then *assumed* to lie midway between the two others, an assumption supported *a posteriori* by the consistency of the results. In order to check this consistency, it is useful to construct a relief diagram of the type represented in fig. 3.41 (at the end of the book). It is interesting to note that the first diagram of this kind drawn up by Wapstra on the same assumptions as M & F about the decay schemes of the various nuclei exhibited at some places a "ruggedness" contrasting with the generally smooth appearance of the energy surface. The occurrence of such irregularities may point to mistakes in the interpretation of some decay scheme, or to some still unknown feature of a decay process. For instance, it was recognized by Wapstra that a much better mutual agreement could be achieved by assuming that the 0,970 MeV γ -ray observed to accompany the β -decay of ^{228}Ac follows the emission of the electron; moreover, the new mass-defect obtained for ^{225}Ra from an improved determination of the starting point for the Np family suggests that the β -decay of this nucleus is followed by the emission of a γ -ray (not yet observed) of about 1 MeV. The adjustment finally adopted by Wapstra is estimated by him to be accurate to about 0,5 MeV.

The latest discoveries of new radioactive series in the U, Th and Ac families may lead to further minor improvements of the mass-defects listed in the table; in particular, the new data might help to fix the position of the sheet for odd mass nuclei more definitely.

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A MeV		
									10 ⁻⁴ MU	MeV			
1	2	3	4	5	6	7	8	9	10	11	12		
1	0	n	1	1	30 m	β ⁻		89,45 ± 0,25	0,0	0,00	0,00		
	1	H	-1	0				81,31 ± 0,03	0,0	0,00	0,00		
2	1	H	0	1				147,24 ± 0,06	23,5	2,19	1,09		
3	1	H	1	2	12,1 ± 0,5 a	β ⁻	0,011 ± 0,002	170,04 ± 0,20	90,2	8,39	2,80		
	2	He	-1	1				169,88 ± 0,20	82,2	7,65	2,55		
4	2	He	0	2				38,60 ± 0,31	302,9	28,20	7,05		
5	2	He	1	3	≈ 6 · 10 ⁻²⁰ s	α + n		137	294	27,4	5,48		
6	2	He	2	4	0,85 ± 0,05 s	β ⁻	3,5 ± 0,6	207,1 ± 7	313,5	29,19	4,86		
	3	Li	0	3				169,17 ± 0,51	343,1	31,94	5,32		
7	3	Li	1	4	43 d	K (γ)	γ 0,474 ± 0,004	181,63 ± 0,57	420,1	39,11	5,59		
	4	Be	-1	3				190,89 ± 0,66	402,7	37,49	5,36		
8	3	Li	2	5	0,89 ± 0,02 s very short	β ⁻ 2 α	12 0,116 ± 0,010	249,67 ± 0,66	441,5	41,10	5,18		
	4	Be	0	4				78,07 ± 0,39	605,0	56,31	7,039		
9	4	Be	1	5	very short	2 α + p		149,58 ± 0,62	622,9	57,98	6,442		
	5	B	-1	4				161,04 ± 0,68	603,3	56,16	6,240		
10	4	Be	2	6	2,5 · 10 ⁶ a	β ⁻ (no γ)	0,560 ± 0,010	166,22 ± 1,36	695,7	64,76	6,476		
	5	B	0	5				161,69 ± 0,70	692,1	64,43	6,443		
	6	C	-2	4				208,6 ± 1,2	637,1	59,30	5,930		
11	5	B	1	6	20,5 m	β ⁺	0,981 ± 0,005	129,01 ± 0,50	814,2	75,80	6,709		
	6	C	-1	5				150,17 ± 0,75	784,9	73,07	6,643		
12	5	B	2	7	0,022 ± 0,002 s	β ⁻	12	168	864,7	80,49	6,707		
	6	C	0	6				38,80 ± 0,25	985,8	91,76	7,515		
13	6	C	1	7	9,93 ± 0,03 m	β ⁺ (no γ)	1,21 ± 0,005	75,61 ± 0,43	1038,4	96,66	7,435		
	7	N	-1	6				99,04 ± 0,44	1006,8	93,72	7,209		
14	6	C	2	8	5100 ± 200 a	β ⁻ (no γ)	0,1563	77,41 ± 0,43	1126,1	104,82	7,487		
	7	N	0	7				75,30 ± 0,16	1120,0	104,26	7,447		
15	7	N	1	8	126 ± 5 s	β ⁺	1,7	48,70 ± 0,72	1236,1	115,06	7,671		
	8	O	-1	7				78	1198,6	111,58	7,439		
16	7	N	2	9	7,35 ± 0,05 s	20% β ⁻ 40% β ⁻ (γ ₁) 40% β ⁻ (γ ₂)	10,3 ± 0,3 4,3 3,8	110 ± 4	1263	117,6	7,350		
	8	O	0	8				γ ₁ γ ₂	6,2 ± 0,1 6,7 ± 0,3	0,00	1366,1	127,16	7,948
17	7	N	3	10	4,14 ± 0,04 s	β ⁻ (n)	2,7 ± 0,4	119	1345	125,2	7,36		
	8	O	1	9				45,0 ± 0,6	1410,5	131,30	7,724		
	9	F	-1	8				75,8	1371,6	127,68	7,511		
18	8	O	2	10	112 ± 4 m	β ⁺	0,72 ± 0,02	48,5 ± 1,8	1496,5	139,30	7,739		
	9	F	0	9				67,0 ± 1,9	1469,8	136,82	7,601		

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
19	8	O	3	11	27 ± 0,5 s	30 % β^- 70 % $\beta^- (\gamma?)$	4,5 ± 0,3 2,9 ± 0,3	93,7 ± 3,5	1540,7	143,45	7,550
	9	F	1	10				45,4 ± 1,2	1580,9	147,16	7,745
	10	Ne	-1	9	20,3 ± 0,5 s	β^+	2,20	79,8	1538,4	143,20	7,537
20	9	F	2	11	12 s	$\beta^- (\gamma)$	5,0	65,4	1650,3	153,63	7,682
	10	Ne	0	10		γ	2,2	-11,05 ± 0,61	1718,7	159,98	7,999
21	10	Ne	1	11				7,4 ± 1,9	1789,7	166,60	7,933
	11	Na	-1	10	23 ± 2 s	β^+		35	1754	163,3	7,775
22	10	Ne	2	12			1,30 ± 0,03	-14,2 ± 3,0	1900,7	176,93	8,042
	11	Na	0	11	3,0 ± 0,2 a	$\beta^+ (\gamma_1)$	0,575	17,0 ± 3,1	1861,4	173,30	7,877
23	10	Ne	3	13	40,7 ± 0,8 s	β^-	4,1 ± 0,3	16,7 ± 4,8	1959,2	182,53	7,932
	11	Na	1	12				-28,5 ± 2,0	1996,3	185,83	8,080
	12	Mg	-1	11	11,8 ± 0,3 s	β^+	2,82	12,7 ± 3,9	1947,0	181,24	7,880
24	11	Na	2	13	14,8 h	$\beta^- (\gamma_1 \gamma_2)$	1,390 ± 0,005	-19,0 ± 3,2	2076,3	193,32	8,055
	12	Mg	0	12		γ_1	1,380	-78,9 ± 3,1	2128,0	198,09	8,254
						γ_2	2,758				
25	11	Na	3	14	58,2 ± 1,3 s	55 % β^- 45 % $\beta^- (\gamma?)$	3,7 ± 0,3 2,7	-22,9 ± 3,7	2169,6	201,96	8,078
	12	Mg	1	13				-62,7 ± 1,8	2201,3	204,91	8,196
	13	Al	(-1)	(12)	≈ 8 s	β^+					
26	12	Mg	2	14				-94,0 ± 2,4	2322,0	216,15	8,313
	13	Al	0	13	7,25 ± 0,2 s	β^+	2,99	-51,0 ± 4	2270,9	211,39	8,130
27	12	Mg	3	15	10,1 ± 0,1 m	$\beta^- (\gamma_1)$	1,75 ± 0,05	-71,9 ± 1,3	2389,4	222,42	8,238
						25 % $\beta^- (\gamma_2 \gamma_3)?$					
	13	Al	1	14		γ_1	1,05 ± 0,08	-102,0 ± 0,8	2411,3	224,46	8,313
						γ_2	0,84 ± 0,03				
28	13	Al	2	15			0,64 ± 0,03				
	14	Si	-1	13	4,92 s	β^+	3,54	-52,0 ± 1,8	2353,2	219,05	8,113
29	13	Al	3	16	6,7 m	β^-	2,5	-126,1 ± 5	2614,3	243,36	8,392
	14	Si	1	15				-153,1 ± 1,4	2633,2	245,12	8,452
30	14	Si	2	16				-103,1 ± 1,6	2575,1	239,70	8,266
	15	P	-1	14	4,6 ± 0,2 s	β^+	3,63 ± 0,07				
31	14	Si	3	17				-169,0 ± 3,2	2738,5	254,92	8,497
	15	P	0	15	130,6 ± 1,5 s	β^+	3,0 ± 0,1	-126,0 ± 3,4	2687,4	250,16	8,339
32	14	Si	3	17	157,3 ± 1,3 m	$\beta^- (\text{no } \gamma)$	1,8	-155,1 ± 4	2814,1	261,96	8,450
	15	P	1	16				-174,4 ± 1,9	2825,3	262,99	8,484
	16	S	-1	15	3,18 ± 0,04 s	β^+	3,85 ± 0,07	-121,7 ± 3,7	2764,4	257,33	8,301
32	15	P	2	17				-172,5 ± 0,7	2912,8	271,14	8,473
	16	S	0	16	14,30 ± 0,03 d	$\beta^- (\text{no } \gamma)$	1,712 ± 0,008	-191,1 ± 0,7	2923,3	272,12	8,504

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation	W—A	δ		δ /A
									MeV	10 ⁻⁴ MU	
1	2	3	4	5	6	7	8	9	10	11	12
33	16	S	1	17	2,4 ± 0,2 s	β^+	4,13 ± 0,07	-198,6 ± 4	3020,2	281,14	8,519
	17	Cl	-1	16				-143,2 ± 4	2956,7	275,23	8,340
34	15	P	4	19	12,40 ± 0,12 s	75 % β^- 25 % $\beta^- (\gamma)$	5,1 3,2	-176	3095	288,2	8,476
	16 17	S Cl	2 0	18 17	33,2 ± 0,5 m	γ' 80 % β^+ 20 % $\beta^+ (\gamma')$	3,4 5,1 ± 0,3 2,4	-230,1 ± 2,0 -163 ± 3,6	3141,2 3066	292,40 285,5	8,600 8,397
35	16	S	3	19	87,1 d	β^-	0,1691	-210,3 ± 2,0	3210,8	298,88	8,539
	17	Cl	1	18	1,88 ± 0,04 s	β^+	4,38 ± 0,07	-211,6 ± 1,9	3204,0	298,25	8,521
	18	A	-1	17				-153,5 ± 2,1	3137,7	292,08	8,345
36	16	S	4	20	≈ 10 ⁶ a	β^- β^+, K	0,64	-220	3310	308,1	8,559
	17	Cl	2	19				-213,5 ± 3,5	3295,3	306,75	8,521
	18	A	0	18				-221,0 ± 4,0	3294,7	306,69	8,519
37	16	S	(5)	(21)	5,04 ± 0,02 m	10 % β^- 90 % $\beta^- (\gamma_1)$	4,3 1,6 2,8	-177	3356	312,5	8,446
	17	Cl	3	20	34,1 ± 0,3 d	γ_1 K (no γ) $\beta?$	0,33	-223,1 ± 1,0	3394,4	315,97	8,540
	18	A	1	19							
38	17	Cl	4	21	38,5 ± 0,5 m	52 % β^- 16 % $\beta^- (\gamma_1)$ 32 % $\beta^- (\gamma_2 \gamma_1)$	4,99 ± 0,06 2,82 1,15	-200,3 ± 3,2	3461,0	322,18	8,478
	18	A	2	20	7,65 ± 0,1 m	γ_1 γ_2 $\beta^+ (\gamma_1)$	2,15 ± 0,05 1,65 ± 0,05 2,53	-253,9 ± 1,3	3506,5	326,41	8,583
							-191,8	3436,3	319,95	8,420	
39	19	K	1	20				-250 — ϵ	3583,9	333,61	8,554
40	18	A	4	22	2,4 ± 1 · 10 ⁸ a	γ 30 % β^- 70 % K(γ)	1,55 ± 0,05	-245,1 ± 1,2	3676,6	342,24	8,556
	19	K	2	21			1,35 ± 0,05	-244 — ϵ	3667,3	341,38	8,535
	20	Ca	0	20				-249 — ϵ	3672,9	341,98	8,539
41	18	A	5	23	109,4 ± 1 m	0,7 % β^- 99,3 % $\beta^- (\gamma_1)$	2,55 1,18 ± 0,03 1,37 ± 0,06 1,1 ± 0,1	-226,0	3746,9	348,79	8,507
	19	K	3	22		γ_1 γ'	-252,5	3765,3	350,59	8,551	
20	Ca	(1)	(21)	8,5 ± 0,8 d	K(γ')		-260 — ϵ	3764,3	350,49	8,549	
21	Sc	-1	20	0,87 ± 0,03 s	β^+	4,94 ± 0,07					
42	19	K	4	23	12,4 ± 0,2 h	70 % β^- (no γ) 30 % $\beta^- (\gamma?)$	3,50 ± 0,12 2,04 1,51	-244 — ϵ	3846,6	358,16	8,528
	20	Ca	2	22		γ		-283 — ϵ	3877,1	360,91	8,593
43	20	Ca	3	23	3,92 ± 0,02 h	γ 80 % β^+ 20 % K(γ)	1,65	-278 — ϵ	3961,2	368,83	8,577
	21	Sc	1	22			1,13 ± 0,05	-255 — ϵ	3930,0	365,92	8,510

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A MeV
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
44	20	Ca	4	24			1,33				
	21	Sc	2	23	3,92 ± 0,03 h	25 % β + 75 % $K(\gamma)$	1,33				
	21	Sc*	2	23	2,44 ± 0,03 d	I. T. γ	0,268 ± 0,005				
45	20	Ca	5	25	152 d	β -(no γ)	0,260 ± 0,005	-321 ± 6	4183,4	389,52	8,656
	21	Sc	3	24				-325 ± 6	4179,3	389,04	8,645
	22	Ti	1	23	3,08 ± 0,06 h	β +	1,24	-302,4 ± 7	4148,6	386,18	8,582
46	20	Ca	6	26							
	21	Sc	4	25	85 ± 1 d	6 % β -(γ_1) 94 % β -($\gamma_2 \gamma_1$)	1,49 ± 0,01 0,36 ± 0,01	-318,5 ± 6,3	4262,3	396,86	8,627
	22	Ti	2	24		γ_1 γ_2	0,88 ± 0,02 1,12 ± 0,02	-345,1 ± 6,2	4280,7	398,48	8,663
47	20	Ca	(7)	(27)	5,8 d	β -(γ)	1,1				
	21	Sc	(5)	(26)	3,40 d	γ	1,3	-346,9	4380,1	407,79	8,676
	22	Ti	3	25		β -(no γ)	0,46	-352,6 ± 9,4	4377,7	407,50	8,670
48	20	Ca	8	28							
	21	Sc	6	27	1,81 ± 0,02 d	β -($\gamma_2 \gamma_1$)	0,640 ± 0,007	-359,0	4481,7	417,24	8,694
	22	Ti	4	26		γ_1 γ_2	1,33 ± 0,03 0,98 ± 0,02	-366,4 ± 4,7	4480,9	417,12	8,690
	23	V	2	25	16,0 ± 0,2 d	58 % β + ($\gamma_2 \gamma_1$) 42 % K	0,716 ± 0,015	-321,8	4428,2	412,31	8,590
49	20	Ca	9	29	2,5 ± 0,1 h	β -(γ)	2,3 ± 0,1				
	20	Ca(*)	(9)	(29)	30 m	β -					
	21	Sc	7	28	57 ± 2 m	γ	0,8 ± 0,1	-332,4 ± 5,3	4544,6	423,05	8,634
	22	Ti	5	27		β -(no γ)	1,8 ± 0,1				
	23	V	3	26	600 ± 50 d	γ γ	0,19 1,55	-351,8 ± 5,2	4555,8	424,08	8,655
	23	V*	3	26	33 ± 1 m	K (no β + no γ)					
	24	Cr	1	25	41,9 ± 0,3 m	β + β + (γ)	1,9 1,45				
50	22	Ti	6	28				-376,8 ± 3,8	4670,2	434,74	8,695
	23	V	4	27	3,7 ± 0,2 h	β +					
	24	Cr	2	26							
51	22	Ti	7	29	72 ± 2 d	β -(γ_3)	0,36	-405,3	4788,2	445,78	8,741
	22	Ti(*)	7	29	2,9 ± 0,1 m	β -(γ)					
	23	V	5	28		γ_3 γ_1 γ_2	1,02 0,237 ± 0,001 0,330 ± 0,001	-420,9 ± 6,9	4795,6	446,41	8,753
	24	Cr	3	27	26,4 ± 1,0 d	97 % K (no γ) ≈ 0,3 % $K(\gamma_1)$ ≈ 3 % $K(\gamma_2)$		-420	4787	445,6	8,737
	25	Mn	1	26	46 ± 2 m	β +	2,0				

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
52	23	V	6	29	3,74 ± 0,01 m	$\beta^-(\gamma_1)$	2,05	-438,8 ± 7,5 -475,4	4903,0	456,40	8,777
	24	Cr	4	28		γ_1	1,46 ± 0,03		4931,5	459,17	8,830
				γ_2		0,734 ± 0,015					
				γ_3		0,940 ± 0,021					
	25	Mn	2	27	6,5 ± 1 d	$\beta^+(\gamma_2 \gamma_3 \gamma_1)$ 65 % K(γ)	0,582 ± 0,015	-424,5	4872,4	453,67	8,724
25	Mn*	2	27	21 ± 2 m	$\beta^+(\gamma_1)$ 0,5 % I.T. γ	2,66 ± 0,05 0,392 ± 0,008					
26	Fe	0	26	7,8 h	β^+	0,55					
53	24	Cr	5	29	8,9 ± 0,2 m	β^+		-440	4985	464,0	8,755
	26	Fe	1	27				-336 ± 22	4865	453,0	8,547
54	24	Cr	6	30	310 ± 20 d	K(γ)	γ 0,835 ± 0,015	-397 ± 19	5032	468,4	8,674
	25	Mn	4	29							
	26	Fe	2	28						-400 ± 20	5019
55	24	Cr	(7)	(31)	≈ 2 h	β^-	$\gamma^?$ γ γ γ γ 0,070 0,16 0,21 0,8 1,2	-403 ± 17	5118,9	476,62	8,666
	25	Mn	5	30	≈ 4 a						
	26	Fe	3	29							
	27	Co	1	28	18,2 h	K($\gamma^?$) $\beta^+(\gamma)$	1,50				
56	25	Mn	6	31	2,59 ± 0,02 h	60 % $\beta^-(\gamma_1)$	2,86 ± 0,05	-388 ± 17	5193,5	483,57	8,635
				25 % $\beta^-(\gamma_2 \gamma_1)$		1,05 ± 0,03					
				15 % $\beta^-(\gamma_3 \gamma_1)$		0,75 ± 0,1					
				γ_1		0,845 ± 0,015	-429 ± 17	5226,6	486,52	8,688	
				γ_2		1,81 ± 0,04					
				γ_3		2,13 ± 0,05					
				γ_4		1,24 ± 0,04					
				γ		1,74					
				γ		2,01					
				γ		2,55					
				γ		3,25					
	27	Co	2	29		72 d	$\beta^+(\gamma_4 \gamma_1)$ K(γ)	1,50 ± 0,05	-378 ± 17	5167,7	481,16
	57	26	Fe	5	31	270 d	γ	0,117 ± 0,001	-430	5317	495,1
				γ	0,130 ± 0,001						
				γ	0,202 ± 0,004						
				γ	0,215 ± 0,004						
27	Co	3	30	36,9 ± 1,5 h	β^+ K(γ)	0,26					
28	Ni	1	29		β^+	0,67 ± 0,1					

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation	W—A	δ		δ /A					
									MeV	10 ⁻⁴ MU		10 ⁻⁴ MU	MeV			
1	2	3	4	5	6	7	8	9	10	11	12					
58	26	Fe	6	32	72 d	γ_1	0,805 ± 0,012	-402,9 ± 4,1	5363,1	499,23	8,607					
	27	Co	4	31		10 % β+ (γ_1) 90 % K (γ_1)	0,470 ± 0,015									
	28	Ni	2	30	10 m	β+										
	29	Cu	(0)	(29)												
59	26	Fe	7	33	44 d	50 % β- (γ_1) 50 % β- (γ_2)	0,257 ± 0,008 0,460 ± 0,007	-475,2 ± 4	5532,9	515,17	8,732					
	27	Co	5	32	γ_1 γ_2	1,30 ± 0,02 1,10 ± 0,02										
	28	Ni	(3)	(31)	15 a	β+	0,05									
	29	Cu	(1)	(30)	81 ± 2 s	β+										
60	27	Co	6	33	5,3 a	β- ($\gamma_1 \gamma_2$)	0,308 ± 0,008	-471,3 ± 3,5	5618,5	523,14	8,719					
	27	Co*	6	33	10,7 m	90 % I.T. γ 10 % β- (γ_3)	0,056 ± 0,003 1,25 ± 0,06									
	28	Ni	4	32	24,6 ± 0,3 m	γ_1 γ_2 γ_3	1,30 ± 0,03 1,10 ± 0,02 1,50 ± 0,05	-501,9 ± 3,4	5641,0	525,10	8,752					
	29	Cu	2	31		5 % β+ β+ (γ_3)	3,3 1,8									
61	27	Co	7	34	1,75 ± 0,05 h	β- (no γ)	1,1	-447	5684	529,2	8,68					
	28	Ni	5	33	3,4 ± 0,1 h	β+ (no γ) K (no γ)	1,225 ± 0,015	-460 ± 15	5688,5	529,53	8,681					
	29	Cu	3	32				-435 ± 15	5654,9	526,53	8,632					
62	28	Ni	6	34	10,1 ± 0,1 m	β+	2,6	-504,1 ± 3,9	5822,1	541,96	8,741					
29	Cu	4	33													
63	28	Ni	7	35	long	γ_1 γ_2 γ 90 % β+ (no γ) 7 % β+ (γ_1) 3 % β+ (γ_2) K (γ)	0,960 ± 0,008 1,89 ± 0,06 2,60 ± 0,08	-512,2 ± 6,3	5911,5	550,28	8,735					
	29	Cu	5	34	38,3 ± 0,5 m		2,36 ± 0,04 1,40 0,5									
	30	Zn	3	33												
64	28	Ni	8	36	12,8 ± 0,3 h	γ	1,20 ± 0,05	-525,6 ± 5,6	6022,5	560,61	8,760					
	29	Cu	6	35		65 % β- (no γ) 32 % β+ (no γ) 3 % K (γ)	0,578 ± 0,003 0,659 ± 0,003	-507,4 ± 5,6	5996,1	558,16	8,721					
	30	Zn	4	34	48 ± 2 m	β+										
	31	Ga	(2)	(33)												
65	28	Ni	9	37	2,6 ± 0,3 h	β- (γ)	1,9	-532 ± 30	6110	568,8	8,750					
	29	Cu	7	36		γ γ γ γ'	0,280 0,65 0,93 1,14 ± 0,02									

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
	30	Zn	5	35	250 ± 5 d	γ γ 53 % K (no γ) 45 % K (γ') 2,2 % β^+ K (γ)	0,054 0,117 0,47	-516	6086	566,7	8,718
	31	Ga	3	34	15 m						
66	28	Ni	10	38	56 h	β^-					
	29	Cu	8	37	5,05 ± 0,05 m	$\beta^- (\gamma_1)$	2,58				
	30	Zn	6	36		γ_1	1,32 ± 0,04				
	31	Ga	4	35	9,2 ± 0,2 h	β^+	3,9 ± 0,2				
67	29	Cu	(9)	(38)	56 h	β^-					
	30	Zn	7	37		γ γ γ	0,297 0,180 0,0925				
	31	Ga	5	36	3,26 ± 0,03 d	K (γ)					
68	30	Zn	8	38							
	31	Ga	6	37	66 ± 3 m	β^+	1,85				
	32	Ge	4	36	195 d	K					
69	30	Zn	9	39	57 ± 2 m	β^- (no γ)	0,86 ± 0,04	-455	6382	594,1	8,610
	30	Zn*	9	39	13,8 ± 0,4 h	I.T. γ	0,439				
	31	Ga	7	38				-463	6383	594,2	8,612
	32	Ge	(5)	(37)	1,66 d	β^+					
70	30	Zn	10	40							
	31	Ga	8	39	19,8 ± 0,4 m	$\beta^- (\gamma)$	1,68				
	32	Ge	6	38							
71	31	Ga	9	40		γ	0,6	-476	6575	612,0	8,620
	32	Ge	7	39	11 d	K (γ)					
	32	Ge*	7	39	37 ± 1,5 h	β^+	1,2				
	33	As	5	38	52 m	β^+					
	34	Se	3	37	44 m	β^+					
72	30	Zn	12	42	49 h	5 % β^- 95 % $\beta^- (\gamma)$	1,6 0,3				
	31	Ga	10	41	14,25 ± 0,20 h	9,5 % $\beta^- (\gamma_1)$ 8 % $\beta^- (\gamma_2 \gamma_1)$ 10,5 % $\beta^- (\gamma_3 \gamma_2 \gamma_1)$ 32 % $\beta^- (\gamma_4 \gamma_1)$ 40 % $\beta^- (\gamma_5 \gamma_1)$	3,15 2,52 1,48 0,955 0,64				
	32	Ge	8	40		γ_1 γ_2 γ_3 γ_4 γ_5	0,835 0,631 1,05 2,20 2,51				
	32	Ge*	8	40	5 ± 0,5 · 10 ⁻⁷ s	I.T. γ'	0,691				
	33	As	6	39	26 h	$\beta^+ (\gamma)$	2,78 ± 0,10				
	34	Se	4	38	9,5 d	K					

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
73	30	Zn	(13)	(43)	< 2 m	β^-					
	31	Ga	(11)	(42)	5 h	β^- (no γ)	1,4				
	32	Ge	9	41		γ	0,052 $\pm$ 0,003				
	33	As	7	40	90 $\pm$ 10 d	K(γ)					
	34	Se	5	39	7,0 h	β^+ (no γ) K	1,29				
74	32	Ge	10	42							
	33	As	8	41	17,5 $\pm$ 0,1 d	β^+ (γ ?) β^- (γ ?) γ'	0,9 1,3 0,582				
	34	Se	6	40							
	35	Br	4	39	106 m	β^+ (no γ)	1,5				
75	32	Ge	11	43	89 $\pm$ 6 m	β^- (γ)	1,2				
	33	As	9	42		γ'	0,35				
	34	Se	7	41	127 $\pm$ 2 d	K(γ')	0,18				
76	32	Ge	12	44							
	33	As	10	43	26,75 $\pm$ 0,15 h	60% β^- (no γ) 25% β^- (γ_1) 15% β^- ($\gamma_2 \gamma_1$)	2,76 2,16 0,94				
						γ_1	0,57 $\pm$ 0,02				
						γ_2	1,25 $\pm$ 0,03				
						γ	1,84				
						γ	2,15				
77	32	Ge	13	45	12 h	β^- (γ)	2,0				
	32	Ge(*)	(13)	(45)	59 $\pm$ 2 s	β^-	2,8				
	33	As	(11)	(44)	40 h	β^-	0,7				
	34	Se	9	43							
	34	Se*	9	43	17,5 $\pm$ 0,3 s	I.T. γ	0,15				
	35	Br	7	42	48 h						
	36	Kr	5	41	65 m	β^+ K(γ)	1,7				
78	33	As	12	45	65 $\pm$ 3 m	β^- (γ')	1,4				
	34	Se	10	44		γ'	0,27				
						γ	0,046				
						γ	0,108				
	35	Br	8	43	6,4 $\pm$ 0,1 m	β^+ (γ)	2,3				
	36	Kr	6	42				-550 $\pm$ 16	7234	673,4	8,633
79	34	Se	(11)	(45)	< 10 m or > 7 $\cdot$ 10 ⁸ a	β^-					
	35	Br	9	44							
	36	Kr	7	43	1,30 d	β^+ (γ)	0,4 $\pm$ 0,1				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A MeV
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
80	34	Se	12	46	18,5 ± 0,5 m 4,54 ± 0,10 h	β ⁻ (γ) 3% β ⁺ I.T. γ' γ' γ	2,00 ± 0,10 0,73 ± 0,10 0,037 0,049 < 0,5				
	35	Br	10	45							
	35	Br*	10	45							
	36	Kr	8	44							
81	33	As	(15)	(48)	< 10 m 18 m 57 ± 1 m	β ⁻ β ⁻ (no γ) I.T. γ	1,5 0,098				
	34	Se	(13)	(47)							
	34	Se*	(13)	(47)							
	35	Br	11	46							
82	34	Se	14	48	33,9 ± 0,3 h 20 m	β ⁻ (γ) γ γ γ	0,465 0,547 0,787 1,35	-610 ± 12	7652	712,3	8,687
	35	Br	12	47							
	36	Kr	10	46							
	37	Rb	(8)	(45)							
83	34	Se	15	49	30 m 67 ± 3 s 140 ± 10 m	β ⁻ (γ) β ⁻ (γ') γ γ γ	1,5 3,4 0,17 0,37 1,1				
	34	Se*	15	49							
	35	Br	13	48							
	36	Kr	11	47							
	36	Kr*	11	47	113 m	I.T. γ γ	0,029 0,046				
84	34	Se	16	50	≈ 2 m 33 m 6,5 h 40 d	β ⁻ β ⁻ (γ) β ⁺	5,3	-618 ± 13	7839	729,7	8,687
	35	Br	14	49							
	36	Kr	12	48							
	37	Rb	(10)	(47)							
	37	Rb(*)	(10)	(47)							
	38	Sr	8	46							
85	35	Br	15	50	3,0 ± 0,5 m ≈ 10 a 4,5 h	β ⁻ β ⁻ (no γ) β ⁻ (γ) γ γ γ'	0,74 0,94 0,17 0,37 0,8				
	36	Kr	13	49							
	36	Kr*	13	49							
	37	Rb	11	48							
	38	Sr	9	47	65 d	K(γ')					
	38	Sr*	9	47	70 m	I.T. γ	0,170				
86	36	Kr	14	50	19,5 d	β ⁻ (γ)	1,60 ± 0,03	-616 ± 13	8016	746,2	8,67
	37	Rb	12	49							
	38	Sr	10	48							

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A				
									10 ⁻⁴ MU	MeV					
1	2	3	4	5	6	7	8	9	10	11	12				
87	35	Br	17	52	50 ± 10 s	β^-	≈ 4 0,132 0,25 0,034 0,053 0,082 0,102 0,129 0,386 0,5								
	36	Kr	15	51	74 m	β^-									
	37	Rb	13	50	5,8 ± 1,0 · 10 ¹⁰ a	$\beta^-(\gamma)$									
	38	Sr	11	49		$\beta^-(\gamma)$									
						γ									
						γ									
						γ									
						γ									
						γ									
	38	Sr*	11	49	2,75 ± 0,1 h	I.T.						γ			
39	Y	9	48	80 ± 3 h	K(no γ)										
39	Y*	(9)	(48)	14 ± 2 h	I.T.	γ									
88	36	Kr	16	52	2,92 ± 0,16 h	β^-	2,5								
	37	Rb	14	51	17,5 ± 0,5 m	β^-	5,06 ± 0,15								
	38	Sr	12	50		γ	2,8								
						γ	1,87 ± 0,05								
						γ	0,908								
	39	Y	(10)	(49)	87 d	K(γ)	1,2								
	39	Y(*)	10	49	2,0 ± 0,2 h	β^+									
89	36	Kr	17	53	2,6 m	β^-	4,5 ± 0,3 1,51								
	37	Rb	15	52	15,4 ± 0,2 m	β^-									
	38	Sr	13	51	54 d	$\beta^-(\text{no } \gamma)$									
	39	Y	11	50	78 ± 1 h	$\beta^+(\text{no } \gamma)$	1,0								
	40	Zr	9	49		$\beta^+(\text{no } \gamma)$									
	40	Zr*	9	49	4,5 m	I.T. or K(γ)									
	90	36	Kr	18	54	33 s	β^-					0,6 2,45			
37		Rb	16	53	short	β^-									
38		Sr	14	52	25 a	$\beta^-(\text{no } \gamma)$									
39		Y	12	51	62 h	$\beta^-(\text{no } \gamma)$									
40		Zr	10	50	21 h	β^+	1								
41		Cb	(8)	(49)											
91	36	Kr	19	55	≈ 8 s	β^-	3,2 1,3 1,53 $\approx 1,3$ 0,61 0,15 0,94								
	37	Rb	17	54	short	β^-									
	38	Sr	15	53	9,7 h	60 % β^-									
	39	Y	13	52	57 ± 3 d	40 % $\beta^-(\gamma)$									
						$\beta^-(\text{no } \gamma)$									
						γ									
	39	Y*	13	52	51 m	I.T.						γ'			
	40	Zr	11	51		γ									
						γ									
						γ									
	41	Cb	(9)	(50)	55 d	K(γ)									
92	36	Kr	20	56	3 s	β^-	3,5 0,6 1,38 0,59 $\approx 1,0$								
	37	Rb	18	55	short	β^-									
	38	Sr	16	54	2,7 h	β^-									
	39	Y	14	53	3,5 h	$\beta^-(\gamma)$									
	40	Zr	12	52	11 ± 1 d	γ									
	41	Cb	10	51		β^-									
	42	Mo	8	50		$\beta^-(\gamma)$									
						γ									

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
93	36	Kr	21	57	2,0 s	β^-					
	37	Rb	19	56	short	β^-					
	38	Sr	17	55	7 m	β^-					
	39	Y	15	54	11,5 h	$\beta^-(\gamma)$	3,1				
		(Zr)			?	γ	0,7				
	41	Cb	11	52		γ'	1,6				
	41	Cb*	11	52	55 $\pm$ 5 d	I.T.	γ	0,94			
						γ	0,15				
	42	Mo	9	51	6,70 $\pm$ 0,05 h	$\beta^+(\gamma')$	0,7				
						$\beta^+(\gamma')$	0,3				
94	36	Kr	(22)	(58)	1,4 s	β^-					
	37	Rb	(20)	(57)	short	β^-					
	38	Sr	(18)	(56)	$\approx$ 2 m	β^-					
	39	Y	(16)	(55)	20 m	$\beta^-(\gamma)$					
	40	Zr	14	54							
	41	Cb	12	53	> 100 a						
	41	Cb*	12	53	6,6 m	I.T.	γ	$\approx$ 0,05			
						$\beta^-(\gamma')$		1,4			
	42	Mo	10	52		γ'	0,380 $\pm$ 0,004				
						γ	0,873 $\pm$ 0,004				
						γ	1,48 $\pm$ 0,01				
						γ	1,85 $\pm$ 0,01				
						γ	2,74 $\pm$ 0,01				
	43	Tc	(8)	(51)	< 50 m	30 % β^+ 70 % $K(\gamma\gamma')$	2,45 $\pm$ 0,03				
95	43	Tc*	(8)	(51)	50 $\pm$ 2 m	I.T.	γ	0,0334			
	39	Y	17	56	< 3 h	β^-					
	40	Zr	15	55	65 d	2 % β^- 98 % $\beta^-(\gamma)$	1,0 0,394				
	41	Cb	13	54	36,5 d	γ	0,73				
						$\beta^-(\gamma')$	0,154				
	41	Cb*	13	54	80 h	I.T.	γ	0,24			
	42	Mo	11	53		γ'	0,776	-618	8773	816,9	8,599
						γ_1	0,201 $\pm$ 0,002				
						γ_2	0,810 $\pm$ 0,005				
						γ_3	0,570 $\pm$ 0,002				
						γ_4	1,017 $\pm$ 0,010				
						γ	0,95				
	43	Tc	9	52	20,0 $\pm$ 0,5 h	$K(\gamma')$					
	43	Tc(*)	(9)	(52)	62 d	0,8 % β^+ (no γ) 29 % K (no γ) 29 % $K(\gamma_2\gamma_1)$ 38 % $K(\gamma_3\gamma_1)$ 3 % $K(\gamma_4)$	$\approx$ 0,4				
	44	Ru	7	51	1,65 $\pm$ 0,05 h	β^+ $K(\gamma)$	1,1				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A MeV
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
96	40	Zr	16	56							
	42	Mo	12	54		γ	0,92				
	43	Tc	10	53	4,2 $\pm$ 0,1 d	K(γ)					
	43	Tc(*)	(10)	(53)	2,7 $\pm$ 0,4 h	β^+					
	44	Ru	8	52							
97	36	Kr	25	61	short	β^-					
	37	Rb	23	60	short	β^-					
	38	Sr	21	59	short	β^-					
	39	Y	19	58	short	β^-					
	40	Zr	17	57	17,0 $\pm$ 0,2 h	$\beta^-(\gamma)$	2,1				
	41	Cb	15	56	75 $\pm$ 3 m	γ	$\approx$ 0,8				
						$\beta^-(\gamma)$	1,4				
	42	Mo	13	55		γ	0,78	-631	8965	834,8	8,606
						γ'	0,097				
	43	Tc	11	54	93 $\pm$ 5 d	γ	0,23				
98	42	Mo	14	56		K(γ')					
	43	Tc	(12)	(55)	2,8 $\pm$ 0,1 d	K(γ)					
	44	Ru	10	54		γ	1,3 $\pm$ 0,2 0,9 $\pm$ 0,1				
99	42	Mo	15	57	67 $\pm$ 2 d	90 % $\beta^-(\text{no } \gamma)$	1,03				
						10 % $\beta^-(\gamma_2\gamma_1)$	0,24				
	43	Tc	13	56	9,4 $\cdot$ 10 ⁵ a	$\beta^-(\text{no } \gamma)$	0,32				
	43	Tc*	13	56	6,6 $\pm$ 0,4 h	γ_2	0,71				
	44	Ru	11	55		I.T. γ_1	0,136	-643 $\pm$ 31	9140	850,8	8,594
100	42	Mo	16	58							
	44	Ru	12	56		γ	0,6				
						γ	1,2				
	45	Rh	(10)	(55)	19,4 h	γ'	1,8				
						γ'	0,090				
101						5 % β^+	3,0				
						95 % K(γ)					
	46	Pd	(8)	(54)	4,0 d	K(γ')					
101	42	Mo	17	59	14,6 $\pm$ 0,3 m	β^-	2,2				
						$\beta^-(\gamma)$	1,0				
	43	Tc	15	58	14,0 $\pm$ 0,3 m	γ	0,3				
						γ	0,9				
						$\beta^-(\gamma')$	1,2				
	44	Ru	13	57		γ'	0,30	-655 $\pm$ 31	9332	868,7	8,601
	45	Rh	(11)	(56)	4,3 d	K(γ')					
102	46	Pd	(9)	(55)	9 h	10 % β^+	2,3 $\pm$ 0,2				
						90 % K(no γ)					

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ 10 ⁻⁴ MU MeV		δ /A MeV	
1	2	3	4	5	6	7	8	9	10	11	12	
102	44	Ru	14	58	200 ± 10 d	β+(γ) β-	1,13 ± 0,08 1,04 ± 0,08					
	45	Rh	12	57								
	46	Pd	10	56								
103	44	Ru	15	59	42 d	5% β- 95% β-(γ)	0,80 0,2	-550 ± 38	9397	874,7	8,493	
	45	Rh	13	58	53 ± 3 m 17,0 d	I.T. K	γ γ'					0,55 0,020 ± 0,001
	45	Rh*	13	58								
	46	Pd	11	57								
104	44	Ru	16	60	41,8 ± 0,7 s 4,34 ± 0,06 m	β- I.T.	γ	2,58 ± 0,05 0,069				
	45	Rh	14	59								
	45	Rh*	14	59								
	46	Pd	12	58								
105	42	Mo	21	63	short	β-	1,35 0,76 0,60 0,33					
	43	Tc	19	62	short	β-						
	44	Ru	17	61	4,4 ± 0,1 h	β-(γ)						
	45	Rh	15	60	36,5 h	β-(γ)						
	46	Pd	13	59		γ						
106	44	Ru	18	62	1,0 a	β-(no γ)	≈ 0,03	-570 ± 32	9678	900,8	8,499	
	45	Rh	16	61	30 s	82 % β-(no γ) 18 % β-(γ')	3,55 ± 0,10 2,30 ± 0,10					
	46	Pd	14	60		γ' γ' γ' and γ γ γ	0,51 ± 0,02 1,25 ± 0,05 0,72 ± 0,02 1,06 ± 0,03 1,63 ± 0,04					
	47	Ag	12	59	8,2 ± 0,2 d	K(γ)	2,04 ± 0,05					
	47	Ag(*)	12	59	25,0 ± 0,5 m	β+(no γ)						
	48	Cd	10	58								
	107	46	Pd	15	61	very short or > 3 · 10 ⁸ a	β-					0,846 0,0935 0,320 ± 0,010
47		Ag	13	60	44,3 ± 0,2 s	γ ₂ I.T. γ ₁						
47		Ag*	13	60								
48		Cd	11	59	6,7 h	0,31 % β+(γ ₁) 0,42 % K(γ ₁) 99,27%K(γ ₂ γ ₁)						
108	46	Pd	16	62	2,44 ± 0,06 m ≈ 5 h	β- K(γ)	2,06 0,65					
	47	Ag	14	61								
	48	Cd	12	60								
	49	In	(10)	(59)								

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
109	45	Rh	19	64	< 1 h	β^-					
	46	Pd	17	63	13,2 h	β^- (no γ)	1,03				
	47	Ag	15	62				-552 $\pm$ 20	9919	923,3	8,471
	47	Ag*	15	62	40,5 $\pm$ 0,7 s	I.T. γ	0,0884				
	48	Cd	13	61	158 d	γ	0,5				
	49	In	11	60	6,5 h	K(γ) $\approx$ 2% β^+ K(γ)	2				
110	46	Pd	18	64		γ	1,51 $\pm$ 0,04	-592 $\pm$ 33	10057	936,1	8,511
						γ	0,925 $\pm$ 0,03				
						γ	0,650 $\pm$ 0,02				
	47	Ag	(16)	(63)	225 $\pm$ 10 d	K(γ)					
	47	Ag(*)	16	63	24,5 $\pm$ 0,3 s	β^-	2,7				
	48	Cd	14	62							
49	In	12	61	66 $\pm$ 5 m	β^+	1,6 $\pm$ 0,3					
111	46	Pd	19	65	26 m	β^-	3,5				
	47	Ag	17	64	7,6 d	β^- (no γ)	0,80				
	48	Cd	15	63		γ'	0,173				
						γ'	0,247				
	48	Cd*	15	63	48,7 $\pm$ 0,3 m	I.T. γ	0,145				
						γ	0,230				
49	In	13	62	2,7 d	K(γ')						
112	46	Pd	20	66	21 h	β^- (no γ)	0,2				
	47	Ag	18	65	3,2 h	β^-	3,6				
						β^- (γ)	2,2				
	48	Cd	16	64		γ	0,86				
	49	In	14	63	9 m	β^-	1				
						β^+	1,7				
						K(γ')					
	49	In*	14	63	23 m	I.T. γ	0,16 $\pm$ 0,01				
50	Sn	12	62								
113	48	Cd	17	65							
	48	Cd*	17	65	2,3 m						
	49	In	15	64		γ'	0,085 $\pm$ 0,01				
	49	In*	15	64	104 $\pm$ 2 m	I.T. γ	0,39 $\pm$ 0,02				
	50	Sn	13	63	105 $\pm$ 15 d	K(γ')					
114	48	Cd	18	66							
	49	In	16	65	72 s	β^-	1,98 $\pm$ 0,03				
	49	In*	16	65	48,5 $\pm$ 2 d	I.T. γ	0,192 $\pm$ 0,001				
	50	Sn	14	64							
115	48	Cd	19	67	2,33 d	40% β^-	1,20				
						60% β^- (γ_1)	0,56				
	48	Cd*	19	67	43 $\pm$ 3 d	β^- (γ_1)?	1,7				
	49	In	17	66		γ_1	0,65 $\pm$ 0,02				
	49	In*	17	66	4,53 $\pm$ 0,03 h	I.T. γ	0,338 $\pm$ 0,001				
	50	Sn	15	65							

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
116	48	Cd	20	68							
	49	In	18	67	13 s	β^- (no γ)	2,8				
	49	In ^(*)	18	67	54,05 $\pm$ 0,16 m	β^- (γ)	0,85 $\pm$ 0,01				
	50	Sn	16	66		γ	2,32 $\pm$ 0,07	-609 $\pm$ 46	10578	984,7	8,489
						γ	1,85 $\pm$ 0,05				
						γ	1,35 $\pm$ 0,04				
						γ	1,08 $\pm$ 0,04				
117	48	Cd	21	69	2,72 $\pm$ 0,2 h	β^-	$\approx$ 1,5				
	49	In	19	68	1,95 $\pm$ 0,05 h	β^- (no γ)	1,73				
	50	Sn	17	67							
	51	Sb	(15)	(66)	2,8 $\pm$ 0,3 h	K					
118	50	Sn	18	68		γ	1,5	-620 $\pm$ 47	10768	1002,4	8,495
	51	Sb	(16)	(67)	5,1 $\pm$ 0,3 h	K (γ)					
	51	Sb ^(*)	16	67	3,5 m	β^+	3,1 $\pm$ 0,2				
	52	Te	14	66	6,0 d	K (no γ)					
119	50	Sn	19	69				-625 $\pm$ 48	10862	1011,1	8,497
	51	Sb	17	68	39 $\pm$ 1 h	γ	0,2				
						γ	0,5				
						γ	1,4				
120	50	Sn	20	70		K (no γ)					
	51	Sb	18	69	6,0 d	K (γ)	1,1				
	51	Sb ^(*)	18	69	15 $\pm$ 0,8 m	β^+	1,53				
	52	Te	16	68							
121	50	Sn	21	71	28 h	β^- (no γ)	0,4				
	51	Sb	19	70		γ	0,61				
	52	Te	17	69	17 $\pm$ 1 d	K (γ)					
	52	Te [*]	17	69	5 $\pm$ 2 $\cdot$ 10 ⁻⁸ s	I.T.	0,233				
	52	Te ^{**}	17	69	143 $\pm$ 5 d	I.T.	0,050				
122	50	Sn	22	72				-543 $\pm$ 50	11049	1028,5	8,430
	51	Sb	20	71	63 h	β^-	1,76 $\pm$ 0,10				
						β^- (γ)	1,19 $\pm$ 0,05				
	51	Sb [*]	20	71	3,5 m	I.T.	0,146				
123	52	Te	18	70		γ	0,57				
	51	Sb	21	72							
	52	Te	19	71							

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
124	50	Sn	24	74				-552 ± 51	11237	1046.0	8,453
	51	Sb	22	73	60 d	21% $\beta^-(\gamma_1)$ 8% $\beta^-(\gamma_2\gamma_1)$ 9% $\beta^-(\gamma_3\gamma_2\gamma_1)$ 44% $\beta^-(\gamma_4\gamma_1)$ 18% $\beta^-(\gamma)$	2.37 1.62 1.00 0.65 0.42				
	51	Sb*	22	73	1.3 m	$\beta^-(\gamma')$	3.2 ± 0.2				
	51	Sb**	22	73	21 m	I.T.	γ''				
	52	Te	20	72		γ_1 γ_2 γ_3 γ_4 γ	0.603 0.714 0.650 1.708 2.062				
	52	Te*	20	72	12 ± 4 · 10 ⁻⁴ s	I.T.	γ'''				
	53	I	18	71	4.0 ± 0.3 d	β^+	0.069 ?				
	54	Xe	16	70							
125	50	Sn	(25)	(75)	11.8 ± 0.5 m	β^-					
	51	Sb	(23)	(74)	≈ 2.7 a	35 % β^- 65 % $\beta^-(\gamma)$	0.7 0.3 0.56				
	52	Te	21	73		γ					
	53	I	(19)	(72)	56 d	K(no β no γ)					
126	52	Te	22	74							
	53	I	20	73	13.3 ± 0.3 d	$\beta^-(\gamma)$ K	1.20 ± 0.03				
	54	Xe	18	72		γ	0.5				
127	51	Sb	25	76	93 h	$\beta^-(\gamma)$	1.15				
	52	Te	23	75	9.3 ± 0.5 h	γ	0.72				
						$\beta^-(\text{no } \gamma)$	0.70				
	52	Te*	23	75	90 ± 2 d	I.T.	γ	0.086			
	53	I	21	74		γ	0.9				
	54	Xe	(19)	(73)	34 d	K?(γ)					
	54	Xe*	(19)	(73)	75 s	I.T.	γ	0.175 0.125			
128	52	Te	24	76							
	53	I	22	75	24.98 ± 0.02 m	93 % β^- 7 % $\beta^-(\gamma_1)$	2.02				
	54	Xe	20	74		γ_1	0.428				
129	51	Sb	27	78	4.2 h	β^-					
	52	Te	25	77	72 ± 3 m	$\beta^-(\gamma)$	1.75 ± 0.1				
	52	Te*	25	77	32 ± 2 d	I.T.	γ'	0.102			
	53	I	23	76	≈ 10 ⁸ a	γ	γ	0.8 0.3			
						β^-					
	54	Xe	21	75				-546 ± 13	11645	1084.0	8,403

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
130	52	Te	26	78							
	53	I	24	77	12,5 ± 0,5 h	60%β ⁻ (γ ₁ γ ₂ γ ₃) 40%β ⁻ (γ ₄ γ ₁ γ ₂ γ ₃)	1,03 ± 0,02 0,61 ± 0,02 γ ₁ 0,537 ± 0,005 γ ₂ 0,667 ± 0,008 γ ₃ 0,744 ± 0,010 γ ₄ 0,417 ± 0,005				
	54	Xe	22	76							
	55	Cs	(20)	(75)	30 m						
	56	Ba	18	74							
131	52	Te	27	79	25 ± 5 m	β ⁻					
	52	Te*	27	79	29 ± 5 h	I.T. γ	0,177				
	53	I	25	78	8,0 ± 0,2 d	β ⁻ (γ ₁ γ ₂)	0,595 ± 0,010				
	54	Xe	23	77		γ ₁	0,367 ± 0,007				
						γ ₂	0,080 ± 0,001				
						γ'	0,145 ± 0,010				
	55	Cs	21	76	10,2 d	γ	0,22 ± 0,01				
						γ	0,500 ± 0,015				
						γ	1,7 ± 0,1				
	56	Ba	19	75	11,7 ± 0,3 d	K(γ')					
						K(γ)					
132	51	Sb	30	81	≈ 5 m	β ⁻					
	52	Te	28	80	70 h	β ⁻ (γ)	0,28				
	53	I	26	79	2,3 h	γ	0,22				
						50 % β ⁻	2,1				
						50 % β ⁻ (γγ')	1,0				
	54	Xe	24	78		γ	1,4	-550	11918	1109,4	8,405
						γ'	0,68				
	55	Cs	(22)	(77)	7,1 d	K(γ')					
	56	Ba	20	76							
133	51	Sb	31	82	< 10 m	β ⁻					
	52	Te	29	81	60 m	β ⁻					
	53	I	27	80	22 h	β ⁻ (γ)	1,2				
	54	Xe	25	79	5,4 d	γ	0,55				
						β ⁻ (γ)	0,33				
	55	Cs	23	78		γ	0,084				
						γ'	0,320 ± 0,010				
	56	Ba	21	77	> 20 a	K(γγ')					
	56	Ba*	21	77	38,8 h	I.T. γ	0,2764				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A MeV
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
134	51	Sb	(32)	(83)	< 10 m	β ⁻					
	52	Te	(30)	(82)	43 m	β ⁻					
	53	I	(28)	(81)	54 m	β ⁻ (γ)					
	54	Xe	26	80		γ	> 1				
	55	Cs	24	79	1,7 ± 0,1 a	75% β ⁻ (γ ₁ γ ₂ γ ₃) 25% β ⁻ (γ ₄ γ ₁ γ ₂)	0,645 ± 0,02				
	55	Cs*	24	79	3,0 ± 0,3 h	I.T. γ β ⁻	0,16 0,8				
	56	Ba	22	78		γ ₁ γ ₂ γ ₃ γ ₄	0,602 ± 0,012 0,794 ± 0,015 1,35 ± 0,03 0,568 ± 0,015				
135	52	Te	31	83	< 2 m	β ⁻					
	53	I	29	82	6,6 ± 0,3 h	β ⁻	1,35				
	54	Xe	27	81	9,4 h	γ	1,5				
	54	Xe*	27	81	15,6 m	β ⁻ (γ) I.T. γ	0,94 0,54				
	55	Cs	25	80	> 2,5 · 10 ⁴ a	γ β ⁻	0,26				
	56	Ba	23	79							
136	54	Xe	28	82							
	56	Ba	24	80							
	58	Ce	20	78							
137	53	I	(31)	(84)	30 ± 6 s	β ⁻					
	54	Xe	(29)	(83)	3,4 m	β ⁻	≈ 4				
	55	Cs	27	82	33 a	β ⁻ (γ ₁)	0,550 ± 0,005				
	56	Ba	25	81		γ'	0,88 ± 0,1				
	56	Ba*	25	81	158 ± 5 s	I.T. γ ₁	0,663 ± 0,006				
	57	La	(23)	(80)	17,5 ± 0,5 h	K(γ')					
138	54	Xe	(30)	(84)	17 ± 1 m	β ⁻					
	55	Cs	(28)	(83)	32 ± 0,5 m	β ⁻ (γ)	2,6				
	56	Ba	26	82		γ	1,2				
	57	La	24	81							
	58	Ce	22	80							
139	54	Xe	31	85	41 s	β ⁻					
	55	Cs	29	84	7 m	β ⁻					
	56	Ba	27	83	85,6 m	β ⁻ (γ)	2,2				
	57	La	25	82		γ	0,6	-478 ± 35	12448	1158,7	8,336
						γ'	0,21				
	58	Ce	(23)	(81)	140 ± 1 d	K(γ')					
140	54	Xe	32	86	9,8 s	β ⁻					
	55	Cs	30	85	short	β ⁻					
	56	Ba	28	84	12,8 d	75 % β ⁻ 25 % β ⁻ (γ)	1,05 ± 0,05 ≈ 0,4				

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
	57	La	26	83	40,0 $\pm$ 0,3 h	γ 10 % β^- 70 % $\beta^-(\gamma_1)$ 20 % $\beta^-(\gamma)$	0,535 2,12 $\pm$ 0,08 1,40 $\pm$ 0,03 0,90 $\pm$ 0,03				
	58	Ce	24	82		γ γ γ γ_1 γ	0,333 0,505 0,832 1,61 2,52				
	59	Pr	22	81	3,4 $\pm$ 0,1 m	β^+	2,40 $\pm$ 0,15				
	141	54 Xe	33	87	1,7 s	β^-					
	55 Cs	31	86	short	β^-						
	56 Ba	(29)	(85)	18 m	$\beta^-(\gamma)$						
	57 La	(27)	(84)	3,7 h	$\beta^-(\gamma)$	2,8					
	58 Ce	25	83	30,6 $\pm$ 0,7 d	$\beta^-(\gamma\gamma')$	0,65					
	59 Pr	23	82		γ	0,1371					
	59 Pr*	23	82	7 $\pm$ 2 $\cdot$ 10 ⁻⁵ s	I.T.	γ'	0,2				
	60 Nd	(21)	(81)	1,6 $\pm$ 0,3 h	β^+	0,78					
142	58 Ce	26	84		γ	1,9					
	59 Pr	24	83	19,3 $\pm$ 0,1 h	$\beta^-(\gamma')$ K γ (γ)	2,14 $\pm$ 0,02					
	60 Nd	22	82		γ'	1,0					
143	54 Xe	35	89	1 s	β^-						
	55 Cs	33	88	short	β^-						
	56 Ba	31	87	< 0,5 m	β^-						
	57 La	29	86	19 m	β^-						
	58 Ce	27	85	33 h	$\beta^-(\gamma)$	1,35					
	59 Pr	25	84	13,8 d	γ	0,5					
					$\beta^-(\text{no } \gamma)$	0,95					
	60 Nd	23	83								
144	54 Xe	36	90	short	β^-						
	55 Cs	34	89	short	β^-						
	56 Ba	32	88	short	β^-						
	57 La	30	87	short	β^-						
	58 Ce	28	86	275 d	$\beta^-(\text{no } \gamma)$	0,348					
	59 Pr	26	85	17,5 m	$\beta^-(\gamma)$	3,07					
	60 Nd	24	84		γ	0,135					
	62 Sm	20	82		γ	1,25					
145	60 Nd	25	85								
	62 Sm	21	83	$\geq$ 72 d							
146	60 Nd	26	86					-387 $\pm$ 30	12958	1206,2	8,262

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation	W—A	g		g /A	
									MeV	10 ⁻⁴ MU		10 ⁻⁴ MU
1	2	3	4	5	6	7	8	9	10	11	12	
147	60	Nd	27	87	11,0 d	60 % β- 40 % β-(γ)	0,90 ≈ 0,4					
	61	Pm	25	86	3,7 a	γ	0,58					
	62	Sm	23	85		β-(no γ)	0,20					
148	60	Nd	28	88	5,3 d	β-(γ)	2,5 0,8	-371 ± 20	13122	1221,5	8,253	
	61	Pm	26	87								
	62	Sm	24	86								
149	61	Pm	27	88	47,5 ± 1,5 h	β-(γ)	1,1					
	62	Sm	25	87	γ	0,25						
150	60	Nd	30	90				-356 ± 21	13285	1236,6	8,244	
	62	Sm	26	88								
151	62	Sm	27	89	≈ 20 a							
	63	Eu	25	88								
152	62	Sm	28	90	3,0 · 10 ¹¹ a	α	2,4					
	63	Eu	26	89	≈ 5 a	β-(γ)	1,885 ± 0,012					
	63	Eu(*)	26	89	9,2 h		γ					0,123 ± 0,001
	64	Gd	24	88	γ		0,163 ± 0,001					
						γ	0,725 ± 0,003					
153	61	Pm	31	92	< 5 m	β-	0,73 0,694 0,103					
	62	Sm	29	91	47 ± 1 h	β-(γ)						
	63	Eu	27	90	γ							
	64	Gd	25	89	> 72 d	γ						
154	62	Sm	30	92	22 a	β-(γ)	0,93					
	63	Eu	28	91		β-(γ)	0,62					
	64	Gd	26	90		γ	0,1224					
						γ	0,2473					
						γ	0,2860					
						γ	0,3428					
				γ	0,4078							
				γ	1,23							
155	62	Sm	(31)	(93)	21 m	β-(γ)	1,8					
	63	Eu	29	92	2 ... 3 a	γ	≈ 0,3					
	64	Gd	27	91	β-(γ)	0,23						
						γ	0,0844	-426	13770	1281,8	8,270	
156	61	Pm	(34)	(95)	< 5 m	β-	≈ 0,8					
	62	Sm	(32)	(94)	≈ 10 h	β-	2,4					
	63	Eu	30	93	15,4 d	40 % β- 60 % β-(γ ₁)	≈ 0,5 2,0					
	64	Gd	28	92	γ ₁							

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A MeV
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
157	63	Eu	(31)	(94)	15,4 h	25% β^- 75% $\beta^-(\gamma)$	$\approx 1,7$ $\approx 1,0$				
	64	Gd	29	93		γ γ	0,2 0,6	-432	13954	1298,9	8,273
158	64	Gd	30	94				-303	13916	1295,4	8,199
	66	Dy	26	92							
159	65	Tb	29	94							
160	64	Gd	32	96	3,9 h	β^-		-307	14098	1312,3	8,202
	65	Tb	30	95	72 d	$\beta^-(\gamma)$	0,70				
	65	Tb(*)	30	95			0,0856				
	66	Dy	28	94		γ γ γ γ γ	0,1947 0,2132 0,2980 1,10				
161	66	Dy	29	95							
162	66	Dy	30	96							
	68	Er	26	94							
163	66	Dy	31	97							
164	66	Dy	32	98							
	68	Er	28	96							
165	66	Dy	33	99	156 $\pm$ 3 m	$\beta^-(\gamma_1)$ $\beta^-(\gamma_2)$ $\beta^-(\gamma_3)$	1,24 $\pm$ 0,03 0,88 0,42				
	66	Dy*	33	99	1,25 m	I.T. γ	0,18				
	67	Ho	31	98		γ_1 γ_2 γ_3	0,0914 0,36 0,76				
166	67	Ho	32	99	27,3 $\pm$ 0,5 h	$\beta^-(\gamma)$	1,8				
	68	Er	30	98							
167	68	Er	31	99							
168	68	Er	32	100							
	70	Yb	28	98							
169	68	Er	(33)	(101)	9,4 $\pm$ 0,2 d	$\beta^-(\text{no } \gamma)$	0,33 $\pm$ 0,01				
	69	Tm	31	100		γ	0,4				
	69	Tm*	(31)	(100)	10 ⁻⁸ s	I.T. γ'	0,2				
	70	Yb	(29)	(99)	33 $\pm$ 1,5 d	K($\gamma\gamma'$)					
170	68	Er	34	102	127 $\pm$ 5 d	$\beta^-(\text{no } \gamma)$	1,1				
	69	Tm	32	101							
	70	Yb	30	100							

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
171	68	Er	(35)	(103)	7,5 h	6 % β^- (no γ) 72 % β^- ($\gamma_2 \gamma_1$) 22 % β^- (γ_3)	1,49 $\pm$ 0,05 1,05 $\pm$ 0,03 0,67 $\pm$ 0,03				
	69	Tm	(33)	(102)	500 $\pm$ 100 d	γ_3	0,805 $\pm$ 0,025				
	69	Tm*	(33)	(102)	2,5 $\cdot$ 10 ⁻⁶ s	γ_2	0,305 $\pm$ 0,010				
	70	Yb	31	101		I.T. γ_1	0,113 $\pm$ 0,005				
172	70	Yb	32	102							
173	70	Yb	33	103							
174	70	Yb	34	104							
	72	Hf	30	102							
175	70	Yb	35	105	4,2 $\pm$ 0,2 d	β^- $\beta^-(\gamma)$	0,45 $\pm$ 0,07 0,13				
	71	Lu	33	104		γ	0,35				
176	70	Yb	36	106							
	71	Lu	34	105	2,4 $\cdot$ 10 ¹⁰ a	K $\beta^-(\gamma)$	0,215 $\pm$ 0,015				
	71	Lu*	(34)	(105)	3,67 $\pm$ 0,03 h	β^-	1,25 $\pm$ 0,07				
	72	Hf	32	104		γ	0,260				
177	70	Yb	(37)	(107)	1,9 $\pm$ 0,2 h	β^-	1,15 $\pm$ 0,10				
	71	Lu	35	106	6,6 $\pm$ 0,1 d	β^-	0,440				
	72	Hf	33	105							
178	72	Hf	34	106							
179	72	Hf	35	107							
180	72	Hf	36	108							
	73	Ta	34	107	8,2 $\pm$ 0,2 h	K(γ)					
	73	Ta*	34	107	17 m						
	74	W	32	106							
181	72	Hf	37	109	55 $\pm$ 7 d	$\beta^-(\gamma\gamma')$	0,63 $\pm$ 0,03				
	73	Ta	35	108		γ	0,3455				
						γ	0,4787				
						γ	0,600				
	73	Ta*	35	108	2,2 $\cdot$ 10 ⁻⁵ s	γ''	1,83				
						I.T. γ'	0,1325				
182	74	W	(33)	(107)	140 d	K($\gamma''\gamma'$)					
	73	Ta	36	109	99 $\pm$ 1 d	$\beta^-(\gamma)$	0,52 $\pm$ 0,03				
	73	Ta*	(36)	(109)	16,2 $\pm$ 0,5 m	$\beta^-(\gamma)$					
	74	W	34	108		γ	0,15				
						γ	0,22				
						γ	1,13				
						γ	1,22				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A
									10 ⁻⁴ MU	MeV	
1	2	3	4	5	6	7	8	9	10	11	12
183	74	W	35	109							
184	74 75 76	W Re Os	36 (34) 32	110 (109) 108	54 ± 2 d	γ K(γ)	0,85 ± 0,1				
185	74 75 76	W Re Os	(37) 35 (33)	(111) 110 (109)	73,2 ± 0,5 d 94,7 ± 2 d	β ⁻ (no γ) γ K(γ)	0,430 ± 0,010 0,75				
186	74 75 76	W Re Os	38 36 34	112 111 110	92,8 ± 0,2 h	β ⁻ (no γ)	1,07				
187	74 75	W Re	(39) 37	(113) 112	24,1 ± 0,1 h 4 ± 1 · 10 ¹² a	60% β ⁻ (no γ) 40% β ⁻ (γ) γ γ γ γ γ γ γ β ⁻ I.T.	1,40 ± 0,05 0,55 0,086 0,101 0,135 0,48 0,57 0,69 0,79 0,86 0,043				
	75 76	Re* Os	37 35	112 111	10 ⁻⁶ s	γ	0,135 ?				
188	75 76	Re Os	38 36	113 112	18,9 ± 0,2 h	β ⁻ (γ) γ γ γ γ	2,05 0,16 0,48 0,64 0,94 1,43				
189	76	Os	37	113							
190	76 77 78	Os Ir Pt	38 (36) (34)	114 (113) (112)	10,7 ± 0,3 d long	γ K(γ)	0,25	238 ± 63	16139	1502,3	7,907
191	76 77 78	Os Ir Pt	39 37 (35)	115 114 (113)	31,8 ± 0,2 h 3,00 ± 0,02 d	β ⁻ (γ) γ γ' K?(γ')	0,95 1,17 0,57	246 ± 64	16212	1509,1	7,901
192	76 77 78	Os Ir Pt	40 38 36	116 115 114	75 ± 3 d	β ⁻ (γ) γ γ γ	0,59 ± 0,01 0,307 0,467 0,603	240 ± 63	16316	1518,8	7,910

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation	W—A	δ		δ /A		
							MeV	10 ⁻⁴ MU	10 ⁻⁴ MU	MeV	MeV		
1	2	3	4	5	6	7	8	9	10	11	12		
193	76	Os	(41)	(117)	16,1 ± 0,2 d	β-(γ)	0,142 ± 0,003	249 ± 64	16388	1525,5	7,904		
	77	Ir	39	116			0,039 ± 0,001						
							0,1291						
							0,17						
							1,7						
	78	Pt	(37)	(115)	4,33 ± 0,02 d	K(γ')							
	79	Au	(35)	(114)	15,8 ± 0,3 h	K							
194	77	Ir	40	117	19,0 ± 0,2 h	80% β- 20% β-(γ ₁ γ ₂)	2,07 ± 0,03						
	78	Pt	38	116			γ ₁ γ ₂					0,38 1,65	239 ± 80
195	78	Pt	39	117	195 ± 5 d	γ γ	0,17 1,7	273 ± 78	16535	1539,2	7,893		
	79	Au	(37)	(116)			K(γ)						
196	78	Pt	40	118	13 h 5,55 ± 0,12 d	γ' γ' β- 6% β-(γ ₁) 24% β-(γ ₂) 70% K(γ')	0,139 0,358	235 ± 80	16662	1551,0	7,913		
	79	Au	(38)	(117)			0,43						
	79	Au(*)	(38)	(117)			0,27						
							γ ₁ γ ₂					0,173 0,334	
	80	Hg	36	116									
197	78	Pt	(41)	(119)	19 ± 2 h 3,3 d	β- β-(γ)	0,68	400 ± 40	16579	1543,3	7,834		
	78	Pt*	(41)	(119)			γ ₄ γ ₆					0,127 0,159	
	79	Au	39	118			I.T.(γ ₁) γ ₁ I.T.(γ ₂ γ ₃) γ ₂ γ ₃					0,38 0,25 0,077	
	79	Au*	39	118			K(γ ₃) K(γ ₄ γ ₆)						
	80	Hg	37	117									
	80	Hg*	37	117									
198	78	Pt	42	120	2,76 ± 0,02 d	K	0,970 ± 0,005	317 ± 80	16759	1560,0	7,879		
	79	Au	40	119								85% β-(γ ₁) 15% β-(γ ₂ γ ₃ γ ₁) γ ₁ γ ₂ γ ₃	0,605 0,4112 0,208 0,157
	80	Hg	38	118									
199	78	Pt	43	121	31 m 3,3 d	β- β-(γ) γ	1,8						
	79	Au	41	120			0,38						
	80	Hg	39	119			0,18						
200	80	Hg	40	120									
201	80	Hg	41	121									
202	80	Hg	42	122	11,8 ± 1,2 d	γ K(γ)	0,40						
	81	Tl	(40)	(121)									

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A	Sym- bol
									10 ⁻⁴ MU	MeV	MeV	
1	2	3	4	5	6	7	8	9	10	11	12	13
203	80	Hg	(43)	(123)	51,5 d	β^- $\beta^-(\gamma)$	0,46 0,11	438	17061	1588,1	7,823	
	81	Tl	41	122		γ γ'	0,270 0,470					
	82	Pb	(39)	(121)	52 $\pm$ 0,5 h	$K(\gamma \gamma')$						
	82	Pb(*)	(39)	(121)	10,25 m	β^+	1,66					
204	80	Hg	44	124	3,5 a	β^- (no γ)	0,87	432	17148	1596,2	7,825	
	81	Tl	(42)	(123)								
	82	Pb	40	122	68 $\pm$ 2 m	γ'	$\approx$ 0,2					
	82	Pb*	40	122		I.T.	γ					
	83	Bi	38	121	12 h	$K(\gamma \gamma')$	0,90					
205	80	Hg	45	125	5,5 m	β^-	1,62	449	17229	1603,8	7,823	
	81	Tl	43	124								
206	81	Tl	44	125	4,23 $\pm$ 0,03 m	β^- (no γ)	1,77	461,5	17305,9	1611,35	7,822	RaE''
	82	Pb	42	124	6,4 d	γ	1,1	443,1	17316,1	1612,30	7,827	RaG
	83	Bi	40	123		γ	0,4					
						γ	0,8					
	84	Po	(38)	(122)	9 d	$K(\gamma)$ 90 % $K(\gamma)$ 10 % α	5,3					
207	81	Tl	45	126	4,76 $\pm$ 0,02 m	99,5 % β^- 0,5 % $\beta^-(\gamma)$	1,47 $\approx$ 0,4	478,7	17378,1	1618,07	7,817	AcC''
	82	Pb	43	125	5,7 h	γ	0,87	462,9	17385,8	1618,79	7,820	AcD
	84	Po	(39)	(123)		99,99 % $K(\gamma')$ 0,01 % α	5,2					
208	81	Tl	46	127	3,1 m	10 % β^- ($\gamma_2 \gamma_1$) 18 % β^- ($\gamma_3 \gamma_1$) 72 % β^- ($\gamma_4 \gamma_1$)	1,820 1,586 1,514	525,9	17420,4	1622,01	7,798	ThC''
	82	Pb	44	126	long $\approx$ 3 a	γ_1	2,62	472,0	17466,1	1626,27	7,819	ThD
						γ_2	0,277					
						γ_3	0,511					
						γ_4	0,583					
	83	Bi	(42)	(125)				508	17422	1622,2	7,799	
	84	Po	(40)	(124)		α	5,24					
209	81	Tl	47	128	< 1 h	β^-		562,1	17473,6	1626,97	7,785	
	82	Pb	45	127	3,30 $\pm$ 0,03 h	β^-	0,68 $\pm$ 0,03	521,3	17506,3	1630,01	7,799	
	83	Bi	43	126				514,0	17505,4	1629,93	7,799	
210	81	Tl	48	129	1,32 m	$\beta^-(\gamma?)$	1,80	625,5	17500,6	1629,48	7,759	RaC''
	82	Pb	46	128	22 a	$\beta^-(\gamma)$	0,035	553,6	17563,4	1635,33	7,787	RaD
	83	Bi	44	127	5,0 d	γ	0,012	553,1	17555,8	1634,62	7,784	RaE
						β^- (no γ) 5 $\cdot$ 10 ⁻⁸ % α	1,22					
	84	Po	42	126	140 d	α	5,403	539,9	17560,9	1635,09	7,786	RaF, Po

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A	Sym- bol
									10 ⁻⁴ MU	MeV	MeV	
1	2	3	4	5	6	7	8	9	10	11	12	13
211	82	Pb	47	129	36,1 $\pm$ 0,2 m	80% β^- 20% $\beta^- (\gamma_1)$	1,40 0,571	605,0	17601,5	1638,87	7,767	AcB
	83	Bi	45	128	2,16 m	γ_1 0,32% $\beta^- (\gamma)$ 99,68% α	0,829 6,739	589,9	17608,4	1639,52	7,770	AcC
	84	Po	43	127	5 $\cdot$ 10 ⁻³ s	α	7,581	583,0	17607,2	1639,40	7,769	AcC'
	85	At	41	126	7,5 h	40% K 60% α	6,05					
212	82	Pb	48	130	10,6 h	$\beta^- (\gamma_1)$	0,362	637,5	17658,4	1644,17	7,756	ThB
	83	Bi	46	129	60,5 m	γ_1 66,3% β^- 33,7% α	0,238 2,25 6,200	631,1	17656,7	1644,01	7,755	ThC
	84	Po	44	128	3 $\cdot$ 10 ⁻⁷ s	α	8,947	606,9	17672,8	1645,51	7,762	ThC'
213	83	Bi	47	130	46 m	98% β^- 2% α	$\approx$ 1,25 5,97	664,8	17712,4	1649,20	7,743	
	84	Po	45	129	4,4 $\cdot$ 10 ⁻⁶ s	α	8,496	651,3	17717,8	1649,70	7,745	
214	82	Pb	50	132	26,8 m	$\beta^- (\gamma_1)$	0,65	726,8	17748,1	1652,52	7,722	RaB
	83	Bi	48	131	19,7 m	γ_1 99,96% $\beta^- (\gamma_2)$ 0,04% α	0,294 3,15 5,612	716,8	17749,9	1652,69	7,723	RaC
	84	Po	46	130	1,5 $\cdot$ 10 ⁻⁴ s	γ_2 α	0,607 7,829	676,4	17782,1	1655,69	7,737	RaC'
215	84	Po	47	131	1,83 $\cdot$ 10 ⁻³ s	5 $\cdot$ 10 ⁻⁴ % β^- α	7,509	724,4	17823,6	1659,55	7,718	AcA
	85	At	45	130	$\approx$ 10 ⁻⁴ s	α	8,15	716,4	17823,5	1659,56	7,719	
216	84	Po	48	132	0,158 s	α	6,904	750,1	17887,4	1665,49	7,711	ThA
	85	At	46	131	10 ⁻³ s	α	7,94	755,1	17874,2	1664,26	7,705	
217	85	At	47	132	0,020 s	α	7,157	780,5	17938,3	1670,23	7,697	
218	84	Po	50	134	3,05 m	α	6,1124	831,1	17985,2	1674,60	7,682	RaA
	86	Em	46	132	0,019 $\pm$ 0,002 s	α	7,25	793,1	18007,0	1676,63	7,691	
219	86	Em	47	133	3,92 s	α	6,953	837,8	18051,7	1680,79	7,675	An
	87	Fr	45	132	$\approx$ 10 ⁻⁴ s	α	7,44	834,9	18046,5	1680,31	7,673	
220	86	Em	48	134	54,5 s	α	6,400	857,9	18121,1	1687,25	7,669	Tn
	87	Fr	46	133	$\approx$ 30 s	α	6,81	867,3	18103,6	1685,64	7,662	
221	87	Fr	47	134	4,8 m	α	6,43	888,3	18172,0	1691,99	7,656	
222	86	Em	50	136	3,825 d	α	5,5887	929,9	18227,9	1697,20	7,645	Rn
	88	Ra	46	134	38,0 s	α	6,63	903,1	18238,5	1698,19	7,650	
223	87	Fr	(49)	(136)	21 m	β^-	1,20	952,3	18286,9	1702,69	7,635	AcK
	88	Ra	47	135	11,2 d	α	5,823	939,1	18291,9	1703,16	7,639	AcX
	89	Ac	45	134	$\approx$ 2 m	α	6,76	947,5	18276,5	1701,74	7,630	

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation	W—A	δ		δ /A	Sym- bol	
									MeV	10 ⁻⁴ MU			10 ⁻⁴ MU
1	2	3	4	5	6	7	8	9	10	11	12	13	
224	88	Ra	48	136	3,64 d	α	5,7858	958,8	18361,6	1709,65	7,632	TnX	
	89	Ac	46	135	$\approx 2,5$ h	10 % α 90 % K	6,28	975,5	18338,8	1707,55	7,623		
225	88	Ra	49	137	14,8 d	$\beta^-(\gamma)$	$\approx 0,05$	998,7	18411,3	1714,27	7,619		
	89	Ac	47	136	10,0 d	α	5,906	994,2	18407,6	1713,93	7,617		
226	88	Ra	50	138	1590 a	α	4,879	1021,1	18478,3	1720,51	7,613	Ra	
	90	Th	46	136	30,9 m	α	6,41	1010,6	18472,5	1719,97	7,610		
227	89	Ac	49	138	13,5 a	99 % β^- 1 % α	0,22 5,10	1046,5	18534,2	1725,72	7,602	Ac	
	90	Th	47	137	18,9 d	α	6,159	1045,0	18528,5	1725,19	7,600	RdAc	
	91	Pa	45	136	38 m	83 % α 17 % K	6,58	1056,0	18508,5	1723,35	7,592		
228	88	Ra	52	140	6,7 a	β^-	0,053	1084,7	18593,6	1731,25	7,593	MsTh ₁	
	89	Ac	50	139	6,13 h	$\beta^-(\gamma)$	1,55	1084,0	18586,1	1730,55	7,590	MsTh ₂	
	90	Th	48	138	1,90 a	γ	0,970	1056,9	18605,1	1732,32	7,598	RdTh	
	91	Pa	46	137	22 h	α 2 % α 98 % K	5,517 6,20	1079,0	18575,0	1729,54	7,586		
229	90	Th	49	139	7 · 10 ⁸ a	α	4,94	1094,1	18657,3	1737,18	7,585		
	91	Pa	(47)	(138)		K α							
230	90	Th	50	140	8,3 · 10 ⁴ a	α	4,76	1110,9	18729,9	1743,94	7,582	Io	
	91	Pa	48	139	17 d	β^-							
	92	U	46	138	20,8 d	α	5,96	1113,4	18711,2	1742,20	7,567		
231	90	Th	51	141	24,6 h	β^-	$\approx 0,2$	1150,4	18780,0	1748,60	7,570	UY	
	91	Pa	49	140	3,2 · 10 ⁴ a	α	5,142	1141,4	18780,8	1748,68	7,570	Pa	
232	90	Th	52	142	1,39 · 10 ¹⁰ a	α	4,28	1169,4	18850,4	1755,16	7,565	Th	
	91	Pa	50	141	1,33 d	β^-							
	92	U	48	140	30 a	α	5,41	1153,6	18849,8	1755,10	7,565		
233	90	Th	53	143	23 m	β^-							
	91	Pa	51	142	27,4 d	$\beta^-(\gamma)$	0,23						
	92	U	49	141	1,63 · 10 ⁶ a	γ	0,084	1185,6	18907,3	1760,46	7,556		
						γ	0,298						
						γ	0,338						
						γ	0,310						
						α	4,915						

A	Z	Sym- bol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10 ⁻⁴ MU	δ		δ /A	Sym- bol
									10 ⁻⁴ MU	MeV	MeV	
1	2	3	4	5	6	7	8	9	10	11	12	13
234	90	Th	54	144	24,1 d	β^-	0,300	1229,5	18969,2	1766,22	7,548	UX ₁
	91	Pa	52	143	6,7 h	10% $\beta^- (\gamma')$	1,2	1222,0	18968,6	1766,16	7,548	UZ
	91	Pa*	52	143	1,14 m	90% $\beta^- (\gamma' \gamma'')$	0,45 $\pm$ 0,03	1227,2	18964,4	1765,77	7,546	UX ₂
						0,15% I.T. γ	0,394 $\pm$ 0,005					
						98% β^-	2,32					
						1% $\beta^- (\gamma_1)$	1,54					
						1% $\beta^- (\gamma_2)$	1,50					
	92	U	50	142	2,30 . 10 ⁵ a	0,2% $\beta^- (\gamma_3)$	0,80 $\pm$ 0,05	1201,2	18981,2	1767,34	7,553	U II
						$\gamma' \approx \gamma''$						
						γ_1						
						γ_2						
	93	Np	48	141	4,4 d	γ_3	1,5					
						α	4,79					
	93	Np	48	141	4,4 d	K(γ)						
235	92	U	51	143	7,07 . 10 ⁸ a	α	4,59	1238,4	19033,4	1772,20	7,541	AcU
	93	Np	49	142	240 d	K(γ)						
236	93	Np	50	143	20 h	β^-						
	94	Pu	48	142		α	5,85	1255,3	19089,7	1777,44	7,531	
237	92	U	53	145	7,0 $\pm$ 0,2 d	$\beta^- (\gamma)$	0,26					
	93	Np	51	144	2,25 . 10 ⁶ a	γ	0,5					
	94	Pu	49	143		α	4,85					
						K						
238	92	U	54	146	4,51 . 10 ⁹ a	α	4,21	1313,5	19226,7	1790,20	7,522	U I
	93	Np	52	145	2,0 d	$\beta^- (\gamma)$	1,35					
	94	Pu	50	144	60 a	γ	0,1	1300,4	19223,5	1789,90	7,521	
						γ	0,25					
						γ	1,3					
						α	5,590 $\pm$ 0,005					
239	92	U	55	147	23,5 m	$\beta^- (\gamma_1)$	1,12	1359,3	19270,4	1794,26	7,507	
	93	Np	53	146	2,3 d	γ_1	0,073	1346,3	19275,2	1794,71	7,509	
	94	Pu	51	145	2,41 . 10 ⁴ a	β^-	1,179	1333,6	19279,8	1795,14	7,511	
						$\beta^- (\gamma_1 \gamma_2)$	0,676					
						$\beta^- (\gamma_1 \dots \gamma_4)$	0,403					
						$\beta^- (\gamma_1 \dots \gamma_6)$	0,288					
						γ_1	0,275					
						γ_2	0,227					
						γ_3	0,206					
						γ_4	0,067					
						γ_5	0,061					
						γ_6	0,057					
						α	5,227 $\pm$ 0,005					
	96	Cm	48	144	30 d	α	6,36	1362,4	19324,2	1799,27	7,496	
241	94	Pu	53	147	long	β^-						
	95	Am	51	146	500 a	α	5,56					
						K						
242	95	Am	52	147	18 h	β^-						
	96	Cm	50	146	150 d	α	6,1	1403,5	19462,0	1812,10	7,488	

REFERENCES

- | | | |
|---------------------|--|---|
| <i>Arkiv</i> | | Arkiv för Matematik, Astronomi och Fysik |
| <i>Bull.</i> | | Bulletin of the American Physical Society |
| <i>F&P</i> 1948 | | G. Friedlander and M. L. Perlman, Chart of the Isotopes, General Electric Research Laboratory, April 1948 |
| <i>Helv.</i> | | Helvetica Physica Acta |
| <i>P.R.</i> | | Physical Review |
| <i>Segrè</i> 1946 | | E. H. Segrè, Chart of the Isotopes, Radiation Laboratory, University of California, Sept. 1946 |
-
- | | | |
|------------------|---------|--|
| ¹ n | 6, 7 | A. H. Snell and L. C. Miller, <i>Bull.</i> 23, no 3, F 12. 1948 |
| ³ H | 6 | R. J. Watts and D. Williams, <i>P.R.</i> 70, 640. 1946 |
| | 8 | A. Novick, <i>P.R.</i> 72, 972. 1947 |
| | 8 | N. Goldblatt, E. S. Robinson and R. W. Spencer, <i>P.R.</i> 72, 973. 1947 |
| ⁵ He | 9, disc | from ⁷ Li(d, α) ⁵ He + 13.43 MeV: C. M. G. Lattes, P. H. Fowler and P. Cuer, <i>Proc. phys. Soc.</i> 59, 883. 1947 |
| ⁶ He | 6, 8, 9 | H. S. Sommers and R. Sherr, <i>P.R.</i> 72, 973. 1947 |
| ⁷ Li | 8 | D. J. Zaffarano, B. D. Kern and A. C. G. Mitchell, <i>P.R.</i> 74, 105. 1948 |
| ⁸ Li | 6 | D. J. Hughes et al., <i>P.R.</i> 72, 646. 1947 |
| ⁸ Be | 8 | A. Hemmendinger, <i>P.R.</i> 73, 806. 1948 |
| ¹⁰ Be | 6, 8 | E. McMillan, <i>P.R.</i> 72, 591. 1947 |
| ¹³ N | 8 | L. M. Langer, C. S. Cook and M. B. Sampson, <i>P.R.</i> 71, 906. 1947 |
| ¹⁴ C | 6 | L. D. Norris and M. G. Inghram, <i>P.R.</i> 73, 350. 1948 |
| | 8 | L. M. Langer, C. S. Cook and H. C. Price, <i>Bull.</i> 23, no 4, J 3. 1948 |
| ¹⁶ N | | E. Bleuler et al., <i>Helv.</i> 20, 96. 1947 |
| ¹⁷ N | | N. Knable et al., <i>Bull.</i> 23, no 3, F 9. 1948 |
| | | L. W. Alvarez, <i>Bull.</i> 23, no 3, F 11. 1948 |
| ¹⁹ O | | E. Bleuler and W. Zünti, <i>Helv.</i> 20, 195. 1947 |
| ²¹ Ne | 9 | F. R. Elder, H. T. Motz and P. W. Davison, <i>P.R.</i> 71, 917. 1947 |
| ²¹ Na | 9 | <i>Segrè</i> 1946 |
| | | H. Bradner and J. D. Gow, <i>Bull.</i> 23, no 5, E 5. 1948 |
| ²² Na | 7 | H. Maier-Leibnitz, <i>Z. Physik</i> 122, 233. 1944 |
| | 8 | W. M. Good, D. Peaslee and M. Deutsch, <i>P.R.</i> 69, 313. 1946 |
| ²³ Ne | 6 | O. Huber et al., <i>Helv.</i> 17, 139. 1944 |
| | 9 | from ²³ Ne(β) ²³ Na |
| ²³ Na | 9 | see ²¹ Ne |
| ²³ Mg | 9 | from ²³ Mg(β) ²³ Na |
| ²⁴ Na | 7, 8, 9 | see ³² P |
| | | M. L. Wiedenbeck, <i>P.R.</i> 72, 429. 1947 |
| ²⁵ Na | | see ¹⁹ O |
| ²⁵ Al | | H. Bradner and J. D. Gow, <i>Bull.</i> 23, no 5, E 5. 1948 |
| ²⁶ Al | 9 | from ²⁶ Al(β) ²⁶ Mg |
| | | from ²⁷ Al(γ, n) ²⁶ Al: R. A. Becker, A. O. Hanson and B. C. Diven, <i>P.R.</i> 71, 466. 1947 |
| ²⁶ Mg | 9 | see ¹⁹ O |
| ²⁷ Mg | 8 | S. Eklund and N. Hole, <i>Arkiv</i> 29 A, no 26. 1943 |
| | 7, 8, 9 | see ¹⁹ O |
| | disc | J. Benes, A. Hedgran and N. Hole, <i>Arkiv</i> 35 A, no 12. 1948 |
| ²⁷ Si | 9 | see ¹⁹ O |
| ²⁸ Al | 6 | S. Eklund and N. Hole, <i>Arkiv</i> 29 A, no 26. 1943 |
| | | see ¹⁹ O |
| | disc | J. Benes, A. Hedgran and N. Hole, <i>Arkiv</i> 35 A, no 12. 1948 |
| ²⁸ Si | 9 | see ¹⁹ O |
| | | J. Benes, A. Hedgran and N. Hole, <i>Arkiv</i> 35 A, no 12. 1948 |
| ³² P | 8 | K. Siegbahn, <i>P.R.</i> 70, 127. 1946 |
| ³⁴ P | | E. Bleuler and W. Zünti, <i>Helv.</i> 19, 137. 1946 |
| ³⁴ Cl | 7, 8 | Ho Zah-Wei, <i>P.R.</i> 70, 782. 1946 |
| ³⁵ S | 8 | C. S. Cook, L. M. Langer and H. C. Price, <i>P.R.</i> 74, 548. 1948 |
| ³⁶ S | 9 | <i>Segrè</i> 1946 |

³⁶ Cl	6	Manhattan Project, <i>Science</i> 103, 697. 1946
³⁷ S		see ³⁴ P
³⁷ A	6, 7, 8	P.K. Weimer, J. D. Kurbatov and M. L. Pool, <i>P.R.</i> 66, 209. 1944
³⁸ Cl	6	N. Hole and K. Siegbahn, <i>Arkiv</i> 33 A, no 9. 1946
	7, 8	see ⁶⁰ Ga
³⁸ K	7, 8, 9	M. M. Ramsey, J. L. Meem and A. C. G. Mitchell, <i>P.R.</i> 72, 639. 1947
⁴⁰ K	6, 7	E. Bleuler and M. Gabriel, <i>Helv.</i> 20, 67. 1947
	8	B. Dzelepov, M. Kopjova and E. Vorobjov, <i>P.R.</i> 69, 538. 1946
	8	E. Gleditsch and T. Graf, <i>P.R.</i> 72, 640. 1947
⁴⁰ Ca	9	from ⁴⁰ K(β) ⁴⁰ Ca
⁴¹ A	6, 8	E. Bleuler, W. Bollmann and W. Zünti, <i>Helv.</i> 19, 419. 1946
⁴¹ K	9	from ⁴¹ A(β) ⁴¹ K
⁴² K	7, 8, 9	see ¹⁸ O
		K. Siegbahn and A. Johanson, <i>Arkiv</i> 34 A, no 10. 1947
⁴³ Ca	9	from ⁴³ Sc(β) ⁴³ Ca
⁴³ Sc	6, 7, 8	see ⁴⁴ Sc
	9	from ⁴⁰ Ca(α, p) — 4.27 MeV
⁴⁴ Sc, Sc*	6, 7, 8	C. T. Hibdon, M. L. Pool and J. D. Kurbatov, <i>P.R.</i> 67, 289. 1945
⁴⁵ Ca	6	see ¹⁰³ Pd
	8, 9	A. K. Solomon and L. E. Glendenin, <i>P.R.</i> 73, 415. 1948
⁴⁶ Sc	7, 8, 9	W. C. Peacock and R. G. Wilkinson, <i>P.R.</i> 72, 521. 1947
⁴⁷ Ca		see ¹⁰³ Pd
^{47, 48} Sc	6, 7, 8, 9	C. T. Hibdon and M. L. Pool, <i>P.R.</i> 67, 313. 1945
⁴⁸ V	7, 8, 9	see ⁵² Mn
	7	W. M. Good, D. Peaslee and M. Deutsch, <i>P.R.</i> 69, 313. 1946
⁴⁹ V*		O. Huber, O. Lienhard and H. Wäffler, <i>Helv.</i> 17, 195. 1944
⁵¹ Ti	7, 8	C. E. Mandeville and M. V. Scherb, <i>P.R.</i> 73, 340, 655. 1948
⁵¹ Cr	7, 8	H. Bradt <i>et al.</i> , <i>Helv.</i> 18, 259. 1945
	9	Segrè 1946
⁵² V	6	C. Martelly, <i>C.R.</i> 216, 767, 838. 1943
	7, 8	see ⁵² V
⁵² Cr	9	from ⁵² V(β) ⁵² Cr
⁵² Mn	7, 8, 9	W. C. Peacock and M. Deutsch, <i>P.R.</i> 69, 306. 1946
⁵² Mn*	7, 8	R. K. Osborne and M. Deutsch, <i>P.R.</i> 71, 467. 1947
⁵² Fe		B. B. Cunningham <i>et al.</i> , <i>P. R.</i> 72, 73. 1947
	8	D. R. Miller, R. C. Thompson and B. B. Cunningham, <i>P.R.</i> 74, 347. 1948
⁵³ Cr	9	Segrè 1946
⁵³ Fe	9	from ⁵⁴ Fe(γ, n) ⁵³ Fe — 14.2 $\pm$ 0.4 MeV
		see ⁴⁹ V*
⁵⁴ Cr, Fe	9	H. E. Duckworth, <i>P.R.</i> 62, 19. 1942
⁵⁵ Mn	9	from ⁵⁵ Mn(d, p) ⁵⁶ Mn + 4.76 MeV: A. B. Martin, <i>P.R.</i> 72, 378. 1947
⁵⁵ Fe	7, 8	H. Bradt <i>et al.</i> , <i>Helv.</i> 19, 222. 1946
⁵⁶ Mn	9	from ⁵⁶ Mn(β) ⁵⁶ Fe
⁵⁶ Co	9	from ⁵⁶ Co(β) ⁵⁶ Fe
⁵⁷ Fe	9	Segrè 1946
⁵⁸ Cu		C. E. Leith, A. Bratenahl and B. J. Moyer, <i>P.R.</i> 72, 732. 1947
⁵⁹ Fe	6	see ³⁶ Cl
⁵⁹ Co	9	from ⁵⁹ Co(d, p) ⁶⁰ Co + 5.78 MeV
⁵⁹ Ni		see ³⁶ Cl
⁶⁰ Cu		see ⁵⁸ Cu
⁶⁰ Co, Co*	8, 9	M. Deutsch, L. G. Elliott and A. Roberts, <i>P.R.</i> 68, 193. 1945
⁶⁰ Cu		see ⁵⁸ Cu
⁶¹ Co		T. J. Parmley and B. J. Moyer, <i>P.R.</i> 72, 82. 1947
⁶¹ Cu	8, 9	H. Bradt <i>et al.</i> , <i>Helv.</i> 18, 252. 1945
		see ⁵⁸ Cu
⁶² Cu	6	see ⁵⁸ Cu
⁶³ Ni		F&P 1948
⁶³ Zn	6, 7, 8	H. Bradt <i>et al.</i> , <i>Helv.</i> 19, 221. 1946
	7, 8	O. Huber <i>et al.</i> , <i>Helv.</i> 20, 494. 1947
⁶⁴ Cu	7, 8	H. Bradt <i>et al.</i> , <i>Helv.</i> 19, 219. 1946
	8	M. Deutsch, <i>P.R.</i> 72, 729. 1947
⁶⁵ Ni		J. A. Swartout <i>et al.</i> , <i>P.R.</i> 70, 232. 1946
⁶⁵ Cu	9	see ⁵⁴ Cr
⁶⁵ Zn	9	from ⁶⁵ Cu(p, n) ⁶⁵ Zn — 2.130 MeV
		W. E. Shoupp, B. Jennings and W. Jones, <i>P.R.</i> 73, 421. 1948
		W. M. Good and W. C. Peacock, <i>P.R.</i> 69, 680. 1946

- ⁶⁶Ni H. H. Hopkins, jr. and B. B. Cunningham, *P.R.* 73, 1406. 1948
⁶⁶Cu 7, 8 L. Meitner, *Arkiv* 33 A, no 3. 1946
⁶⁶Ga 8 E. Bleuler and W. Zünti, *Helv.* 19, 375. 1946
⁶⁷Cu see ⁶⁶Ni
⁶⁷Ga 6 D. A. McCown *et al.*, *Bull.* 23, no 3, Z 1. 1948
⁶⁸Ga see ⁶⁶Ni
⁶⁹Zn 8 see ⁶⁶Ga
⁶⁹Ge see ⁶⁷Ga
⁷¹As, Se see ⁶⁶Ni
⁷²Ga 7, 8 S. K. Haynes, *P.R.* 73, 187. 1948
A. C. G. Mitchell, D. J. Zaffarano and B. D. Kern, *P.R.* 73, 1424. 1948
⁷²Ge* J. C. Rowe *et al.*, *P.R.* 73, 1219. 1948
⁷²As 7, 8 A. C. G. Mitchell, E. T. Jurney and M. Ramsey, *P.R.* 71, 825. 1947
⁷²Se see ⁶⁶Ni
⁷³As see ⁶⁷Ga
⁷³Se W. S. Cowart *et al.*, *P.R.* 73, 1454. 1948
⁷⁴As 6 see ⁶⁷Ga
⁷⁴Br L. L. Woodward *et al.*, *Bull.* 23, no 4, J 5. 1948
⁷⁵Se H. N. Friedlander, L. Seren and S. H. Turkel, *P.R.* 72, 23. 1947
8 see ³⁰Cl
6 see ⁷³Se
⁷⁶As 8 L. C. Miller and L. F. Curtiss, *P.R.* 70, 983. 1947
7, 8 K. Siegbahn, *Arkiv* 34 A, no 7. 1947
8 M. V. Scherb and C. E. Mandeville, *P.R.* 73, 418. 1948
⁷⁷Ge* J. R. Arnold and N. Sugarman, *J. Chem. Physique* 15, 703. 1947
⁷⁷Se* M. Goldhaber and C. O. Muelhause, *Bull.* 23, no 3, Z 5. 1948
⁷⁷Br, Kr see ⁷⁴Br
⁷⁹Kr see ⁷⁴Br
⁷⁹Se *F&P* 1948
⁸⁰Br 7, 8 W. C. Barber, *P.R.* 72, 1156. 1947
⁸³Se* see ⁷⁷Ge*
⁸⁴Rb* see ⁸⁰Br
⁸⁷Rb S. Eklund, *Arkiv* 33 A, no 14. 1945
⁸⁰Rb see ⁶⁶Ga
⁹⁰, ⁹¹Cb *F&P* 1948
⁹²Kr-Y S. Katcoff, J. A. Miskel and C. W. Stanley, *Bull.* 23, no 3, I 13. 1948
⁹³Cb* M. L. Wiedenbeck, *P.R.* 68, 1. 1945
⁹³Mo 6, 7, 8 D. N. Kundu and M. L. Pool, *P.R.* 70, 111. 1946
⁹³, ⁹⁴Kr-Y see ⁹²Kr
⁹⁴Cb, Cb* see ⁷⁷Se*
⁹⁴Tc 6 E. E. Motta and G. E. Boyd, *P.R.* 74, 220. 1948
7, 8 O. Huber *et al.*, *P.R.* 73, 1208. 1948
⁹⁵Tc 7, 8 O. Huber *et al.*, *P.R.* 73, 1211. 1948
⁹⁵Tc*, Ru D. T. Eggen and M. L. Pool, *P.R.* 74, 57. 1948
⁹⁶Tc see ⁹⁵Tc*
⁹⁷Tc E. E. Motta, G. E. Boyd and A. R. Brosi, *P.R.* 71, 210. 1947
⁹⁷Ru W. H. Sullivan, N. R. Sleight and E. M. Gladrow, *P.R.* 70, 778. 1946
see ⁹⁵Tc*
⁹⁸Tc E. E. Motta and G. E. Boyd, *P.R.* 74, 220. 1948
⁹⁹Mo 8 L. C. Miller and L. F. Curtiss, *P.R.* 70, 983. 1947
7, 8 C. E. Mandeville and M. V. Scherb, *P.R.* 73, 848. 1948
6 E. E. Motta, G. E. Boyd and Q. V. Larson, *P.R.* 72, 1270. 1947
¹⁰⁰, ¹⁰¹Rh, Pd M. Lindner and I. Perlman, *P.R.* 73, 1202. 1948
¹⁰²Rh 6, 8 N. Hole, *Arkiv* 32 A, no. 3. 1945
¹⁰³Rh* 6 see ¹⁰²Rh
¹⁰³Pd D. E. Matthews and M. L. Pool, *P.R.* 72, 163. 1947
¹⁰⁴Rh 8 N. Hole, *Arkiv* 34 B, no 5. 1947
¹⁰⁶Ru 6 E. Bohr and N. Hole, *Arkiv* 32 A, no 15. 1945
¹⁰⁶Rh 7, 8 W. C. Peacock, *P.R.* 72, 1049. 1947
¹⁰⁷Ag* see ¹⁰⁶Cd
6, 8 H. Bradt *et al.*, *Helv.* 18, 256. 1945
¹⁰⁷Cd see ¹⁰⁶Cd
8 H. Bradt *et al.*, *P.R.* 68, 57. 1945
¹⁰⁸In S. N. Ghoshal, *P.R.* 73, 417. 1948

$^{109}\text{Ag}^*$		see ^{109}Cd
	6, 8	H. Bradt <i>et al.</i> , <i>Helv.</i> 18 , 256. 1945
^{109}Cd		A. C. Helmholz, <i>P.R.</i> 70 , 982. 1946
$^{109}, ^{110}\text{In}$		see ^{108}In , ^{111}In
^{110}Ag		M. Goldhaber, <i>P.R.</i> 70 , 89. 1946
$^{110}\text{Ag}^*$	6	O. Hirzel and H. Wäffler, <i>Helv.</i> 19 , 214. 1946
^{111}Cd , Cd^*		N. Hole, <i>Arkiv</i> 34 B , no 19. 1947
		see $^{77}\text{Se}^*$
^{111}In		D. J. Tendham and H. L. Bradt, <i>P.R.</i> 72 , 527. 1947
^{112}In , In^*	8	D. J. Tendham and H. L. Bradt, <i>P.R.</i> 72 , 1118. 1947
$^{113}\text{Cd}^*$		A. C. Helmholz and C. L. Ginnis, <i>Bull.</i> 23 , no 5, <i>E</i> 6. 1948
$^{110}\text{In}^*$	6	A. C. Graves and R. L. Walker, <i>P.R.</i> 71 , 1. 1947
	8	see ^{122}Sb
$^{117}, ^{118}, ^{119}\text{Sb}$		K. D. Coleman and M. L. Pool, <i>P.R.</i> 72 , 1070. 1947
$^{118}\text{Sb}^*$		M. Lindner and I. Perlman, <i>P.R.</i> 73 , 1124. 1948
$^{118}, ^{119}\text{Te}$		see $^{118}\text{Sb}^*$
^{120}Sb		see $^{118}\text{Sb}^*$
^{121}Sn		see $^{118}\text{Sb}^*$
^{121}Te , Te^{**}		J. E. Edwards and M. L. Pool, <i>P.R.</i> 69 , 140. 1946
$^{121}\text{Te}^*$		P. T. Bittencourt and M. Goldhaber, <i>P.R.</i> 70 , 780. 1946
^{122}Sb	7, 8	C. E. Mandeville and M. V. Scherb, <i>P.R.</i> 73 , 340. 1948
$^{122}\text{Sb}^*$		E. Der Mateosian <i>et al.</i> , <i>P.R.</i> 72 , 1271. 1947
^{124}Sb	7, 8	B. D. Kern, D. J. Zaffarano and A. C. G. Mitchell, <i>P.R.</i> 73 , 1142. 1948
$^{124}\text{Sb}^*$, Sb^{**}		see $^{122}\text{Sb}^*$
$^{124}\text{Te}^*$		O. Hirzel, P. Stoll and H. Wäffler, <i>Helv.</i> 20 , 241. 1947
^{125}I		L. E. Glendenin and R. R. Edwards, <i>P.R.</i> 71 , 742. 1947
^{128}I	7, 8	K. Siegbahn and N. Hole, <i>P.R.</i> 70 , 133. 1946
^{129}I	6	S. Katcoff, <i>P.R.</i> 71 , 826. 1947
^{130}Cs		K. Lark-Horovitz, J. R. Risser and R. N. Smith, <i>P.R.</i> 72 , 1117. 1947
^{131}Cs , Ba		Fu-Chun Yu, D. Gideon and J. D. Kurbatov, <i>P.R.</i> 71 , 382. 1947
$^{132}\text{Sb-I}$		see ^{92}Kr
^{132}Cs		<i>F&P</i> 1948
^{133}Ba		S. Katcoff, <i>P.R.</i> 72 , 1160. 1947
$^{134}\text{Sb-I}$		Fu-Chun Yu and J. D. Kurbatov, <i>P.R.</i> 74 , 34. 1948
$^{134}\text{Cs}^*$	8	see ^{92}Kr
	7, 8	K. Siegbahn and M. Deutsch, <i>P.R.</i> 71 , 483. 1947
	7, 8	L. G. Elliot and R. E. Bell, <i>P.R.</i> 72 , 979. 1947
^{137}Cs	7, 8	see $^{77}\text{Se}^*$, ^{203}Hg
$^{137}\text{Ba}^*$		J. Townsend <i>et al.</i> , <i>P.R.</i> 74 , 99. 1948
^{138}La		J. Townsend, N. Cleland and A. L. Hughes, <i>P.R.</i> 74 , 499. 1948
^{139}Ce		M. G. Inghram, R. J. Hayden and D. C. Hess, <i>P.R.</i> 72 , 967. 1947
^{141}Ce	6	see ^{103}Pd
	8	M. L. Pool and N. L. Krisberg, <i>P.R.</i> 73 , 1035. 1948
$^{141}\text{Pr}^*$		see ^{100}Tb
^{141}Nd		see $^{124}\text{Te}^*$
^{145}Sm		O. Huber, O. Lienhard and H. Wäffler, <i>Helv.</i> 17 , 251. 1944
^{148}Pm	7, 8	M. G. Inghram, R. J. Hayden and D. C. Hess, <i>P.R.</i> 71 , 643. 1947
^{149}Pm	6	G. W. Parker <i>et al.</i> , <i>P.R.</i> 72 , 85. 1947
	6	M. G. Inghram <i>et al.</i> , <i>P.R.</i> 71 , 743. 1947
	6	W. Bothe, <i>Z. Naturforsch.</i> 1 , 179. 1946
^{151}Sm	6	see ^{145}Sm
^{152}Sm	6, 7	A. J. Dempster, <i>P.R.</i> 73 , 1125. 1948
^{152}Eu		see ^{145}Sm
$^{152}\text{Eu}^*$		R. J. Hayden and M. G. Inghram, <i>P.R.</i> 70 , 89. 1946
^{153}Sm		R. D. Hill, <i>P.R.</i> 74 , 78. 1948
		W. Bothe, <i>Z. Naturf.</i> 1 , 179. 1946
^{153}Gd		see ^{145}Sm
^{154}Eu	8	see ^{208}Hg
	8	J. M. Cork, R. G. Shreffler and C. M. Fowler, <i>P.R.</i> 72 , 1209. 1947;
		73 , 78. 1948
	6	N. L. Krisberg, M. L. Pool and C. T. Hibdon, <i>Bull.</i> 23 , no 3, <i>Z</i> 10. 1948
^{160}Tb	8	J. M. Cork, R. G. Shreffler and C. M. Fowler, <i>P.R.</i> 73 , 1220. 1948
^{165}Dy	7, 8	H. Slätis, <i>Arkiv</i> 33 A , no. 7. 1946
$^{165}\text{Dy}^*$		M. G. Inghram <i>et al.</i> , <i>P.R.</i> 72 , 515. 1947
^{166}Ho	6	M. G. Inghram and R. J. Hayden <i>P.R.</i> 71 , 130. 1947
	6, 8	W. Bothe, <i>Z. Naturf.</i> 1 , 179. 1946
^{169}Er		B. H. Ketelle and W. C. Peacock, <i>P.R.</i> 73 , 1269. 1948

- ¹⁶⁹Tm* F. K. McGowan and S. DeBenedetti, *P.R.* **73**, 1269. 1948
¹⁶⁹Yb W. Bothe, *Z. Naturf.* **1**, 172. 1946
¹⁷⁰Tm 6, 7, 8 see ¹⁶⁹Yb
¹⁷¹Er, Tm see ¹⁶⁹Er
¹⁷¹Tm* see ¹⁶⁹Tm*
¹⁷⁵Yb see ¹⁶⁹Yb, ¹⁷⁷Yb
¹⁷⁶Lu 6, 7 M. G. Inghram, R. J. Hayden and D. C. Hess, *P.R.* **71**, 144. 1947
¹⁷⁷Yb A. Flammersfeld, *Z. Naturf.* **2**, 86. 1947
¹⁷⁷Lu H. Atterling, E. Bohr and Th. Sigurgeirsson, *Arkiv* **32 A**, no 2. 1945
¹⁸¹Hf 6 M. G. Inghram, R. J. Hayden and D. C. Hess, *P.R.* **71**, 270. 1947
¹⁸¹Hf 7, 8 see ¹⁷⁷Yb
¹⁸¹Ta* 8 A. F. Voigt and B. J. Thamer, *Bull.* **23**, no 4, J 10. 1948
¹⁸¹W 8 J. M. Cork, R. G. Shreffler and C. M. Fowler, *P.R.* **72**, 1209. 1947
¹⁸²Ta 7, 8 S. DeBenedetti and F. K. McGowan, *P.R.* **70**, 569. 1946
¹⁸²Ta* G. Wilkinson, *Nature* **160**, 864. 1947
¹⁸⁵W 6, 8 W. Rall and G. Wilkinson, *P.R.* **71**, 321. 1947
¹⁸⁵Os 6, 8 L. Seren, H. N. Friedlander and S. Turkel, *P.R.* **72**, 163. 1947
¹⁸⁶Re 8 D. Saxon, *Bull.* **23**, no 4, J 9. 1948
¹⁸⁶Re 8 see ¹⁸⁶Re
¹⁸⁶Re 6, 7, 8 L. Katzin and M. Pobereskin, *P.R.* **74**, 264. 1948
¹⁸⁷W 8 L. J. Goodman and M. L. Pool, *P.R.* **71**, 288. 1947
¹⁸⁷Re 6, 7, 8 W. M. Schwarz and M. L. Pool, *P.R.* **71**, 122. 1947
¹⁸⁷Re* 6, 7, 8 S. N. Naldrett and W. F. Libby, *P.R.* **73**, 487. 1948
¹⁸⁸Re 6, 7, 8 S. DeBenedetti and F. K. McGowan, *P.R.* **71**, 380. 1947
¹⁸⁸Re 8 see ¹⁸⁵Os
¹⁹⁰Ir 8 L. C. Miller and L. F. Curtiss, *P.R.* **70**, 983. 1946
¹⁹⁰, ¹⁹¹Pt see ¹⁸⁵Os
¹⁹¹Os 6, 7, 8 G. Wilkinson, *P.R.* **73**, 252. 1948
¹⁹²Ir 6, 8 see ¹⁸⁵Os
¹⁹³As 6, 8 see ¹⁸⁵Os
¹⁹³Pt, Au 8 see ¹⁸⁵W
¹⁹⁴Ir 6, 7, 8 J. M. Cork, R. G. Shreffler and G. M. Fowler, *P.R.* **72**, 1209. 1947
¹⁹⁵Au see ¹⁹⁰Pt
¹⁹⁷Au*, Hg, Hg* see ¹⁸⁵Os
¹⁹⁸Au 6 H. Frauenfelder et al., *Helv.* **20**, 338. 1947
¹⁹⁸Au 7 N. Feather and J. Dainty, *Proc. Cambr. phil. Soc.* **40**, 57. 1944
¹⁹⁸Au 7, 8 S. Jnanananda, *P.R.* **70**, 812. 1946
¹⁹⁸Au 8 P. W. Levy and E. Greuling, *P.R.* **73**, 83. 1948
¹⁹⁸Au 8 J. DuMond, D. A. Lind and B. B. Watson, *P.R.* **73**, 1392. 1948
²⁰³Hg 7, 8 C. E. Mandeville and M. V. Scherb, *Bull.* **23**, no 5, I 9. 1948
²⁰⁴Tl 7, 8 M. L. Wiedenbeck and K. Y. Chu, *P.R.* **72**, 1164. 1947
²⁰⁴Pb*, Bi see ²⁰⁸Tl
²⁰⁶Tl see ²⁰⁶Bi
²⁰⁶Bi E. Broda and N. Feather, *Proc. R. Soc.* **190**, 20. 1947
²⁰⁶Bi J. J. Howland, D. H. Templeton and I. Perlman, *P.R.* **71**, 552. 1947
²⁰⁶, ²⁰⁷, ²⁰⁸Po D. H. Templeton, J. J. Howland and I. Perlman, *P.R.* **72**, 758. 1947
²⁰⁸Bi see ²⁰⁶Bi
²⁰⁸Bi 9 J. M. McElkenney, A. O. Hansen and B. B. Duffield, *Bull.* **23**, no 4, D 2. 1948
²⁰⁹Tl *F&P* 1948
²⁰⁹Pb 8 see ¹⁸²Ta
²¹⁰Bi 7 see ²⁰⁰Tl
²¹³Bi, Po F. Hageman et al., *P.R.* **72**, 252. 1947
²¹⁵Po A. C. English et al., *P.R.* **72**, 253. 1947
²¹⁵, ²¹⁶At B. Karlik and T. Bernert, *Z. Physik* **123**, 51. 1944
²¹⁸Em A. Ghiorso, W. W. Meinke and G. T. Seaborg, *P.R.* **74**, 695. 1948
²¹⁸, ²²⁰Fr M. H. Studier and E. R. Hyde, *P.R.* **74**, 591. 1948
²²¹Fr see ²¹⁵At
²²¹Fr see ²¹³Bi
²²²Ra see ²¹⁸Em
²²³, ²²⁴Ac see ²¹⁵At
²²⁵Ra, Ac see ²¹³Bi
²²⁶Th see ²¹⁸Em
²²⁷, ²²⁸Pa see ²¹⁵At
²²⁷Th see ²¹³Bi
²²⁹Pa G. T. Seaborg, *Chem Eng. News* **26**, 1902. 1948
²³⁰U see ²¹⁸Em

$^{230}, ^{232}\text{Pa}$		see ^{218}Em
^{232}U		J. M. Cork, <i>Radioactivity and Nuclear Physics</i> , N. Y. 1947
^{233}Pa	8	P. W. Levy, <i>P.R.</i> 72 , 352. 1947
^{233}U		see ^{213}Bi
$^{234}\text{Pa}, \text{Pa}^*$	7, 8	H. Bradt and P. Scherrer, <i>Helv.</i> 18 , 405. 1945
$^{234}, ^{235}, ^{236}\text{Np}$		see ^{232}U
$^{236}, ^{237}\text{Pu}$		see ^{232}U
^{238}Np		see ^{232}U
^{238}Pu		see ^{232}U
	8	see ^{239}Pu
^{239}U	7, 8	N. Feather, <i>Nature</i> 160 , 749. 1947
^{239}Np	7, 8	H. Slätis, <i>Nature</i> 160 , 579. 1947
^{239}Pu	8	W. P. Jesse and H. Forstat, <i>P.R.</i> 73 , 926. 1948
^{240}Cm		see ^{232}U
$^{241}\text{Pu}, \text{Am}$		see ^{232}U
^{241}Cm		see ^{239}Pa
^{242}Am		G. T. Seaborg, <i>Chem. Eng. News</i> 25 , 358. 1947
^{242}Cm		see ^{232}U

CONTENTS

CONTENTS

Preface	VII—X
Notations	XI—XII
N. 1 Mathematical symbols	
.11 Numbers and observables, XI — .12 Vectors and tensors, XI — .13 Functions of dynamical variables, XI — .14 Legendre polynomials and tesseral harmonics, XII.	
N. 2 System of references	
.21 Numbering of sections, etc., XIII — .22 References to literature, XIII.	
Units and constants	XIV—XV
U. 1 Units	
.11 Relativistic correlation of units, XIV — .12 Mass and energy units, XIV.	
U. 2 Constants	
.21 Fundamental constants, XIV — .22 Nuclear constants, XV.	
Introduction	XVI—XIX

PART I

General features of nuclear forces

Chapter I. Properties of elementary particles

1.1	Leptons	3—5
	.11 Exclusion principle and anti-particles, 3 — .12 Types of β -processes, 3 — .13 Energy balances of β -processes, 4 — .131 Stability with respect to β -transitions, 4 — .132 Existence of the neutrino, 5.	
1.2	Nucleons	5—10
	.21 The proton, 5 — .22 The neutron, 7 — .23 The anti-proton, 8 — .231 The anti-neutron, 10.	
1.3	Mesons	10—21
	.31 The field concept, 10 — .32 The meson field, 11 — .321 The mass-range relation, 12 — .322 Interaction of meson field with leptons, 14 — .33 The mesons of cosmic radiation, 14 — .331 The meson mass, 15 — .332 The meson decay, 16 — .333 The meson spin, 18 — .34 Many-body forces, 18 — .35 Excited states of nucleons, 19.	

Chapter II. Nuclear radii and mass-defects

2.1	Nuclear radius and range of nuclear forces	22—23
2.2	Mass-defects and saturation of nuclear forces	23—27
	.21 Mass-defects, 23 — .22 Saturation properties of nuclear forces, 24.	
2.3	Kinetic effects	27—28

Chapter III. Charge dependence of nuclear forces

3.1	Proton and neutron pairing	29—30
	.11 Classification of nuclei, 29 — .12 Proton and neutron pairing, 29.	

- 3.2 Stability of isobars** 30—33
 .21 Stable isobars, 30 — .22 Atomic weights of isobars, 31 — .221 Comparison with measurements of mass-defects, 33.
- 3.3 Charge symmetry of nuclear forces** 33—35
- 3.4 Building up of stable nuclei** 35—40
 .41 The energy surface, 35 — .42 Stability conditions, 37 — .43 Properties of the critical values of the neutron excess, 39.

PART II

Two-nucleon systems on the hypothesis of central interaction

Chapter IV. Dynamical variables and fundamental equations

- 4.1 Dichotomic variables** 43—50
 .11 Definition and main properties, 43 — .111 Spin, 44 — .12 Transformation of eigenfunctions, 45 — .13 Exchange operators for two identical particles, 46 — .14 Equivalent formulations of the exclusion principle, 47.
- 4.2 Wave-equation of the nucleon** 50—52
 .21 Linearization of wave-equation, 50 — .22 Isotopic variable, 51 — .221 Interaction with electromagnetic field, 52.
- 4.3 Reduction of the wave-equation of the two-nucleon system** 53—62
 .31 The fundamental wave-equation, 53 — .311 Reduction to the barycentric system of reference, 54 — .32 Classification of eigenstates by angular momentum, 55 — .33 Reduction to "large" components, 56 — .331 Properties of the large components, 57 — .34 Central potential, 59 — .341 Exchange potentials, 61.
- 4.4 Electromagnetic interactions** 62—65
 .41 The exchange operators, 62 — .42 The exchange moments, 63 — .43 The electric dipole moment and Siegert's theorem, 64.

Chapter V. The stationary states of the proton-neutron system

- 5.1 Properties of short-range central potentials** 66—75
 .11 Types of short-range central potentials, 66 — .111 Normalization of radial eigenfunctions, 68 — .12 Behaviour of radial eigenfunction at small and large distances, 69 — .121 Some useful identities, 71 — .13 General behaviour of eigenfunctions of *S* states, 71 — .131 Repulsive potential, 72 — .132 Attractive potential, 72 — .133 Attractive potential modified by very short range repulsion, 74 — .14 Eigenfunctions of states of higher orbital momentum, 74.
- 5.2 Stationary states of binding** 75—81
 .21 The *S* states of the potential well, 75 — .211 The *S* states of the exponential potential, 76 — .22 The *S* states of Hulthén's potential, 76 — .23 The variational method, 78 — .231 The *S* states of the meson potential, 79 — .24 The variation-iteration method, 80.
- 5.3 Calculation of the phases** 81—89
 .31 General variational method, 81 — .32 Estimate of *S*-phase for nearly critical values of potential strength, 82 — .321 Calculation of *S*-phases for a potential well, 86 — .3211 Calculation of *S*-phases for an exponential potential, 87 — .322 Calculation of *S*-phases for Hulthén's potential, 87 — .323 Calculation of *S*-phases for the meson potential, 88 — .33 Estimate of phases of states with higher orbital momentum, 88 — .331 More accurate estimate, 89.

Chapter VI. Physical properties of the proton-neutron system

- 6.1 The ground state of the deuteron** 90—98
 .11 Energy and angular momentum, 90 — .12 Electromagnetic properties, 90 — .121 Calculation of the deuteron quadrupole moment, 94 — .122 Para-ortho conversion, 95 — .13 Magnetic interaction of proton and neutron, 95.
- 6.2 Scattering of slow neutrons by protons** 98—111
 .21 Theory of proton-neutron scattering, 98 — .211 Case of the well potential, 101 — .212 Transformation to laboratory system, 101 — .213 Scattering of slow neutrons by bound protons, 103 — .22 Experiments on the scattering of slow neutrons by protons, 106 — .23 The S level of small energy of the deuteron, 110 — .231 Radiative capture of slow neutrons, 111.
- 6.3 Slow neutron scattering by hydrogen molecules** 111—119
 .31 Experiments, 112 — .32 Theory, 114 — .33 Analysis of experiments, 115 — .34 The neutron spin, 116 — .35 Scattering of polarized neutrons by protons, 118.
- 6.4 Scattering of fast neutrons by protons** 119—131
 .41 Total scattering cross-section of protons for fast neutrons, 119 — .411 Scattering of neutrons of energies 0.1 ... 1 MeV, 119 — .412 Scattering of neutrons produced by the $D(d, n)$ reaction, 120 — .413 Scattering of very fast neutrons by protons, 120 — .42 Angular distribution of proton-neutron scattering, 122 — .43 Comparison with theory, 126 — .431 Scattering by potential well, 127 — .432 Scattering by other types of central potential, 130.
- 6.5 Radiative processes** 132—141
 .51 The photodisintegration of the deuteron, 132 — .52 Radiative capture of neutrons by protons, 135 — .53 Disintegration of the deuteron by electron impact, 138.

Chapter VII. Central interaction between protons

- 7.1 Proton-proton scattering** 142—155
 .11 Theory, 142 — .12 Analysis of experiments, 146 — .13 Derivation of nuclear potential parameters from the observed phases, 151 — .131 Comparison of second group of scattering data with theory, 153 — .14 Range of potential from disintegration "stars", 154.
- 7.2 Deuteron formation by proton collisions** 155—157

Chapter VIII. The charge independence of nuclear interaction

- 8.1 Comparison of proton-neutron and proton-proton potential** 158—160
- 8.2 General form of central nuclear potential** 160—162
 .21 Traces of operators related to the nuclear potential, 162
- 8.3 Charge independent interaction and meson field** 162—180
 .31 Neutral and symmetrical meson theories, 162 — .311 Instability of the neutral meson, 167 — .32 Central static interactions on meson theory, 170 — .33 Scattering of fast neutrons by protons, 171 — .34 Photodisintegration of the deuteron and meson theory, 175.
- 8.4 Central nuclear potential: summary of argument** 180—181

PART III

Nuclear models and saturation properties on central force hypothesis

Chapter IX. Survey of nuclear models

9.1	Description of heavy nuclei	185—189
	.11 Bohr's droplet model, 185 — .12 The Fermi gas model, 188.	
9.2	Description of light nuclei	189—191
	.21 The quasi-atomic model, 189 — .22 Nuclear models of the collective type, 190 — .23 The Wigner approximation, 191.	
9.3	A table of nuclear models	191—192
9.4	Elementary properties of Fermi gas model	192—198
	.41 Kinetic energy of ground state (absolute zero), 192 — .42 Kinetic energy of excited states, 194 — .43 Kinetic properties of nuclear matter, 195.	

Chapter X. Nuclear multiplets

10.1	Classification of nuclear states	199—203
	.11 General invariance properties, 199 — .12 Charge multiplets, 199 — .13 Spin multiplets, 200 — .14 Wigner supermultiplets, 200.	
10.2	Stationary states of light nuclei	203—207
10.3	Energies of nuclear states	208—216
	.31 Nuclear potential energy in a supermultiplet state, 208 — .311 Supermultiplet quantum numbers of ground state, 211 — .32 Mass-defects of light nuclei on Wigner's approximation, 212 — .33 Rough estimate of spin dependent interactions, 214 — .34 Coulomb energy of light nuclei, 215.	

Chapter XI. Saturation properties of nuclear forces

11.1	The saturation requirements	217—224
	.11 Decomposition of mean nuclear energy into ordinary and exchange terms, 217 — .12 Saturation properties of exchange interaction terms, 222 — .121 Velocity dependent potentials, 222 — .13 The saturation requirements, 224.	
11.2	The first saturation requirement	224—230
	.21 Necessary saturation conditions, 224 — .22 Sufficient saturation conditions, 227.	
11.3	The second saturation requirement	231—234
	.31 Light α -nuclei, 231 — .32 Heavy nuclei, 231 — .33 Determination of interaction parameters, 233.	
11.4	Stability of isobars	234—237
	.41 Ground state of odd isobar, 237.	

Chapter XII. Properties of heavy nuclei

12.1	Mass-defects and nuclear radii	238—243
	.11 The volume energy, 238 — .12 The surface energy, 239 — .13 Nuclear radii, 241 — .14 Variation of density within a nucleus, 242.	
12.2	Standard heavy nucleus to second approximation	243—258
	.21 The volume energy, 243 — .22 The momentum distribution, 250. — .23 The spatial correlations, 254 — .24 The virial theorem, 257.	

- 12.3 Penetration of fast nucleons into heavy nuclei** 258—268
 .31 Differential collision cross-section, 258 — .32 Total collision cross-section, 261 — .33 Energy loss of a fast nucleon in nuclear matter, 263 — .34 Range of fast nucleon in nuclear matter, 266 — .35 Theory of disintegration stars, 266 — .36 Determination of range of nuclear force, 268.

Chapter XIII. Collective nuclear models

- 13.1 The α -particle model** 269—281
 .11 Empirical evidence on α -particle interaction, 270 — .12 Theory of the interaction between two α -particles, 271 — .13 Binding energies of α -nuclei, 275 — .14 Lability of α -clusters, 276 — .15 Rotations and vibrations of α -nuclei, 277.
13.2 Nuclear states on α -particle model 281—287
 .21 Stationary states of lightest α -nuclei, 281 — .211 Comparison with experiment, 283 — .22 Stationary states of nuclei of mass number $4a \pm 1$, 285.
13.3 The resonating group structure 287—289

Chapter XIV. Properties of light nuclei

- 14.1 The three-nucleon systems** 290—301
 .11 Stationary states on resonating group method, 290 — .111 Doublet states on individual model, 293 — .12 Scattering of neutrons by deuterons, 295 — .13 Scattering of protons by deuterons, 299 — .14 Radiative capture of thermal neutrons by deuterons, 300 — .15 Disintegration of the deuteron by proton or neutron impact, 300.
14.2 Binding energies of the lightest nuclei 301—307
 .21 Data and methods, 301 — .22 Discussion of results, 304 — .23 Argument for like-nucleon forces, 306 — .24 Stationary states of ${}^6\text{Li}$ and ${}^6\text{He}$, 306.

Summary of Part III

308

PART IV

Non-central and non-static couplings

Chapter XV. General types of non-central and non-static nuclear forces

- 15.1 General empirical evidence** 311
15.2 Survey of nuclear interactions 311—319
 .21 Static and first order interactions, 311 — .22 Second order interactions, 314.
15.3 Non-central and non-static couplings on meson theory 319—332
 .31 Static interactions due to meson fields, 320 — .32 Cut-off and mixture, 323 — .33 Non-static interactions due to spin precession and charge exchange, 324 — .34 Velocity dependent coupling on mixed meson theory, 327 — .35 Spin and life-time of mesons, 329.

Chapter XVI. Non-central couplings and two-nucleon systems

- 16.1 General considerations** 333—337
 .11 The ${}^3D$ -admixture to the ground state of the deuteron, 333 — .12 The axial dipole coupling as a perturbation, 335 — .13 Stationary states of not too high energy, 336.

16.2	The Rarita-Schwinger theory	337—342
	.21 The ground state of the deuteron, 338 — .22 Proton-neutron scattering, 338 — .23 Radiative processes, 341.	
16.3	Meson theories with cut-off	342—345
	.31 Symmetrical pseudoscalar theory, 342 — .311 Symmetrical zero spin meson theory, 343 — .32 Neutral vector theory, 344 — .33 Unsymmetrical zero spin meson theory, 345.	
16.4	Mixed theories	346—351
	.41 Schwinger's mixed theory, 346 — .42 Møller and Rosenfeld's mixed theory, 348 — .43 "Large" or "small" axial dipole coupling, 350.	
16.5	General conclusion	351—352
 Chapter XVII. Non-central couplings and properties of heavier nuclei		
17.1	Non-central couplings and saturation properties	353—355
17.2	Non-central couplings and binding energies of light nuclei	355—356
17.3	Fine structure of nuclear levels	356—357
17.4	Fine structure of ${}^5\text{He}$ and ${}^5\text{Li}$	357—368
	.41 Theory of the scattering of nucleons by Helium, 358 — .42 Analysis of scattering experiments, 363 — .43 Fine structure of ${}^5\text{He}$ and spin-orbit coupling, 365.	
17.5	Fine structure of ${}^7\text{Li}$	369—373
	.51 Empirical data, 369 — .52 Discussion of data, 369 — .53 Relativistic doublet splitting of ${}^7\text{Li}$, 372.	

APPENDICES

Appendix I. Allowed β -transitions

A1.1	Allowed transitions on Wigner's approximation	377—382
	.11 General expression of life-time, 377 — .12 Selection rules, 378 — .13 Calculation of matrix-elements, 380 — .131 Matrix-elements for class (a) nuclei of neutron excess —1, 380 — .132 Matrix-elements for class (b) nuclei, 381.	
A1.2	Discussion of empirical data	382—389
	.21 Class (a) nuclei of neutron excess —1, 382 — .211 Life-time of the neutron, 385 — .212 The β -decay of ${}^3\text{H}$ and the mass of the neutrino, 385 — .22 Class (b) nuclei, 387.	

Appendix II. Electromagnetic properties of nuclei

A2.1	Multipole moments of a system of charged particles	390—393
	.11 System in slowly varying external field, 390 — .12 Magnetic moment in spectroscopic sense, 392 — .13 Electric quadrupole moment in spectroscopic sense, 392 — .14 Higher multipole moments, 393.	
A2.2	Magnetic moments of nuclei	393—420
	.21 The empirical data, 393 — .22 General expression for magnetic moment, 397 — .221 Conjugate and self-conjugate nuclei, 401 — .23 The distribution of the magnetic moments of odd mass nuclei, 402 — .24 Isotopes with equal magnetic moments, 404 — .25 Magnetic moments of light odd mass nuclei, 405 — .251 The conjugate nuclei ${}^3\text{H}$ and ${}^3\text{He}$, 405 — .252 The " $2p$ shell" nuclei, 409 — .253 The " $2s$ shell" nuclei, 414 — .26 Magnetic moments of stable odd nuclei, 414 — .261 Unstable odd nuclei, 416 — .27 Relativistic correction to the magnetic moment, 417.	

A2.3	Quadrupole moments of nuclei	420—426
	.31 The empirical data, 420 — .32 Quadrupole moments of odd mass nuclei and nuclear models, 422 — .321 Quadrupole moment due to a single proton, 423 — .322 Quadrupole moments of light nuclei, 425 — .33 Quadrupole moments of stable odd nuclei, 426.	

Appendix III. Strong coupling theory of nuclear forces

A3.1	Excited states of nucleons	427—428
	.11 Theory, 427 — .12 Scattering of mesons by nucleons, 428 — .13 Electromagnetic properties of nucleons, 428.	
A3.2	Properties of the deuteron	428—432
	.21 Static interaction, 428 — .22 The S states of the deuteron, 429 — .23 Fast neutron scattering by protons, 431 — .24 Charge independence of nuclear interactions, 431 — .25 Excitation of isobaric states, 431.	
A3.3	Saturation requirements	432—433
Addenda et Corrigenda		437—456

INDEXES

Author Index	459—476
Subject Index	477—491
List of Tables	492—493

TABLE OF ATOMIC NUCLEI

Introduction	497—500
Description of table	497
List of references	498
Additional remarks	498—500
Mass-defects of light nuclei ($A < 65$), 498 — Mass-defects of heavier nuclei ($A = 65 \dots 205$), 499 — Pairs of neighbouring "stable" isobars, 499 — Mass-defects of the heavy radioactive nuclei, 499.	
Table	501—528
References	529—534

CORRIGENDUM.

D i a g r a m I V. Third line. Read $^{187}_{76}\text{Os}$ instead of $^{187}_{67}\text{Os}$.

DATE OF ISSUE

This book must be returned within 3/7/14 days of its issue. A fine of ONE ANNA per day will be charged if the book is overdue.

[illegible]

**SEVEN
DAY
BOOK**